Jumping and Escaping: Modular Termination and the Abstract Path Ordering☆

Nachum Dershowitz

School of Computer Science, Tel Aviv University, Ramat Aviv, Israel

Abstract

Combinatorial commutation properties for reordering a sequence consisting of two kinds of steps, and for separating the well-foundedness of their combination into well-foundedness of each, are investigated. A weak commutation property, called "jumping", along with a weakened version of the lifting property, called "escaping" and requiring only an *eventual* lifting, are used for proving well-foundedness of a generic, abstract version of the recursive path orderings.

Keywords: termination, commutation, path orderings, well-founded orderings

For Yoshihito Toyama on the happy occasion of his sixtieth birthday.

At the foundation of well-founded beliefs lies belief that is not well-founded.

—Ludwig Wittgenstein, *On Certainty*, §253

1. Motivation

One of the many topics Toyama investigated is the preservation of termination (strong normalization) of combinations of term-rewriting systems [31, 32]. Here we are interested in sufficient conditions for preservation of termination of combinations of abstract relations, that is, for the union of two binary relations to be terminating when each of the two is terminating on its own.

We apply combinatorial properties of combinations of relations to demonstrate well-foundedness of orderings typically used for proving termination of rewriting systems. The various "path orderings" [9] provide a convenient and popular method of proving termination, particularly of term-rewriting systems.

☆This work extends, and corrects details of, "On lazy commutation", in *Languages: From Formal to Natural (Essays Dedicated to Nissim Francez on the Occasion of His 65th Birthday)*, Lecture Notes in Computer Science, vol. 5533, Springer-Verlag, Berlin, 2009, pp. 59–82, and "An abstract path ordering", presented at the *11th International Workshop on Termination (WST)*, Edinburgh, July 2010.

We set out to prove the well-foundedness of a very abstract ordering that includes the usual path orderings as special cases. Abstract versions of the path orderings have already been proposed in [18, 6]; we aim for a more general version yet, and develop new proof techniques for this purpose. In particular, we make use of a remarkably fruitful condition of Doornbos, Backhouse, and van der Woude [13], denominated here "jumping", which guarantees modular termination.

Path orderings are defined by recursion on subterms, usually building a term ordering from a symbol ordering, called a "precedence". To this end, the multiset ordering of unordered tuples [12] is used in the original multiset path ordering [8], as is a lexicographic comparison of ordered tuples in its generalizations [21]. But there is a noticeable difference between these two types of sequence orderings. Given an infinite descending sequence in the multiset ordering, one can be sure that there is an infinite descending sequence of elements of the underlying set that begins with some element of the very first multiset. One says that the element ordering is "lifted" to the level of multisets. In the lexicographic case, on the other hand, one only knows that some tuple along the way has a component that initiates an infinite descending sequence in the underlying set of elements. This distinction leads us to investigate sequences with the latter, weaker property, which we call "escaping".

As the prime application of jumping and escaping, we explain in the next section why the method developed here demonstrates that a quite general abstraction of the path orderings is in fact well-founded. Then, in Section 3, after agreeing on notations and some terminology, we collect some basic observations on modularity. In particular, we explain the method of constriction, pioneered by Plaisted [27], of which we make crucial use in our proofs. Section 4 considers various more powerful commutation properties that are applicable to sequences mixing two kinds of steps. A weaker notion than jumping, defined in Section 5 and dubbed "selection", suffices to prove the main modularity result. Section 6 discusses the lifting of termination of one relation to that of the other in the presence of selection; escaping is also explained and used with jumping for the same purpose. A hint of future possibilities concludes this article.

2. The Path Ordering

The relations we care about in this work are binary; whenever we speak of a "relation", it should be understood to be binary. In what follows, the underlying (finite or infinite) set of "points" V is fixed, with relations A, B, R, S, etc., connecting pairs of points via (directed) "steps" that are "colored" A, B, etc. Points may be connected by steps of more than one color.

The *immortal* elements of a relation R over V are those $v \in V$ for which there exists an infinite R-chain initiated by v and consisting of (not necessarily distinct) elements of V; $v\ R\ v'\ R\ v''\ R \cdots$.[1] *Termination* of R means that no

[1]These are the non-strongly-normalizing elements in R. The term "immortal" was sug-

element is immortal. We will also say in this case that R is *well-founded*, even when R is not a partial order.

2.1. Main Result

The two main properties of a pair of relations that we need for the well-foundedness of the path ordering are jumping and escaping. The first property gives modularity, making the union of relations A and B terminating if both are. The second then reduces termination of B to that of A. Since the path ordering will be composed of A- and B-steps, this means that it is well-founded, as long as A is.

Definition 1 (Jumping [13, 14]). Relation A *jumps over* relation B if

$$BA \quad \subseteq \quad A(A+B)^* + B \; .$$

We are using juxtaposition for composition of binary relations, the plus sign for union, and the star operation for reflexive-transitive closure. So this condition means that if there is a path $s\ B\ t\ A\ u$ from point s to point u via a B-step to some point t followed by an A-step from t, then either there is an alternate route beginning with an A-step followed by any mix of A- and B-steps or else one can get directly from s to u by a lone B-step.

It is shown in [13] (Corollary 55 below) that jumping guarantees that the union $A + B$ is terminating whenever each of A and B is.

Definition 2 (Escaping). Relation A *escapes from* relation B if there is some point in every infinite B-chain from which an A-step leads to a point that is immortal in the union $A + B$.

Consider, for example, $>$ on the natural numbers (which is well-founded), plus $>$ of the negative integers (which is not), but no relation between positive and negative. Let B be $>$ combined with a lexicographic comparison of pairs of integers, and let A give either component. Then, in any infinite B-sequence of pairs, either the first component of the initial pair is immortal (i.e. negative), or else—from some point on—the second is immortal. So A escapes from B.

Our main result (Theorem 63 below) is the following:

Main Theorem. *If relation A jumps over relation B and escapes from B, then their union $A + B$ is well-founded if and only if A is.*

---

gested in a related context by Friedrich Otto ("On the property of preserving regularity for string-rewriting systems", *Proc. Rewriting Techniques and Applications*, 1997) and is much more succinct and perspicacious than "non-strongly normalizing".

2.2. The Abstract Ordering

We apply the main theorem to an abstract version of the path orderings. What distinguishes the various path orderings in the literature is the use of a surface ordering, like a precedence or numeric interpretation, together with a recursive comparison of subterms.

Suppose the set V of points (think terms, if you wish) is equipped with a well-founded partial ordering $\rhd$ that provides some structure on V (think subterms, or "special" subterms of some sort, or something more elaborate). This will provide our A steps. Suppose further that we have some kind of "surface" relation $\gg$ on V (think precedences, lexicographically ordered tuples, limited forms of rewriting, and the like). Let's call t' a *descendant* of V if $t \rhd t'$ and say that u *follows* t if $t \gg u$. We define t to be bigger than u in the *(abstract) path ordering*, written here as $t \succ u$ $(t, u \in V)$, if either of two cases holds, in a "two-dimensional" recursive way. The second case will be our B steps.

Definition 3 (Path Ordering). The *(abstract) path ordering (apo)* is a relation $\succ$ (not necessarily transitive) on some set V, parameterized by three other abstract relations, $\gg$, $\vartriangleright$, and well-founded $\rhd$. It is defined as the smallest relation $\succ$ such that

$$t \succ u \quad \text{if} \quad \begin{cases} t \vartriangleright u \text{ and } t \rhd^{+} \succ^{*} u, \text{ or} & \text{(a)} \\ t \gg u \text{ and } t\, (\rhd^{+} \succ^{*} + \succ)/\rhd\, u\,. & \text{(b)} \end{cases}$$

The division-like notation is defined, for binary relations R and S, as follows:

$$R/S = \{\langle x, y\rangle \colon \forall z.\; ySz \Rightarrow xRz\}\;.$$

In other words, $x(R/S)y$ means that xRz for all z such that ySz. (Division is the upper adjoint [residual mapping] of joining with $\rhd$ with respect to inclusion.)

Strictly speaking, this abstract path ordering need not be an ordering, as it can be non-transitive. It is a generalization of the abstract ordering of [18]. The following is an alternative mutually-recursive definition of $\succ$, which, together with its transitive closure $\succ^{*}$, can be implemented "bottom-up":

$$\succ \;:=\; (\vartriangleright \cap \rhd^{+} \succ^{*}) + \sqsupset \qquad \text{(a}'\text{)}$$

$$\sqsupset \;:=\; \gg \cap\, (\rhd^{+} \succ^{*} + \sqsupset)/\rhd\;\;. \qquad \text{(b}'\text{)}$$

We can have $\sqsupset$ on the right side of the second line instead of $\succ$, as appears in case (b) of the original definition of $\succ$, since case (a) of $\succ$ is subsumed by the first alternative, $\rhd^{+} \succ^{*}$. Note that (by induction on the computation) $\succ \subseteq (\rhd + \sqsupset)^{+}$.

The purpose of $\vartriangleright$ is to optionally constrain the first case, so the ordering need not have the abstract "subterm" property $\rhd \subseteq \succ$. For the defined relation to be useful, we will soon connect the followed-by relation $\gg$ with the bigger-than relation $\succ$ on descendants via $\rhd$, in a fashion that guarantees well-foundedness.

Lemma 4. *For the path ordering, relation $\rhd$ jumps over $\sqsupset$.*

PROOF. By the division in (b′), $\sqsupset \rhd \subseteq \rhd^{+}\succ^{*} + \sqsupset$, exactly as required. $\square$

A simple version of this ordering has $\gg$ transitive and $\rhd$ universal (the full Cartesian product), in which case the definition simplifies to one that does obey the subterm condition, namely:

$$\succ \quad := \quad \rhd\succeq + [\gg \cap \succ/\rhd] \;,$$

where $\succeq$ means $\succ$ or equal, since, in this simple case, $\succ$ is a partial order:[2]

Theorem 5. *The path ordering $\succ$ is transitive if $\gg$ are transitive and $\rhd$ is universal.*

PROOF. Let $\sqsupset$ be short for $\gg \cap \succ /\rhd$. We proceed by induction with respect to well-founded $\rhd$ in any of the three positions, s, t, or u in $s \succ t \succ u$. (i) If $s \rhd s' \succeq t$, then $s' \succ u$ by induction in the first position and $s \succ u$ by definition. (ii) If $s \sqsupset t \rhd t' \succeq u$, then $s \succ t' \succeq u$ on account of the division clause and $s \succ u$ by induction in the second position. (iii) Finally, if $s \sqsupset t \sqsupset u$, then we have $s \gg u$ and $s \succ t' \succ v$ for all $v \lhd u$. By induction in the third position, $s \succ v$ for all $v \lhd u$, from which it follows that $s \sqsupset u$ and, hence, $s \succ u$. $\square$

2.3. Well-Foundedness

Modularity of termination follows from jumping (Corollary 55 below with $\rhd$ for A and $\sqsupset$ for B). In other words, $\succ \subseteq (\rhd + \sqsupset)^{*}$ is well-founded if $\sqsupset$ is (since $\rhd$ is). Of course, $\sqsupset$ is well-founded if the "followed by" relation $\gg$ is. So:

Proposition 6. *A path ordering $\succ$ is well-founded whenever $\gg$ is.*

This works, as is, for simple interpretation-based termination orderings, as when $\gg$ compares terms on the basis of an arbitrary interpretation in the naturals. The problem is that, for path orderings, $\gg$ is normally defined recursively in terms of $\succ$ applied to subterms.

We will demonstrate the following (Corollary 62 using Lemma 4):

Theorem 7. *A path ordering $\succ$ is well-founded if $\rhd$ lifts to $\sqsupset$.*

A minori ad maius this is so if it lifts to $\gg$.

This applies to the nested multiset ordering [12], where $\gg$ is an element ordering combined with its nested multiset extensions, since the first item of an infinite sequence of multisets must have an element that is immortal in the same ordering. So $\gg$ is well-founded if its elemental part is. The general case of such "lifted" definitions was first studied in [21] and pursued further in [15, 18].

Using the more powerful Theorem 63 below, we arrive inescapably at

Theorem 8. *A path ordering $\succ$ is well-founded if $\rhd$ escapes from $\sqsupset$.*

[2] I thank Alfons Geser for this result.

For the multiset [8] path ordering, $\rhd$ is the transitive proper-subterm relation combined with extracting the head symbol; $\unrhd$ is always true; and $\gg$ is the recursive lifting of $\succ$ to first compare precedences and then the multiset of subterms. For the lexicographic [21] and (mixed) recursive path orderings [24, 9], $\gg$ looks at precedence-cum-tuples lexicographically, and at a mixture of tuples and multisets, respectively, from both of which subterm ($\rhd$) escapes.

3. Mortality and Separation

3.1. Mortality

We denote the reflexive, transitive, and reflexive-transitive closures of a relation R by R^ε, R^+, R^*, respectively. The n-fold composition of R is R^n. Let $\varnothing$ be the empty relation and $\mathbf{1}$, the identity relation. So, as usual, $R^0 = \mathbf{1}$, $R^1 = R$, $R^{i+j} = R^i R^j$, and $\mathbf{1}R = R\mathbf{1} = R$, for any R. We will also have recourse to the notation $R^{1:n}$ for $R^1 + \cdots + R^n$.

Define

$$R^\infty = \{\langle u, v\rangle \colon u, v \in V,\ u \text{ is immortal for } R\} ,$$

relating immortal elements to all vertices (as is commonly done in relational program semantics). Thus,

$$R^\infty S \subseteq R^\infty , \tag{1}$$

for any relation S. Relation R is well-founded, or *terminating*, if $R^\infty = \varnothing$. Clearly $R^* R^\infty = R^\infty$.

When we say that a sequence of steps has the "form" $A^* B^+$, for example, we mean that it comprises some A-steps followed by one or more B-steps.

We will make frequent use of *well-founded* (or *noetherian*) induction. If relation A on V is terminating, then, to show a property $R \subseteq S$, one may assume $A^n R \subseteq A^n S$ for all $n > 0$. Even if A is not terminating, we may conclude $R \subseteq S + A^\infty$, or, equivalently, $R \setminus A^\infty \subseteq S$.

3.2. Finite and Infinite Separations

This article explores properties of the (non-disjoint) union $E = A + B$ of two relations, A and B. Throughout, we will regularly use E, without elaboration, to denote the union $A + B$ of the two relations under consideration.

Two main properties will claim our attention.

Definition 9 (Finite Separation). Two relations A and B are *finitely separable* if

$$(A + B)^* = A^* B^* .$$

Definition 10 (Infinite Separation). Two relations A and B are *infinitely separable* if

$$(A + B)^\infty = (A + B)^* (A^\infty + B^\infty) .$$

The point is that infinite separability ensures that the union $E = A + B$ is well-founded if each of A and B is, even without finite separability.

Example 11. (a) An easy example of finite, but not infinite, separability is $s\ A\ t\ B\ s$. (b) A simple example of infinite, but not finite, separability is $s\ A\ s\ BA\ t$.

Nota bene. Finite separation is not a symmetric property: we want the A's before the B's. Infinite separation, on the other hand, is symmetric.

It is easy to see [30, Ex. 1.3.5(ii)] that finite separability (called "postponement" in [30]) is equivalent to the (global) "commutation" property $B^*A^* \subseteq A^*B^*$. Another trivial induction also shows that this is equivalent to an even simpler property:

Proposition 12 ([29]). *Two relations A and B are finitely separable if, and only if, $B^+A \subseteq A^*B^*$.*

This condition cannot be weakened further to the purely local property, $BA \subseteq A^*B^*$, which holds in the inseparable example, $s\ B\ t\ B\ u$ and $t\ A\ u\ A\ v$. (See, e.g., [30, Ex. 1.3.2].)

Definition 13 (Full Separation). Two relations A and B are *fully separable* if

$$(A+B)^\infty \quad = \quad A^*B^*A^\infty + A^*B^\infty \quad [= \ A^*B^*(A^\infty + B^\infty)] \ .$$

With finite separability, infinite separability is equivalent to full separability.

Clearly, if A and B are fully separable, and A is well-founded, then $E^\infty = A^*B^\infty$.

Definition 14 (Partial Separation). Two relations A and B are *partially separable* if

$$(A+B)^* \quad \subseteq \quad A^*B^* + A^*B^*A^\infty + A^*B^\infty \quad [= \ A^*B^*(\mathbf{1} + A^\infty + B^\infty)] \ .$$

It is straightforward to see that

Proposition 15. *Infinite separability plus partial separability give full separability.*

A nice in-between (asymmetric) level of infinite separability is the following:

Definition 16 (Nice Separation). Two relations A and B are *nicely separable* if

$$(A+B)^\infty = (A+B)^*A^\infty + A^*B^\infty \ .$$

3.3. Infinite Chains

Obviously, infinite E-chains either have finitely many A's, with some B's in between, or else infinitely many A's:

$$(A+B)^\infty \quad = \quad (B^*A)^*B^\infty + (B^*A)^\infty \tag{2}$$

and, symmetrically,

$$(A+B)^\infty \quad = \quad (A^*B)^*A^\infty + (A^*B)^\infty \ . \tag{3}$$

It follows that if infinitely many interspersed B-steps are precluded (a property called *relative termination* in [23, Example 1.7 (11)]), then $A+B$ is well-founded if, and only if, A is:

Proposition 17. *If relation A is well-founded, then, for any relation B:*

$$(A+B)^\infty \quad = \quad (A^*B)^\infty \ . \tag{4}$$

In other words, in any infinite E-chain, there are either infinitely many B's, in which case E^∞ must be an infinite sequence of segments of the form A^*B, or else from some point on there are only A's, and A is not well-founded. Cf. [2, Lemma 1] and [6, Lemma 12B].

3.4. Constriction

The point of "constriction" is to consider the impact on immortality of a color preference for steps. This is very useful in arguments for well-foundedness, as we will see.

Definition 18 (Constriction [27]). For two given relations A and B, an infinite sequence of A- and B-steps is B*-constricting* if every possible A-step from the source point (tail end of the arrow) of a B-step in the sequence results in mortality.

In other words, all B-steps are "forced": if $s\ B\ t$ appears in the sequence, then all u such that $s\ A\ u$ are mortal. The use of constriction was first suggested by Plaisted [27] for rewriting and by Sørensen [28] for the lambda calculus.

One can always build a constricting sequence from an immortal element. Simply ignore B-steps whenever an immortalizing A-step is available.

Proposition 19 ([27]). *For relations A and B, if a point s is immortal in $A+B$, then there is an infinite B-constricting sequence in $A+B$ originating with s.*

Proof. Just take an A-step whenever possible, that is, whenever there is one that still leads to immortality. Only take B if there is no choice. □

We define a B-*sharp* step as a B-step from an immortal point from which no A-step leads to an immortal point:

$$B_\sharp \ = \ B \setminus AE^\infty \ .$$

Sharp, constricting steps are the only kind of B-step in a constricting sequence. So,

$$E^\infty \ = \ (A+B_\sharp)^\infty \ = \ (A^*B_\sharp)^*A^\infty + (A^*B_\sharp)^\infty \ . \tag{5}$$

Escaping means is that there cannot be an infinite sequence of B-sharp steps; in other words, $B_\sharp^\infty = \varnothing$.

4. Commutation

We begin with relatively simple cases of separability. We are looking for *local* conditions on double-steps BA that help establish separability, finite or infinite. The point is that these local conditions suggest how to eliminate BA patterns, which are exactly what are prohibited in separated sequences.

4.1. Finite Separation

Note that $CB^\varepsilon = C(B + \mathbf{1}) = CB + C$, and recall the following:

Proposition 20 ([19]). *Relations A and B are finitely separable if*

$$BA \subseteq A^* B^\varepsilon \ .$$

By symmetry (of A and B and left and right):

Proposition 21. *Relations A and B are finitely separable if*

$$BA \subseteq A^\varepsilon B^* \ , \tag{6}$$

When A is well-founded, the following property suffices for separation:

Definition 22 (Quasi-commutation [2]). Relation A *quasi-commutes over* relation B when

$$BA \subseteq A(A + B)^* \ .$$

The right-hand side, AE^*, is equivalent to $(AB^*)^+$ $(A(A+B)^* = AB^*(AB^*)^*)$.

Obviously (by induction):

Lemma 23. *Relation A quasi-commutes over relation B if, and only if, $B^*A \subseteq A(A + B)^*$.*

Theorem 24. *If relation A quasi-commutes over relation B, then*

$$(A + B)^+ \quad \subseteq \quad A^+B^* + B^+ + A^\infty \ .$$

This is a special case of partial separation (Definition 14).

Proof. Consider a nonempty E-sequence. If its first point is mortal in A, then we may reason by well-founded induction with respect to A. If the sequence is not already of the form $A^+B^* + A^*B^+$, then it must contain an occurrence of B^+A somewhere. Using the above lemma to replace the first such occurrence in the sequence with AE^*, yields a sequence of the form A^+E^*. By induction, $A^+E^+ \subseteq A^+(A^+B^* + A^*B^+)$. Putting all the cases together, we get $E^+ \subseteq A^\infty + A^+B^* + A^*B^+ + A^+E^0 + A^+B^* + A^*B^+ = A^+B^+ + B^+ + A^\infty$. □

Corollary 25. *If relation A is well-founded, then quasi-commutation of A over a relation B implies their finite separability.*

Quasi-commutation applies to combinatory logic and to orthogonal (left-linear non-overlapping) term-rewriting systems, where A is a leftmost (outermost) step and B is anything but. This means that leftmost steps may precede all non-leftmost ones. Combined with the fact—in these cases—that $A^-B \subseteq B^*A^-$, where A^- is the inverse (converse) relation, this gives *standardization* (leftmost rewriting suffices for computing normal forms). See [22].

4.2. Infinite Separation

We will make repeated use of the following:

Proposition 26. *If, for relations A, B, and C, one has $B \subseteq A^+B + C$, then $B \subseteq A^*C + A^\infty$.*

Proof. One can readily see by induction on A that $B \setminus A^\infty \subseteq A^*C$, since $B \setminus A^\infty \subseteq (A^+B + C) \setminus A^\infty \subseteq A^+A^*C + C = A^*C$. □

This holds also when B is nonterminating. In particular, if $B^\infty \subseteq A^+B^\infty + A^\infty$, then $B^\infty \subseteq A^\infty$, implying that B is well-founded if A is. For example:

Lemma 27. *If relations A and B are infinitely separable and if $B^\infty \subseteq A^+B^\infty + (A+B)^*A^\infty$, then $(A+B)^\infty = (A+B)^*A^\infty$.*

Proof. The right-to-left inclusion is trivial. Applying the preceding proposition to the premise, with B^∞ for B and E^*A^∞ for C, we get $B^\infty \subseteq E^*A^\infty$. Then, by infinite separation, $E^\infty = E^*(A^\infty + B^\infty) \subseteq E^*(A^\infty + E^*A^\infty) = E^*A^\infty$. □

Quasi-commutation (Definition 22) is a simple local condition guaranteeing infinite separability:

Theorem 28 ([2, Thm. 1])**.** *If relation A quasi-commutes over relation B, then A and B are infinitely separable. More precisely, $(A+B)^\infty = A^*B^\infty + A^\infty$.*

Proof. Consider any infinite E-chain. It is all B or has an A: $E^\infty = B^\infty + B^*AE^\infty$. By quasi-commutation and Lemma 23, we have: $E^\infty = B^\infty + AE^\infty$, and, by the above proposition, we end up with $E^\infty = A^\infty + A^*B^\infty$. □

This form of separability was used in [7] to show that "forward closure" termination suffices for orthogonal term rewriting, where B are "residual" steps (at redexes already appearing *below the top* in the initial term) and A-steps are at "created" redexes (generated by earlier rewrites).

4.3. Production and Promotion

With finite separability, a non-empty cycling sequence ($t\ BA\ t$, as in Example 11a) can be "reordered" to give an empty, separated one. To preclude an empty reordering, one can insist on the following:

Definition 29 (Productive Separation). Relations A and B are *productively separable* if

$$(A+B)^+ \quad = \quad A^+B^* + B^+ \quad [= \ A^+B^* + A^*B^+] \,.$$

Given finite separation, productive separation is equivalent to requiring every "mixed" loop to be productively separable: $E^+ \cap \mathbf{1} \subseteq A^+B^* + B^+$.

We have already seen (Theorem 24) that quasi-commutation provides partial separation, in general, and productive separation, in the situation where A is well-founded.

Definition 30 (Promotion). Relation B *promotes* relation A if

$$BA \subseteq AB^* + B^+ \quad [= (A+B)B^*] .$$

An easy induction shows the following:

Lemma 31. *Relation B promotes relation A if, and only if, $B^+A^n \subseteq A^{1:n}B^* + B^+$.*

Theorem 32. *If relation B promotes relation A, then A and B are productively separable.*

The local condition $BA \subseteq A^+B^*+B^+$, or even $BA \subseteq AA+BB$, is insufficient for separability, as the first example in Note 52 below shows.

Proof. By Proposition 21, promotion gives finite separability, $E^* = A^*B^*$. By the previous lemma, $B^*A = A + B^+A \subseteq A + AB^* + B^+ = A^+B^* + B^+$, whence the proposition follows: $E^+ = E^*(A+B) = (A^*B^*)(A+B) = A^*B^*A + A^*B^*B = A^*(A^+B^* + B^+) + A^*B^+ = A^+B^* + B^+$. □

Note 33. One can have productivity without promotion: $s\ A\ t\ A\ u$ with $s\ B\ t$. Obviously, promotion cannot be weakened to allow the erasure of both the A and B, as can easily be seen from $t\ BA\ t$ (Example 11a), which is finitely, but not productively, separable.

In symmetry with promotion, we also have the following:

Theorem 34. *If, for relations A and B, $BA \subseteq A^*B + A^+$, then A and B are productively separable.*

Here is a somewhat analogous way of obtaining a "productive" version of partial separability (Definition 14), similar in flavor to promotion as just used:

Theorem 35. *If, for relations A and B,*

$$B^+A \quad \subseteq \quad A(A+B)^* + B^+ + A^*B^*A^\infty ,$$

then

$$(A+B)^+ \quad \subseteq \quad A^+ + A^*B^+ + A^*B^*A^\infty .$$

Proof. Let $D = AE^* + A^*B^*A^\infty$. Clearly D absorbs E. By induction on n, we obtain $B^+AE^n \subseteq B^+ + D$, since $(B^+ + D)B \subseteq B^+ + D$, while $(B^+ + D)A = B^+A + D \subseteq B^+ + D$, by hypothesis. Considering that E^+ must look like $A^*(A+B^++B^+AE^*)$, we have $E^+ \subseteq A^*(A+B^++AE^*+A^*B^*A^\infty) = A^+E^+ + A^*(A+B^++B^*A^\infty)$, and, by Proposition 26, $E^+ \subseteq A^*(A+B^++B^*A^\infty)+A^\infty$, as stated. □

5. Selection and Jumping

For the abstract path ordering, we need to reason about reorderings of sequences of two kinds of steps ($\rhd$ and $\sqsupset$). That's where commutation comes into play, but we need something better than quasi-commutation.

5.1. Selection

First, a non-local commutation property, weaker than promotion, for which constriction yields separation.

Definition 36 (Selection). Relation B *selects* relation A if

$$BA^+ \subseteq A(A+B)^* + B^+ \ . \tag{7}$$

Definition 37 (Partial Selection). Relation B *partially selects* relation A if

$$BA^+ \subseteq A(A+B)^* + B^+ + A^\infty + A^* B^\infty \ . \tag{8}$$

Clearly, partial selection is weaker than selection.

A weaker form of partial selection suffices for a weak form of partial separation:

Lemma 38. *If $BA^+ \subseteq A(A+B)^* + B^+ + (A+B)^\infty$, then $(A+B)^* \subseteq A^*B^* + (A+B)^\infty$.*

Proof. We reason that $E^* \setminus E^\infty \subseteq A^*B^*$ by well-founded induction on E. A finite E-sequence is either of the separated form A^* or is of the form A^*BE^*. Assuming the initial point is E-mortal, induction gives $A^*BE^* \subseteq A^*BA^*B^*$, and the premise gives $A^*(BA^*)B^* \subseteq A^*(AE^* + B^+)B^*$. Another application of induction results in $A^*(AA^*B^* + B^+)B^*$, as desired. □

Restricting attention to constricting sequences gives the stronger form of separation:

Theorem 39. *If relation B partially selects relation A, then A and B are partially separable.*

Proof. By the above lemma, $E^* \subseteq A^*B^* + E^\infty$, so we need only examine the E^∞ case more carefully. We know (reordering Eq. 5) that $E^\infty \subseteq A^*(B_\sharp A^*)^* A^\infty + A^*(B_\sharp A^*)^\infty$. By assumption, $B_\sharp A^+ \subseteq AE^* + B^+ + A^\infty + A^*B^\infty$, but in such an infinite constricting sequence, $B_\sharp A^+$ cannot be replaced by anything beginning with A, so only B^+ and B^∞ are possible. We have $E^\infty \subseteq A^*(B^+ + B^\infty)^* A^\infty + A^*(B^+ + B^\infty)^\infty = A^*B^\infty + A^*B^*A^\infty$. □

Note 40. Weaker versions of selection do not yield partial separability. To wit, $s \; BA \; t \; A \; t$ is not partially separable, though $BA^+ \subseteq B^*A^\infty$. And $s \; AB \; s \; BA \; t$ is not, despite the fact that $BA^+ \subseteq AE^* + B^*$.

Theorem 41. *If relation B selects relation A and both are well-founded, then A and B are productively separable.*

Proof. Selection implies partial selection, which, per the previous theorem, gives partial separation, which, with well-foundedness, is finite separation. So selection, itself, becomes $BA^+ \subseteq AE^* + B^+ \subseteq A^+B^* + B^+$, which means that B promotes A^+ and Theorem 32 applies to A^+ and B, which is just as good as A and B for the purpose of productive separability. □

Note 42. Without well-foundedness of A, one does not have separation, as may be seen from the following selecting, but inseparable, example: $s\ A\ s\ BA\ t$. Well-foundedness of B is also needed. To wit: $t\ A\ s\ B\ s\ B\ t\ BA\ u$.

When A is transitive, as it is in many of the cases of relevance to path orderings (where A is some sort of special subterm relation), selection amounts to the pleasant local condition, $BA \subseteq AE^* + B^+$.

5.2. Jumping

A notion of absorption is convenient:

Definition 43. Relation X *absorbs* relation A if $XA \subseteq X$.

For example, A^*, E^*, B^∞, and $\varnothing$ all absorb A. (Recall Eq. 1.) Obviously, if X absorbs A, then it absorbs any number of A's, and CX absorbs A whenever X does.

Proposition 44. *If $BA \subseteq B + X$ for a relation X that absorbs A, then $BA^* \subseteq B + X$.*

Proof. By induction on n, $BA^n \subseteq B + X$, since, trivially, $B \subseteq B + X$, while, by the inductive hypothesis and absorption, $BA^nA \subseteq (B+X)A = BA + XA \subseteq B + X$. □

The following is a corollary, with $X = AE^*$:

Corollary 45 ([14, Eq. 4.5])**.** *If relation A jumps over relation B, then $BA^* \subseteq A(A+B)^* + B$.*

In short, jumping (Definition 1) is stronger than selection (Definition 36).

Proposition 46. *If relation A jumps over relation B, then $(B_\sharp A^*)^\infty = B_\sharp^\infty$.*

Proof. Jumping, per the above corollary, says that $B_\sharp A^* \subseteq AE^* + B_\sharp$, but constriction says that an A-step is impossible at any such point in the infinite sequence. So, it must be that $B_\sharp A^* = B_\sharp$. □

Note 47. Jumping is noticeably weaker than quasi-commutation (Definition 22), which was known from [2] to separate well-foundedness of the union into well-foundedness of each. Corollary 56 below is stronger. As pointed out in [13], jumping is also much weaker than transitivity of the union, which—by a direct invocation of a very simple case of the infinite version of Ramsey's Theorem—gives infinite separation. (See [16, 5] for some of the history of this idea.) Note that replacing BA with $AABB$ is a process that can continue unabated: BBA, $BAABB$, $AABBABB$, $AABAABBBB$, ...

A weaker version of jumping, which allows for infinite exceptions (but which we do not make use of), is the following (cf. [26]):

Definition 48 (Partial Jumping). Relation A *partially jumps over* relation B if

$$BA \quad \subseteq \quad A(A+B)^* + B + A^*B^*(A^\infty + B^\infty)\ .$$

Theorem 49. *If relation A partially jumps over relation B, then A and B are partially separable.*

Proof. Applying Proposition 44 (E^*, A^∞, and B^∞ each absorb A), partial jumping implies $BA^* \subseteq AE^* + B + A^*B^*(A^\infty + B^\infty)$, which when appearing in a constricted setting can only be $B_\sharp A^* \subseteq B + B^*(A^\infty + B^\infty)$. So, an infinite constricting sequence either has no B-steps from some point on or looks like $A^*(B_\sharp A^*)^\infty$. Thus, $E^\infty \subseteq (A+B_\sharp)^\infty \subseteq A^*(B_\sharp A^*)^* A^\infty + A^*(B_\sharp A^*)^\infty \subseteq A^*(B + B^*(A^\infty + B^\infty))^* A^\infty + A^*(B + B^*(A^\infty + B^\infty))^\infty = A^*B^*(A^\infty + B^\infty)$. □

That an infinite A-chain cannot be ruled out can be seen from $s\ B\ t\ A\ t\ BA\ u$. So partial jumping gives only partial separability, rather than finite separability.

With well-foundedness, partial separation turns into finite separation and partial jumping into jumping. In general, jumping gives partial separability; with well-foundedness, it gives finite separation.

Theorem 50. *If relation A jumps over relation B, and both are well-founded, then $BA^+ \subseteq A^+B^* + B$.*

Proof. Since we have finite separation, $BA \subseteq AE^* + B \subseteq AA^*B^* + B = A^+B^* + B$. A simple induction gives the desired outcome: $BA^nA \subseteq A^+B^*A + BA = A^+A^*B^* + A^+B^* + B$. □

Moreover, we have productive separability (Definition 29):

Corollary 51. *If relation A jumps over relation B, and both are well-founded, then A and B are productively separable.*

Proof. Again, using finite separation, $E^+ = EA^*B^* = A^+B^* + BA^+B^* + B^+ = A^+B^* + (A^+B^* + B)B^* + B^+ \subseteq A^+B^* + B^+$. □

Well-foundedness of A and of B are necessary, even in the presence of jumping. Refer to the examples in Note 42.

Note 52. Jumping cannot be weakened to $BA \subseteq AE^* + B^+$. Even $BA \subseteq AA + BB$ causes trouble, as can be seen from the (finitely) inseparable example $s\ B\ t\ B\ u\ A\ v$, and $t\ A\ u$. Jumping also cannot be weakened to $BA \subseteq AE^* + B^\varepsilon$, as was seen earlier (Note 40) from the inseparable example: $s\ AB\ s\ BA\ t$.

5.3. Endless Commutation

We will see presently (Theorem 54) that selection (Definition 36) suffices for the main consequences of jumping, namely finite and infinite separation. But, first, observe the following:

Lemma 53. *If*

$$BA^{+} \subseteq A(A+B)^{*} + B^{+} + (A+B)^{*}(A^{\infty} + B^{\infty}) \ , \tag{9}$$

for relations A and B, then they are infinitely separable.

This will be our main tool for separation. Condition (9) clearly does not suffice for full separation. See the example in Note 40.

Proof. By the premise and constriction, any steps $B_{\sharp}A^{*}$ in an infinite constricting E-chain may be replaced by $B^{+} + E^{*}(A^{\infty} + B^{\infty})$, since AE^{*} is precluded. For infinitely many such steps, we have $(B_{\sharp}A^{*})^{\infty} \subseteq (B^{+} + E^{*}(A^{\infty} + B^{\infty}))^{\infty} \subseteq E^{*}(A^{\infty} + B^{\infty})$. Looking at an infinite constricted sequence, $E^{\infty} \subseteq (A + B_{\sharp})^{\infty} \subseteq A^{*}(B_{\sharp}A^{*})^{*}A^{\infty} + A^{*}(B_{\sharp}A^{*})^{\infty} \subseteq E^{*}A^{\infty} + E^{*}(A^{\infty} + B^{\infty})$. □

The net result is that

Theorem 54. *If relation B selects relation A, then A and B are fully separable.*

Proof. Selection (Definition 7) implies the condition (9) of the previous lemma, giving infinite separation. Theorem 39 gives partial separation from selection; Proposition 15 gives full separation. □

By Corollary 45:

Corollary 55. *If relation A jumps over relation B, then A and B are fully separable.*

The fact that jumping implies infinite separability is the main result of [13].

The example $s \ B \ t \ A \ t$ allows jumping, since $BA \subseteq B$. So, the relations are fully separable, but there must be a B-step before an A-step for immortality. A similar example, with a simple infinite chain, rather than a loop, is $s \ B \ t_i$ and $t_i \ A \ t_{i+1}$, for all i. Jumping is essential: for instance, $s \ BA \ s$ is not fully separable.

The simple case $BA \subseteq AA + BB$ does not provide full separation. Just wrap the counterexample of Note 52 around itself: $s \ B \ t \ B \ u \ A \ s$ and $t \ A \ u$. So, in particular, jumping cannot be weakened to $BA \subseteq AE^{*} + B + BB$.

Combining this theorem with Theorem 41, we can summarize as follows:

Corollary 56.

- *If relations A and B are both well-founded and B selects A, then the union $A + B$ is also well-founded and A and B are productively separable.*

- *If relations A and B are both well-founded and A jumps over B, then the union $A + B$ is also well-founded and A and B are productively separable.*

Actually:

Theorem 57. *Whenever relations A and B are productively separable, they are also fully separable.*

Proof. Productive separability ($E^+ \subseteq A^+B^* + B^+$) implies that A^+ jumps over B^+ (i.e. $B^+A^+ \subseteq A^+E^* + B^+$), which means, thanks to Theorem 54, that A^+ and B^+ are fully separable, which is the same as A and B themselves being fully separable, since $E^\infty = (A^+ + B^+)^\infty = (A^+)^*(B^+)^*(A^+)^\infty + (A^+)^*(B^+)^\infty = A^*B^*A^\infty + A^*B^\infty$. □

In particular,

Corollary 58. *If relation B promotes relation A, then A and B are fully separable.*

Theorem 32 provided productive separability in this case.

Promotion cannot be weakened to allow the erasure of both the A and B, as can be seen from $t\ BA\ t$, which only cycles in the union. Compare Note 33.

As a corollary, if, for relations A and B, $BA \subseteq A^*B$, then A and B are fully separable. When B is the subterm relation, this condition is implied by quasi-monotonicity (weak monotonicity) of A (cf. [8, 1]). This condition, equivalent to $BA \subseteq A^+B + B$, is much more demanding than jumping. With jumping, an occurrence of BA need not *always* leave an A (as in quasi-commutation), nor *always* a B (as here), but might *sometimes* leave one and other times the other. This form of separability was used to show that "forward closure" termination suffices for right-linear term rewriting, where B are "created" steps and A are "residual" ones [7].

6. Lifting and Escaping

For various applications of commutation arguments, the well-foundedness of one relation is dependent on that of the other.

Definition 59 (Lifting). Relation A *lifts to* relation B if

$$B^\infty \quad \subseteq \quad A(A+B)^\infty .$$

Equivalently, $B^\infty \setminus AE^\infty = B_\sharp B^\infty = \varnothing$. Regarding lifting, see [21, 15, 18]. In particular, A lifts to B if E-mortality of all A-neighbors of a point implies its own E-mortality.

Lifting implies that if there is an infinite E-chain, then there is one with infinitely many interspersed A-steps. Somewhat more generally:[3]

[3]Personal communication of Alfons Geser.

Lemma 60. *If $B^\infty \subseteq B^*A(A+B)^\infty$, for relations A and B, then $(A+B)^\infty = (B^*A)^\infty$.*

This is a stronger statement than the one in Proposition 17.

Proof. We are given that $B^\infty \subseteq B^*AE^\infty = (B^*A)^+B^\infty + (B^*A)^\infty$. By Proposition 26, $B^\infty \subseteq (B^*A)^\infty$. Therefore, $E^\infty = (B^*A)^*B^\infty + (B^*A)^\infty = (B^*A)^\infty$. □

Theorem 61. *If relations A and B are nicely separable and A lifts to B, then $(A+B)^\infty = (A+B)^*A^\infty$.*

In other words, $A+B$ is well-founded if A is.

Proof. Combining niceness (Definition 16) and lifting, we have that $E^\infty \subseteq A^*B^\infty + E^*A^\infty \subseteq AE^\infty + E^*A^\infty$. By Proposition 26, $E^\infty = A^\infty + A^*E^*A^\infty = E^*A^\infty$. □

Corollary 62. *If relation A lifts to relation B and B selects A, then $(A+B)^\infty = (A+B)^*A^\infty$.*

Proof. Use Theorems 54 and 61, bearing in mind that full separation is more than just nice separation. □

It turns out, however, that oftentimes we need a weaker alternative to lifting, in which the A-step need only take place *eventually*. This was captured by the crucial notion of *escaping* (Definition 2).

The following is our main result:

Theorem 63. *If relation A escapes from relation B and A jumps over B, then $(A+B)^\infty \subseteq (A+B)^*A^\infty$.*

Under these conditions, too, $A+B$ is well-founded as long as A is. Selection, however, is insufficient, without the stronger lifting, as in the previous corollary. To wit: $s\ A\ t$ and $s\ B\ t\ BB\ s$.

Proof. Consider any infinite constricting sequence. It is either of the form E^*A^∞, in which case we are done, or else looks like $A^*(B_\sharp A^*)^\infty$. By jumping (Proposition 46), $B_\sharp A^* \subseteq B_\sharp$, so $A^*(B_\sharp A^*)^\infty \subseteq A^*B_\sharp^\infty$. But, by escape, that option is impossible. □

To capture the dependency-pair method of [1], let A be the well-founded immediate-subterm relation, I be inner rewriting, and D be dependency pairs, including outer rewrites (and including pairs with terms on the right that are headed by constructors). By the nature of rewriting, $IA \subseteq A + A(I+D)$; by the addition of dependencies, one has $DA \subseteq A + D$. Hence, A jumps over $B = I+D$, and $A+B$ is well-founded if B is. For termination, D-steps are made decreasing in some well-founded order (with constructor-headed terms minimal), while I-steps are non-increasing. So, any infinite B-chain must have a tail I^∞ of inner-only rewriting. But A (subterm) escapes I (inner rewriting); hence, escapes B. So, B (which contains the rewrite relation) must be terminating.

7. Conclusion

We are optimistic that the methods herein will help for advanced path orderings, like the general path ordering [11, 17] and higher-order recursive-path-ordering [20, 4], without recourse to reducibility/computability predicates. As pointed out in [10], there is an analogy between the use of reducibility predicates and the use in proofs of well-foundedness of constricting sequences.

Another avenue perhaps worth pursuing is to demonstrate modular termination by "completing" the two systems so that one jumps over the other. Combinations of transformations and orderings are used for termination proofs in [2, 3], and may benefit from such a perspective.

Acknowledgements. Thank you, Frédéric Blanqui, Ori Brostovski, Jörg Endrullis, Alfons Geser, Jean-Pierre Jouannaud, Georg Struth, and anonymous commenters.

[1] Thomas Arts and Jürgen Giesl, 2000, "Termination of term rewriting using dependency pairs", *Theoretical Computer Science* 236:133–178.

[2] Leo Bachmair and Nachum Dershowitz, July 1986, "Commutation, transformation, and termination", *Proc. Eighth International Conference on Automated Deduction (CADE)*, Oxford, England, Lecture Notes in Computer Science, vol. 230, Springer-Verlag, Berlin, pp. 5–20.

[3] Fran coise Bellegarde and Pierre Lescanne, 1980,"Termination by completion", *Applied Algebra on Engineering, Communication and Computer Science* 1(2):79–96.

[4] Frédéric Blanqui, Jean-Pierre Jouannaud, and Albert Rubio, 2008, "The Computability Path Ordering: The End of a Quest", *Proc. Computer Science Logic (CSL)*, Lecture Notes in Computer Science, vol. 5213, Springer-Verlag, Berlin, pp. 1–14.

[5] Andreas Blass and Yuri Gurevich, June 2008, "Program termination, and well partial orderings", *ACM Transactions on Computational Logic* 9(3), article 18.

[6] Jeremy E. Dawson and Rajeev Gore, 2007, "Termination of abstract reduction systems", *Proceedings of Computing: The Australasian Theory Symposium (CATS 2007)*, Ballarat, Australia, pp. 35–43.

[7] Nachum Dershowitz, July 1981, "Termination of linear rewriting systems (Preliminary version)", *Proc. Eighth International EATCS Colloquium on Automata, Languages and Programming (ICALP)*, Acre, Israel, Lecture Notes in Computer Science, vol. 115, Springer-Verlag, Berlin, pp. 448–458.

[8] Nachum Dershowitz, Mar. 1982, "Orderings for term-rewriting systems", *Theoretical Computer Science* 17(3):279–301.

[9] Nachum Dershowitz, 1987, "Termination of rewriting", *J. of Symbolic Computation* 3:69–116.

[10] Nachum Dershowitz, Sept. 2004, "Termination by abstraction", *Proc. Twentieth International Conference on Logic Programming (ICLP)*, St. Malo, France, Lecture Notes in Computer Science, vol. 3132, Springer-Verlag, Berlin, pp. 1–18.

[11] Nachum Dershowitz and Charles Hoot, May 1995, "Natural termination", *Theoretical Computer Science* 142(2):179–207.

[12] Nachum Dershowitz and Zohar Manna, Aug. 1979, "Proving termination with multiset orderings", *Communications of the ACM* 22(8):465–476.

[13] Henk Doornbos, Roland Backhouse, and Jaap van der Woude, 1997, "A calculational approach to mathematical induction", *Theoretical Computer Science* 179:103–135.

[14] Henk Doornbos and Burghard von Karger, 1998, "On the union of well-founded relations", *Logic Journal of the IGPL* 6(2):195–201.

[15] Maria C. F. Ferreira and Hans Zantema, 1995, "Well-foundedness of term orderings", *Proc. 4th International Workshop on Conditional Term Rewriting Systems (CTRS '94)*, Lecture Notes in Computer Science, vol. 968, Springer-Verlag, Berlin, pp. 106–123.

[16] Alfons Geser, 1990, *Relative Termination*, Ph.D. dissertation, Fakultät für Mathematik und Informatik, Universität Passau, Germany.

[17] Alfons Geser, 1996, "An improved general path order", *Applicable Algebra in Engineering, Communication and Computing* 7(6): 469–511.

[18] Jean Goubault-Larrecq, Sept. 2001, "Well-founded recursive relations", *Proc. 15th International Workshop on Computer Science Logic (CSL '01)*, Paris, France, Lecture Notes in Computer Science, vol. 2142, Springer-Verlag, Berlin, pp. 484–497.

[19] J. Roger Hindley, 1964, *The Church-Rosser Property and a Result in Combinatory Logic*, Ph.D. thesis, University of Newcastle-upon-Tyne.

[20] Jean-Pierre Jouannaud and Albert Rubio, 1999, "Higher-order recursive path orderings", *Proceedings 14th Annual IEEE Symposium on Logic in Computer Science (LICS)*, Trento, Italy, pp. 402–411.

[21] Sam Kamin and Jean-Jacques Lévy, Feb. 1980, "Two generalizations of the recursive path ordering", unpublished note, Department of Computer Science, University of Illinois, Urbana, IL. Available at `http://pauillac.inria.fr/~levy/pubs/80kamin.pdf` (viewed June 18, 2012.)

[22] Jan Willem Klop, 1980, *Combinator1y Reduction Systems*, Mathematical Centre Tracts 127, CWI, Amsterdam, The Netherlands.

[23] Jan Willem Klop, 1987, "Term rewriting systems: A tutorial", *Bulletin of the EATCS*, 32:143–183.

[24] Pierre Lescanne, 1990, "On the recursive decomposition ordering with lexicographical status and other related orderings", *J. Automated Reasoning*, 6(1):39–49.

[25] Crispin St. J. A. Nash-Williams, 1963, "On well-quasi-ordering finite trees", *Proceedings Cambridge Phil. Soc.* 59:833–835.

[26] Vincent van Oostrom, Apr. 2011, "Preponement", unpublished note, Universiteit Utrecht. Available at http://www.phil.uu.nl/~oostrom/publication/pdf/preponement.pdf (viewed March 8, 2012.)

[27] David A. Plaisted, 1993, "Polynomial time termination and constraint satisfaction tests", *Proc. 5th Intl. Conference on Rewriting Techniques and Applications (RTA)*, Lecture Notes in Computer Science, vol. 690, Springer-Verlag, pp. 405–420.

[28] Morten Heine Sørensen, 1996, "Properties of infinite reduction paths in untyped λ-calculus", *Proc. Tbilisi Symposium on Language, Logic, and Computation*, CLSI Lecture Notes.

[29] John Staples, 1975, "Church-Rosser theorems for replacement systems", in: *Algebra and Logic*, Lecture Notes in Mathematics, vol. 450, Springer-Verlag, pp. 291–307.

[30] "Terese" (Marc Bezem, Jan Willem Klop, and Roel de Vrijer, eds.), 2002, *Term Rewriting Systems*, Cambridge University Press.

[31] Yoshihito Toyama, 1987, "Counterexamples to termination for the direct sum of term rewriting system, *Inf. Process. Lett.* 25(3):141–143.

[32] Yoshihito Toyama, Jan Willem Klop, and Hendrik Pieter Barendregt, 1995, "Termination for direct sums of left-linear complete term rewriting systems", *J. ACM* 42(6):1275–1304.