Concepts in Programming Languages
Recitation 4:
Untyped Lambda Calculus

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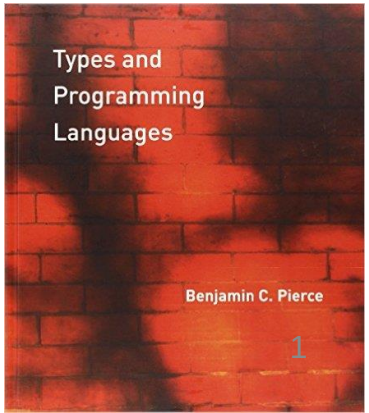
(original slides by Kathleen Fisher, John Mitchell, Shachar Itzhaky, Oded Padon, Mooly Sagiv, S. Tanimoto)

Reference:
Types and Programming Languages
by Benjamin C. Pierce, Chapter 5



Untyped Lambda Calculus - Syntax

t ::=	terms
x	variable
λ x. t	abstraction
t t	application

- Terms can be represented as abstract syntax trees
- Syntactic Conventions:
  - Applications associates to left :
    $e_1\ e_2\ e_3 \equiv (e_1\ e_2)\ e_3$
  - The body of abstraction extends as far as possible:
    $\lambda x.\ \lambda y.\ x\ y\ x \equiv \lambda x.\ (\lambda y.\ (x\ y)\ x)$
- Examples (taken from 2015 exams):
  - (λx. λx. (λx.x) x) ((λx. x x) λx.x)
  - (λt. λf. t) (λx.x) ((λx.x) (λs. λz. s z))

Free vs. Bound Variables

- An occurrence of x in t is bound in λx. t
  - otherwise it is free
  - λx is a binder
- Examples
  - λx. x
  - λy. x (y z)
  - λz. λx. λy. x (y z)
  - (λx. x) x

FV: t $\rightarrow$ P(Var) is the set free variables of t

FV(x) = {x}

FV(λ x. t) = FV(t) – {x}

FV (t_1 t_2) = FV(t_1) $\cup$ FV(t_2)

Semantics: β-reduction, Substitution

- Substitution

  $[x \mapsto s]\ x = s$

  $[x \mapsto s]\ y = y$ if $y \neq x$

  $[x \mapsto s]\ (\lambda y.\ t_1) = \lambda y.\ [x \mapsto s]\ t_1$ if $y \neq x$ and $y \notin FV(s)$

  $[x \mapsto s]\ (t_1\ t_2) = ([x \mapsto s]\ t_1)\ ([x \mapsto s]\ t_2)$

- β-reduction

  $(\lambda\ x.\ t_1)\ t_2 \Rightarrow_\beta [x \mapsto t_2]\ t_1$

Beta-Reduction: Examples

$(\lambda\ x.\ t_1)\ t_2 \Rightarrow_\beta [x \mapsto t_2]\ t_1$ (β-reduction)

redex

$(\lambda\ x.\ x)\ y \Rightarrow_\beta\ y$

$(\lambda\ x.\ x\ (\lambda\ x.\ x)\ )\ (u\ r) \Rightarrow_\beta\ u\ r\ (\lambda\ x.\ x)$

$(\lambda\ x\ (\lambda w.\ x\ w))\ (y\ z) \Rightarrow_\beta\ \lambda w.\ y\ z\ w$

Substitution Subtleties

$(\lambda\ x.\ t_1)\ t_2 \Rightarrow_\beta [x \mapsto t_2]\ t_1$ (β-reduction)

$[x \mapsto s]\ x = s$

$[x \mapsto s]\ y = y$ if $y \neq x$

$[x \mapsto s]\ (\lambda y.\ t_1) = \lambda y.\ [x \mapsto s]\ t_1$ **if $y \neq x$ and $y \notin FV(s)$**

$[x \mapsto s]\ (t_1\ t_2) = ([x \mapsto s]\ t_1)\ ([x \mapsto s]\ t_2)$

$(\lambda x.\ (\lambda x.\ x))\ y \Rightarrow_\beta [x \mapsto y]\ (\lambda x.\ x) = \lambda x.\ y$?

$(\lambda x.\ (\lambda y.\ x))\ y \Rightarrow_\beta [x \mapsto y]\ (\lambda y.\ x) = \lambda y.\ y$?

$(\lambda x.\ (\lambda x.\ x))\ y$ and $(\lambda x.\ (\lambda y.\ x))\ y$ are stuck! They have no β-reduction

Alpha – Conversion

α-conversion:
Renaming of a bound variable and its bound occurrences

$$(\lambda x.\ t) \Rightarrow_\alpha \lambda y.\ [x \mapsto y]\ t \qquad \text{if } y \notin FV(t)$$

$$(\lambda x.\ (\lambda x.\ x))\ y \Rightarrow_\alpha (\lambda x.\ (\lambda z.\ z))\ y \Rightarrow_\beta [x \mapsto y]\ (\lambda z.\ z) = \lambda z.\ z \quad \overset{\alpha}{\Uparrow} \quad \lambda x.\ x \neq \lambda x.\ y$$

$$(\lambda x.\ (\lambda y.\ x))\ y \Rightarrow_\alpha (\lambda x.\ (\lambda z.\ x))\ y \Rightarrow_\beta [x \mapsto y]\ (\lambda z.\ x) = \lambda z.\ y \neq \lambda y.\ y$$