Concepts in Programming Languages

Recitation 1: Inductive Definitions

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Administrative

- Course website
  - https://www.cs.tau.ac.il/~msagiv/courses/pl19.html
  - Forum in Moodle
- Grade: 30% exercises (~5), 70% exam
- Exercise submission in groups of **2-3**
- My reception hour: Thursday, 15-16
  - Email in advance, yotamfe1@mail.tau.ac.il

Concepts in Programming Languages

- Concepts in modern programming languages:
  - Design
  - Implementation
  - Applications in various programming languages
- Today: **Inductive definitions**
  - Syntax, semantics, types, …

(1) Inductively Defined Set

- N: the set of natural numbers
- Context-free grammar (CFG)
  - Tokens (terminals): S,(,),0.
    Non-terminal: N.
  - Notation 1 – production rules:
    $N \rightarrow 0$
    $N \rightarrow S(N)$
  - Notation 2 – Backus-Naur Form (BNF):
    N ::= 0 | S(N)

(1) Inductively Defined Set

- N: the set of natural numbers
- Context-free grammar (CFG)
  - Notation 3 – derivation/proof rules:

$$\textbf{zero}_N \; \frac{}{0 \in N} \qquad\qquad \textbf{succ}_N \; \frac{x \in N}{S(x) \in N}$$

- The set is defined to be the **least** set that is **closed** under the rules
  - Whichever notation used

(2) Inductive proofs

- M ::= 0 | S(0) | S(S(M))
- Equivalently, as derivation rules:

$$\textbf{zero}_M \; \frac{}{0 \in M} \qquad \textbf{one}_M \; \frac{}{S(0) \in M}$$

$$\textbf{succ}_M \; \frac{x \in M}{S(S(x)) \in M}$$

- **Claim**: N=M
  - – Equality between sets of syntactical objects

(2) Inductive proofs

- <u>M⊆N:</u>
- Prove ∀x ∈ M. x ∈ N
  by induction over the derivation of x in M
- Base case – axioms:
  - x derived by **zero_M**. Then x=0.
    To show x ∈ N:
  - x derived by **one_M**. Then x=S(0).
    To show x ∈ N:
- Step – rules:
  - x derived by **succ_M**. So there exists y:

By the **induction hypothesis, y ∈ N**.
To show x ∈ N:

$$\text{zero}_N \quad \frac{}{0 \in N}$$

$$\text{zero}_N \quad \frac{}{0 \in N}$$
$$\text{succ}_N \quad \frac{0 \in N}{S(0) \in N}$$

$$\text{succ}_M \quad \frac{y \in M}{S(S(y)) \in M}$$

x → S(S(y)) ∈ M

$$\text{succ}_N \quad \frac{y \in N}{S(y) \in N}$$
$$\text{succ}_N \quad \frac{S(y) \in N}{S(S(y)) \in N}$$

(2) Inductive proofs

- <u>N⊆M:</u>
- Prove $\forall x \in N.\ \ x \in M$
  by induction over the derivation of x in N
  (also say “by structural induction over x”)
- **Fails!**
- Induction step:

$$\textbf{succ}_N \ \frac{x \in N}{S(x) \in N}$$

- By the induction hypothesis, $x \in M$
- Now what?
  - Can derive S(S(x)), but not S(x)!
- Need to **strengthen** inductive argument

(2) Inductive proofs

- N⊆M:
- Prove $\forall x \in N.\ x \in M \wedge$ **$S(x) \in M$**
  by induction over the derivation of x in N
- Base case – axioms:
  - **zero_N.** $0 \in M$ by **zero_M**. $S(0) \in M$ by **one_M**
- Step – rules:
  - **succ_N**. $x=S(y)$:

$$\textbf{succ}_N\ \frac{y \in N}{S(y) \in N}$$

By the induction hypothesis, **$y \in M$ and $S(y) \in M$**

$x \in M$

$$S(y) \in M$$

$S(x) \in M$

$$\textbf{succ}_M\ \frac{y \in M}{S(S(y)) \in M}$$

(2) Inductive proofs

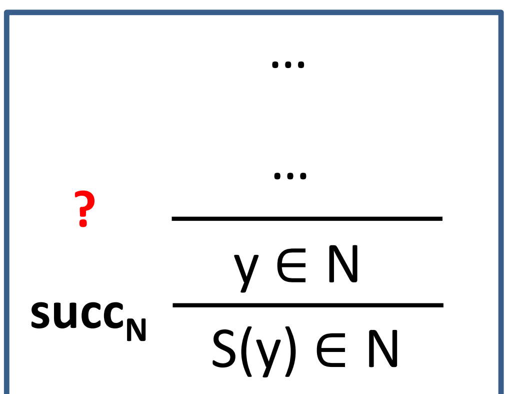
- <u>N⊆M,</u> alternative description:
- ∀x ∈ N. x ∈ M by induction over the **derivation tree** of x in N
- Base case – tree of depth 1:
  - $\mathbf{zero_N}$. 0 ∈ M by $\mathbf{zero_M}$.
- Step:
  - Last rule was $\mathbf{succ_N}$
  - Need to show S(y) ∈ M
  - Split to cases on previous rule in the tree

$$\mathbf{zero_N}\ \frac{}{0 \in N}$$

$$\begin{array}{c} \dots \\ \color{red}{?}\ \dfrac{\dots}{y \in N} \\ \mathbf{succ_N}\ \dfrac{}{S(y) \in N} \end{array}$$

(2) Inductive proofs

- ∀x ∈ N. x ∈ M by induction over the depth of the **derivation tree**
- Step: Last rule was $\textbf{succ}_N$. Split to cases on previous rule:
  - If ?= $\textbf{zero}_N$, y=0 and S(y)=S(0), need to prove S(0) ∈ M: follows by $\textbf{one}_M$



(2) Inductive proofs

- $\forall x \in N.\ x \in M$ by induction over the depth of the **derivation tree**
- Step: Last rule was $\textbf{succ}_N$. Split to cases on previous rule:
  - If ?= $\textbf{zero}_N$, y=0 and S(y)=S(0), need to prove $S(0) \in M$: follows by $\textbf{one}_M$
  - If ?= $\textbf{succ}_N$, y=S(z) with $z \in N$.

    By the induction hypothesis, $z \in M$.

    Now apply $\textbf{succ}_M$ to obtain $x=S(S(z)) \in M$.

$$\dfrac{\dfrac{\dfrac{\cdots}{z \in N}}{y \in N}\ \textbf{succ}_N}{S(y) \in N}\ \textbf{succ}_N$$

$$\dfrac{\dfrac{\cdots}{z \in M}}{S(S(z)) \in M}\ \textbf{succ}_M$$

(3) Syntax: Arithmetic Expressions

- E ::= n | E+E | E*E | (E)
- Equivalently:

1. $$\frac{}{n \in E}$$

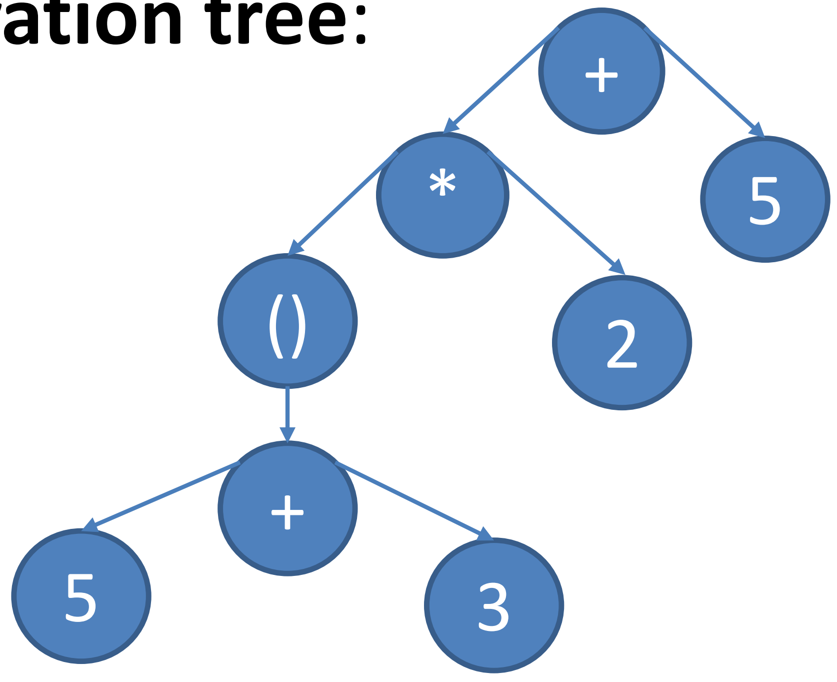
2. $$\frac{e_1 \in E \qquad e_2 \in E}{e_1 + e_2 \in E}$$

3. $$\frac{e_1 \in E \qquad e_2 \in E}{e_1 * e_2 \in E}$$

4. $$\frac{e \in E}{(e) \in E}$$

(3) Syntax: Arithmetic Expressions

- E ::= n | E+E | E*E | (E)
- <u>Claim</u>: (5+3)*2+5 ∈ E
- <u>Proof</u>: By a **derivation tree**:



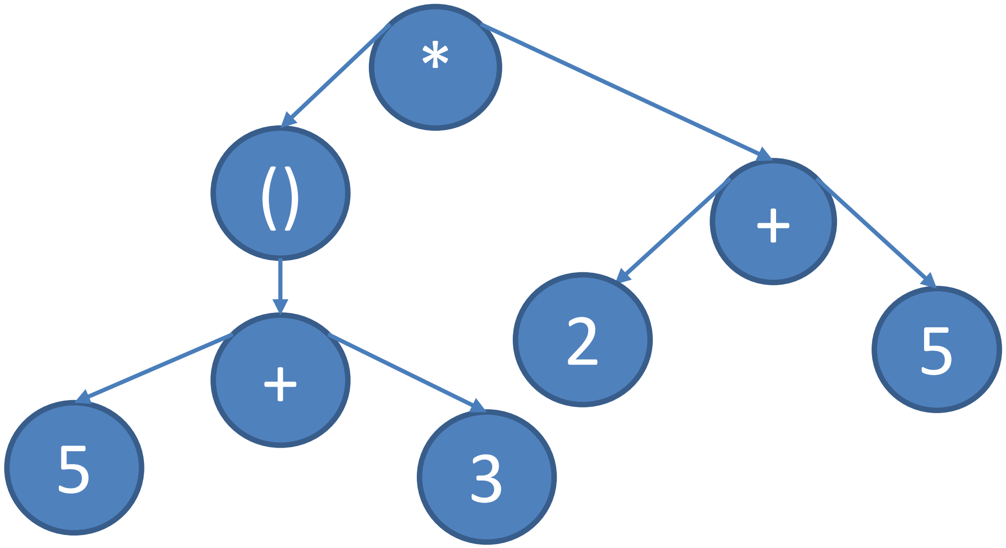
(3) Syntax: Arithmetic Expressions

- E ::= n | E+E | E*E | (E)
- Claim: (5+3)*2+5 ∈ E
- Proof: By a **derivation tree**:

$$+\;\dfrac{*\;\dfrac{()\;\dfrac{+\;\dfrac{\dfrac{}{5\in E}\qquad\dfrac{}{3\in E}}{5+3\in E}}{(5+3)\in E}\qquad\dfrac{}{2\in E}}{(5+3)*2\in E}\qquad\dfrac{}{5\in E}}{(5+3)*2+5\in E}$$

(3) Syntax: Arithmetic Expressions

- E ::= n | E+E | E*E | (E)
- Claim: (5+3)*2+5 ∈ E
- The grammar is ambiguous!



(3) Inductive Functions

- E ::= n | (E+E) | (E*E)
- Define by structural induction/recursion:

lits: E → N	**ops: E → N**
lits(n) = 1	ops(n) = 0
$lits(e_1+e_2)=lits(e_1)+lits(e_2)$	$ops(e_1+e_2)=ops(e_1)+ops(e_2)+1$
$lits(e_1*e_2)=lits(e_1)+lits(e_2)$	$ops(e_1*e_2)=ops(e_1)+ops(e_2)+1$

syntactical semantical

(3) Inductive Functions

- $E ::= n \mid (E+E) \mid (E*E)$

lits: E → N **ops: E → N**

- <u>Claim</u>: $\forall e \in E.\ lits(e) \geq ops(e)$
- How to prove?

- Need to strengthen the claim:

$$\forall e \in E.\ lits(e) > ops(e)$$