Some Principles of Induction

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Goals

- Mathematical Induction
- Inductively Defined Sets
- Derivation Trees
- Structural Induction

Syntax vs. Semantics

- The pattern of formation of sentences or phrases in a language
- Examples
  - Regular expressions
  - Context free grammars

- The study or science of meaning in language
- Examples
  - Interpreter
  - Compiler
  - Better mechanisms will be given in the course

Induction in Programming Languages

- The syntax of programing languages is inductively defined
  - A number is expression
  - If e_1 and e_2 are expressions to so is $e_1 + e_2$
- Types are inductively defined
  - int is a type
  - if $t_1, t_2, \ldots, t_k$ are types then
    struct { t_1 i_1; t_2 i_2; …, t_k i_k;} is a type where $i_1, i_2, \ldots, i_k$ are identifiers
- Recursive functions
  - fac(n) = if n = 1 then 1 else n * fac(n-1)
    fac_1=1, facn=n*fac_{n-1}
- Semantics is inductively defined

Mathematical Induction

- P(n) is a property of natural number n
- To show that P(n) holds for every n, it suffices to show that:
  - P(0) is true
  - If P(m) is true then P(m+1) is true for every number m
- In logic
  - $(P(0) \wedge \forall m \in N.\ P(m) \Rightarrow P(m+1)) \Rightarrow \forall n \in N.\ P(n)$

Simple Example

- Show that $0 + 1 + 2 + \ldots + n = n (n + 1)/2$

Fibonachi Numbers

- A sequence
  - $fib_0=1$
  - $fib_1=1$
  - $fib_n = fib_{n-2}+fib_{n-1}$

1 1 2 3 5

The Golden Ratio

- fib(n) = $(a^n - b^n)/(a-b)$ where $a=(1+\sqrt{5})/2$ and $b=(1-\sqrt{5})/2$
- How can this be proved?

Course of values Induction

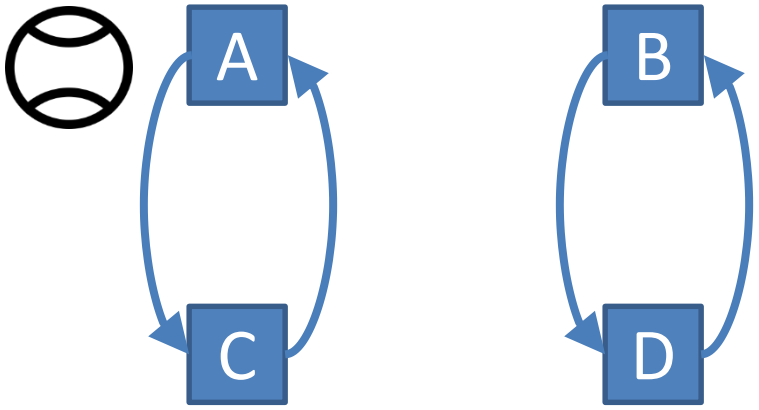
- P(n) is a property of natural number n
- To show that P(n) holds for every n, it suffices to show that for all m if for all k <m, P(k) holds then P(m)
- In logic
  - $\forall m \in N.\ (\forall k < m.\ P(k)) \Rightarrow P(m)) \Rightarrow \forall n \in N.\ P(n)$

The Induction Hypothesis

- The art of inductive proofs is coming up with induction hypothesis
- Given a property Q find a stronger property P which is inductive $P \Rightarrow Q$ or $P \subseteq Q$

Induction on a ball game

- Four players pass a ball:
  - A will pass to C
  - B will pas to D
  - C will pass to A
  - D will pass to B
- The ball starts at player A
- Can the ball get to D?



Induction on a ball game

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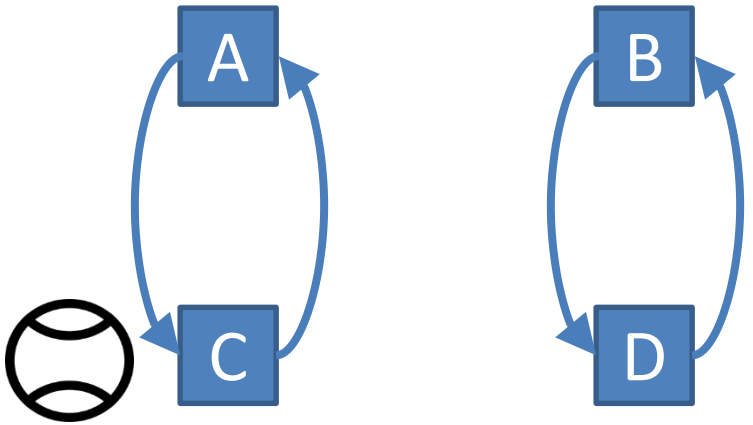
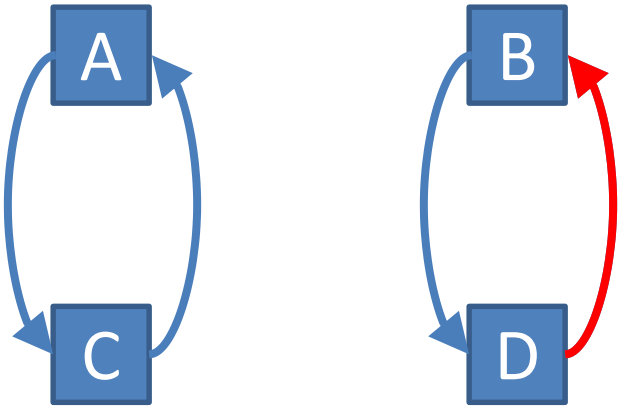


Formalizing with induction

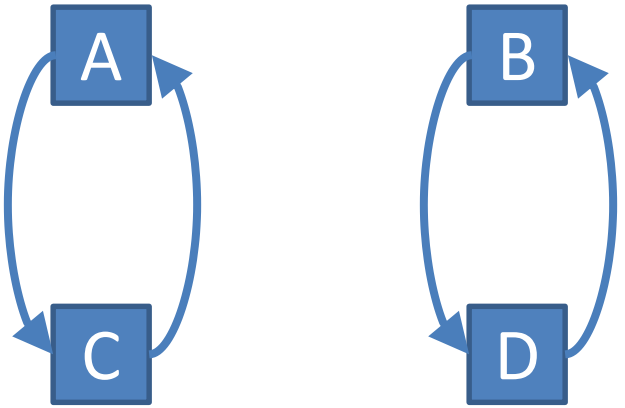
- $x_0 = A$
- $x_{n+1} = \begin{cases} C \ if \ x_n = A \\ D \ if x_n = B \\ A \ if x_n = C \\ B \ if x_n = D \end{cases}$



- Prove by induction $\forall n. x_n \neq D$
  - $x_0 \neq D$?
  - $x_m \neq D \Rightarrow x_{m+1} \neq D$?

Formalizing with induction

- $x_0 = A$
- $x_{n+1} = \begin{cases} C \ if \ x_n = A \\ D \ if x_n = B \\ A \ if x_n = C \\ B \ if x_n = D \end{cases}$



- Prove a stronger claim by induction $\forall n. x_n \neq B \wedge x_n \neq D$
  - $x_0 \neq B \wedge x_0 \neq D$
  - $x_m \neq B \wedge x_m \neq D \Rightarrow x_{m+1} \neq B \wedge x_{m+1} \neq D$

Simple Program

- Consider the sequence
  - $x_0 = 0, y_0=0$
  - $x_n = x_{n-1} + y_{n-1}, y_n = y_{n-1}+2$
- How can we show that
  for all n: x_n is even
- What is the inductive claim?

```
{
x =0, y =0;
while (1) {
      assert x%2 =0 ;
      x = x + y;
      y = y + 2;
            }
}
```

Inductively Defined Sets

Inductively Defined Sets

- It is convenient to define sets by rules
  - Base elements
  - Inference rules for more elements
  - Define the minimal set that is closed under rule application

The Natural Numbers

zero-axiom

$$0 \in \mathbb{N}$$

succ-rule

$$\frac{m \in \mathbb{N}}{succ(m) \in \mathbb{N}}$$

The Even Numbers

zero-even

$$0 \in \text{Even}$$

even-rule

$$\frac{m \in \text{Even}}{\text{succ}(\text{succ}(m)) \in \text{Even}}$$

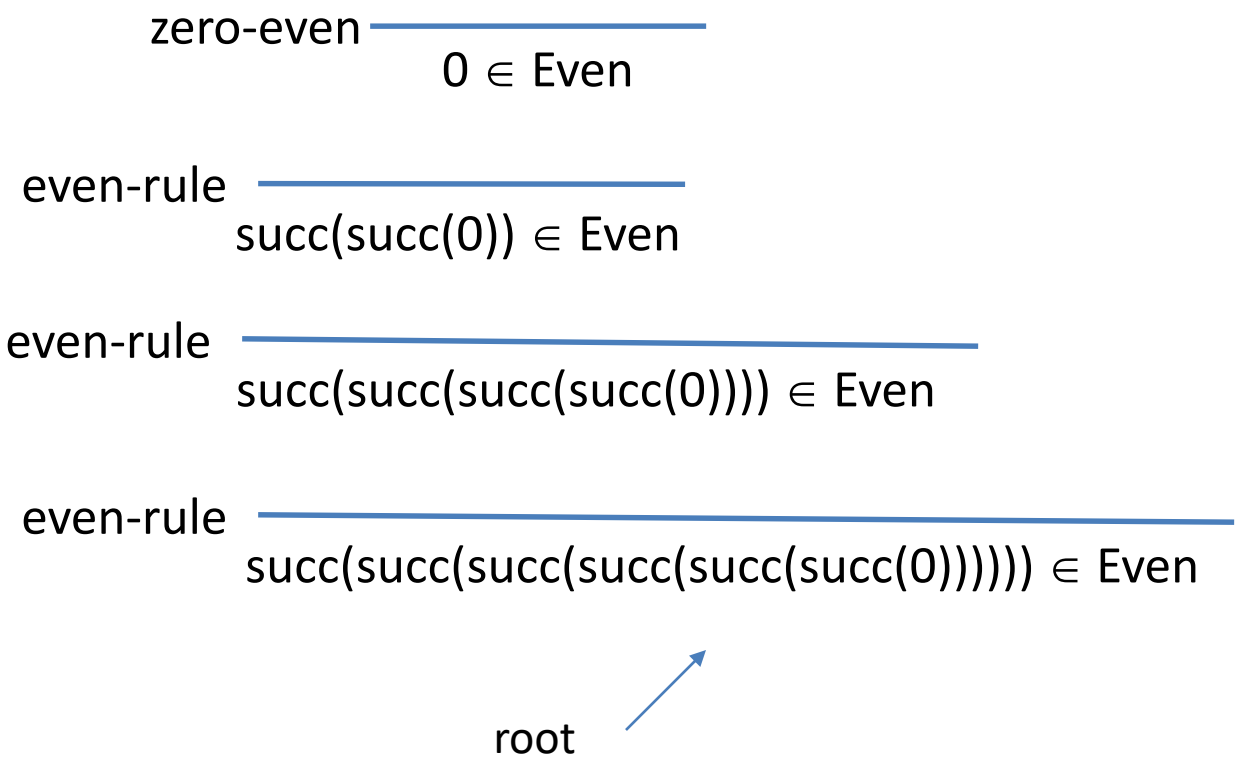
How can we show that “6” $\in$ Even

How can we show that “3” $\notin$ Even

Driven (Proof) Tree

- A proof that element is inductive set
- The leafs are axioms
- Internal nodes are inference rules

A Proof Tree for “6” $\in$ Even



Proving that “3” $\notin$ Even

Why is this relevant to programming languages?

Expression Definition

- A number is an expression
- If e is an expression then so is (e)
- If e1 and e2 are expressions then e1+e2 is an expression
- This is an inductively defined set

Rules for expressions

exp-num-axiom

$$\text{number} \in \text{Exp}$$

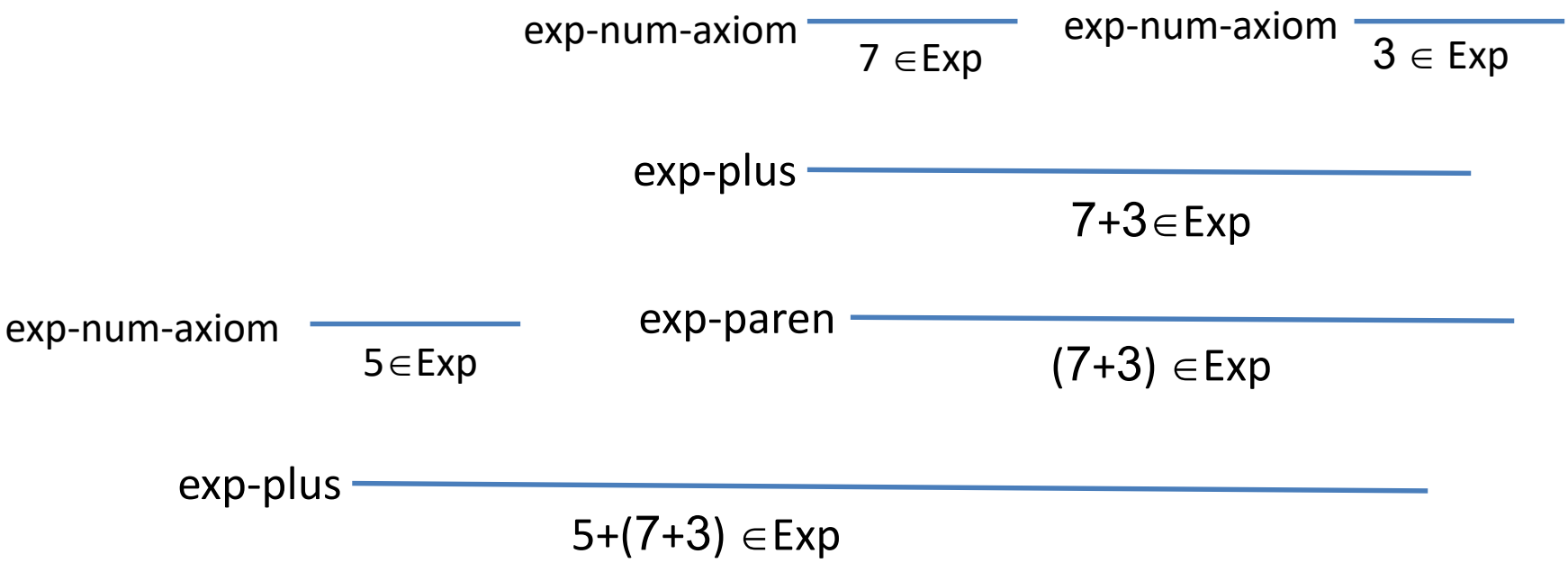
exp-paren

$$\frac{e \in \text{Exp}}{(e) \in \text{Exp}}$$

exp-plus

$$\frac{e_1 \in \text{Exp} \quad e_2 \in \text{Exp}}{e_1 + e_2 \in \text{Exp}}$$

Example 5 + (7+3)


exp-num-axiom
7 ∈Exp
exp-num-axiom
3 ∈ Exp
exp-plus
7+3∈Exp
exp-num-axiom
5∈Exp
exp-paren
(7+3) ∈Exp
exp-plus
5+(7+3) ∈Exp

Benefits of formal definitions

- Intellectual
- Better understanding
- Formal proofs
- Mechanical checks by computer
- Tool generation
  - Consistency
  - Entailment
  - Query evaluation

What is a good formal definition?

- Natural
- Concise
- Easy to understand
- Permits effective mechanical reasoning

Benefits of formal syntax for programming language

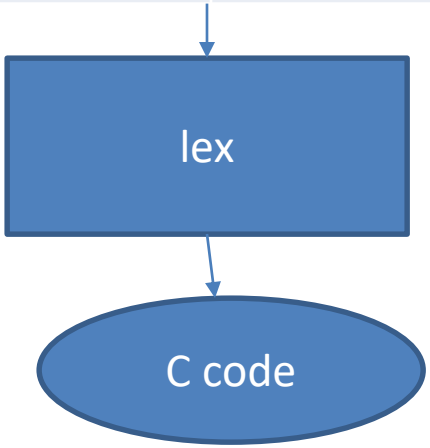
- Intellectual
- Simplicity
- Better understanding
  - Interaction between different parts
- Abstraction
  - Portability
- Tool generations
  - Parser

How can the syntax of programming language be formally defined?

Tokens

- Basic units of the programming language
- Usually defined by regular expressions
- Good tools: lex, awk

Regexp	action
[0-9]+	printf(“number”);
“if”	printf(“if”);



Example Tokens

- Keywords
  - “if”
  - “while”
- Identifier {letter}({letter}|{digit}|_)*
- Numbers [0-9]+

Example Tokens

Type	Examples
ID	foo n_14 last
NUM	73 00 517 082
REAL	66.1 .5 10. 1e67 5.5e-10
IF	if
COMMA	,
NOTEQ	!=
LPAREN	(
RPAREN	)

Example Non Tokens

Type	Examples
comment	/* ignored */
preprocessor directive	#include <foo.h>
	#define NUMS 5, 6
macro	NUMS
whitespace	\t \n \b

Recursive Syntax Definitions

- The syntax of programming languages is naturally defined recursively
- Can be also defined using inductive definitions
- Valid program are represented as syntax trees

Expression Definitions

- Every identifier is an expression
- If E1 and E2 are expressions and op is a binary operation then so is 'E_1 op E_2' is an expression

<E> → id | <E> <op> <E>
<op> → + | - | * | /

plus-ax $\dfrac{}{+ \in Op}$

minus-ax $\dfrac{}{- \in Op}$

mul-ax $\dfrac{}{* \in Op}$

div-ax $\dfrac{}{/ \in Op}$

id-E $\dfrac{id \in ID}{id \in E}$

bin-E $\dfrac{e_1 \in E \quad e_2 \in E \quad op \in Op}{e_1 \; op \; e_2 \in E}$

Statement Definitions

- If id is a identifier and E is an expression then 'id := E' is a statement
- If S_1 and S_2 are statements and E is an expression then
  - 'S_1 ; S_2' is a statement
  - 'if (E) S_1 else S_2' is a statement

<S>→ id := <E>
<S> → <S> **;** <S>
<S> → **if (**<E>**)** <S> **else** <S>

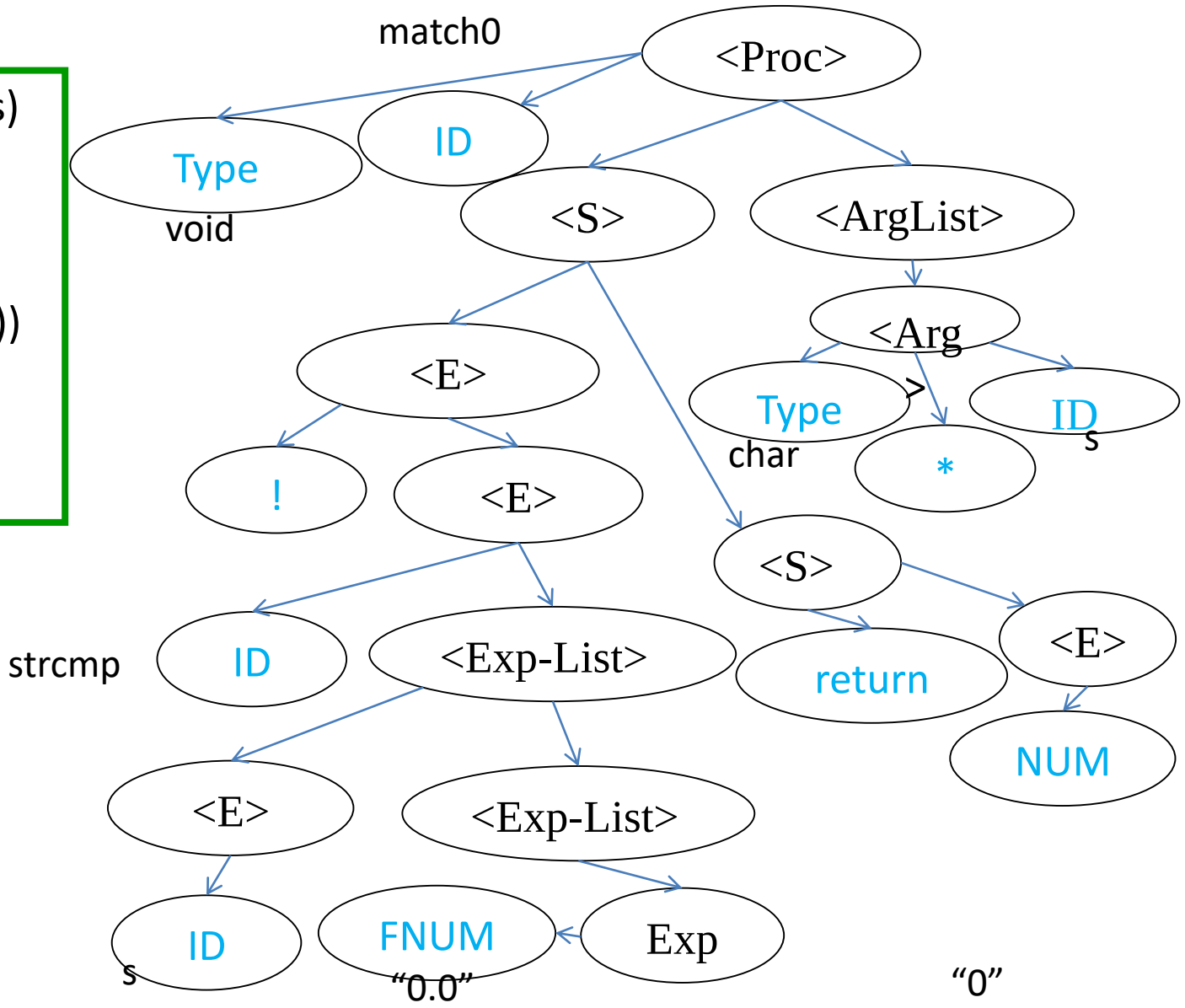
assign-rule $\dfrac{id \in Id,\ e \in E}{id := E \in S}$

seq-s $\dfrac{S_1 \in S \quad S_2 \in S}{S_1 ; S_2 \in S}$

conditional-s $\dfrac{e \in E \quad S_1 \in S \quad S_2 \in S}{\text{if (E) } S_1 \text{ else } S_2 \in S}$

C Example

```
void match0(char *s)
 /* find a zero */
{
  if (!strcmp(s, “0.0”))
     return 0 ;
}
```



Context Free Grammars

- Non-terminals
  - Start non-terminal
- Terminals (tokens)
- Context Free Rules
  <Non-Terminal> $\rightarrow$ Symbol Symbol … Symbol

Example Context Free Grammar

1 <S> → <S> ; <S>
2 <S> → id := <E>
3 <S> → print (<L>)
4 <E> → id
5 <E> → num
6 <E> → <E> + <E>
7 <E> → (<S>, <E>)
8 <L> → <E>
9 <L> → <L>, <E>

Derivations

- Show that a sentence is in the grammar (valid program)
  - Start with the start symbol
  - Repeatedly replace one of the non-terminals by a right-hand side of a production
  - Stop when the sentence contains terminals only
- Rightmost derivation
- Leftmost derivation

Parse Trees

- The trace of a derivation
- Every internal node is labeled by a non-terminal
- Each symbol is connected to the deriving non-terminal

Example Parse Tree

<<S>>

<S> ; <S>

<S> ; id := E

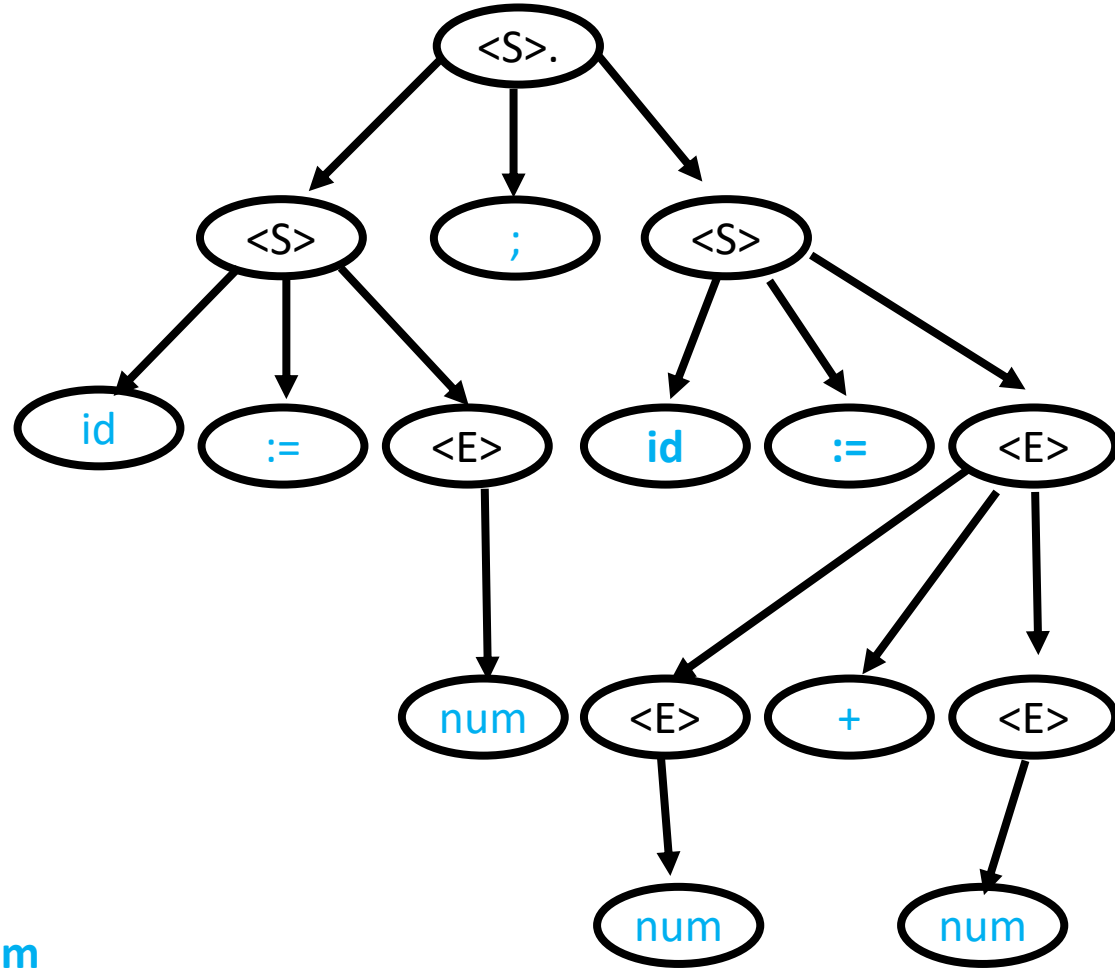
id := <E> **; id :=** <E>

id := num ; id := <E>

id := num ; id := <E> + <E>

id := num ; id := <E> + num

id := num ; id :=num + num



Ambiguous Grammars

- Two leftmost derivations
- Two rightmost derivations
- Two parse trees

A Grammar for Arithmetic Expressions

1 <E> → <E> + <E>
2 <E> → <E> * <E>
3 <E> → id
4 <E> → (<E>)

Drawbacks of Ambiguous Grammars

- Ambiguous semantics
- Parsing complexity
- But how can we express the syntax of PL using non-ambiguous grammars?

Non Ambiguous Grammar for Arithmetic Expressions

Ambiguous grammar

1 <E> → <E> + <E>
2 <E> → <E> * <E>
3 <E> → id
4 <E> → (<E>)

1 <E> → <E> + <T>
2 <E> → <T>
3 <T> → <T> * <F>
4 <T> → <F>
5 <F> → id
6 <F> → (<E>)

How do we show that the above grammars are the same?

Non Ambiguous Grammars for Arithmetic Expressions

Ambiguous grammar

1 <E> → <E> + <E>
2 <E> → <E> * <E>
3 <E> → id
4 <E> → (<E>)

1 <E> → <E> + <T>
2 <E> → <T>
3 T → <T> * <F>
4 T → <F>
5 F → id
6 F → (<E>)

1 <E> → <E> * <T>
2 <E> → <T>
3 <T> → <F> + <T>
4 <T> → <F>
5 <F> → id
6 <F> → (<E>)

Language Equivalence

- What does it mean for two grammars to be equivalent?
- How do we show that two grammars are equivalent?

Structural Induction

- Prove that the property holds for all simple trees
- Prove that the property holds for all composite trees using the fact that it holds for subtress
  - For each rule assume that the property holds for its premises (induction hypothesis) and prove it holds for the conclusion of the rule

Rule Induction

- Prove that the property holds for all simple derivation trees by showing it holds for axioms
- Prove that the property holds for all composite trees:
  - For each rule assume that the property holds for its premises (induction hypothesis) and prove it holds for the conclusion of the rule

Inductive Rules for Interpretation

Operational Semantics

Expression Evaluation

- The rules for expression evaluation can also be defined inductively
- $\text{Val} \subseteq \text{Exp} \times \text{Int}$
  - Denote $e,k \in \text{Val}$ by $e \to k$
- Leads to simple evaluation rules
- Easy to implement

Rules for expression evaluation

vexp-num-ax

$$\frac{}{\text{number} \rightarrow \text{number}}$$

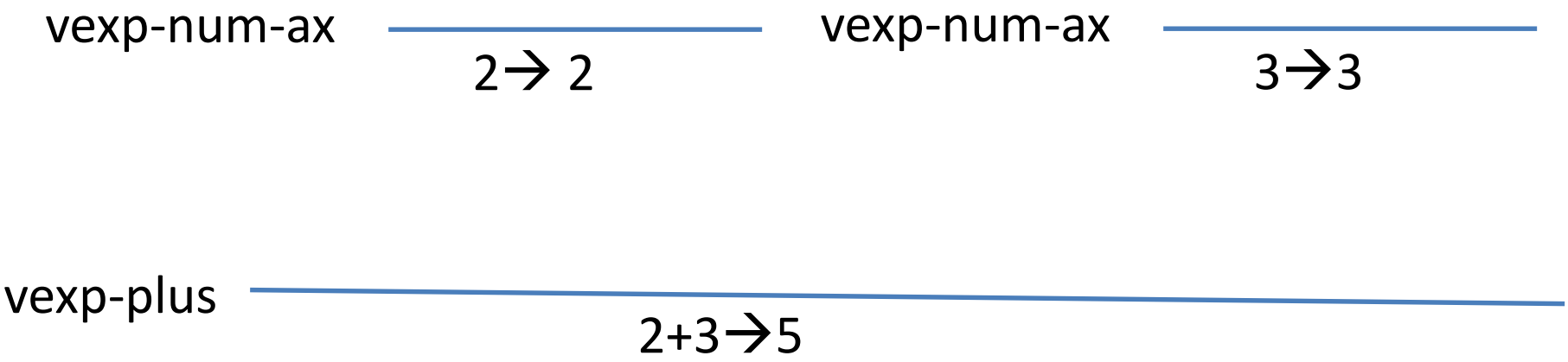
vexp-paren

$$\frac{e \rightarrow v}{(e) \rightarrow v}$$

vexp-plus

$$\frac{e_1 \rightarrow v_1 \qquad e_2 \rightarrow v_2}{e_1 + e_2 \rightarrow v_1 + v_2}$$

Example 2+3



Theorem: Values are deterministic

- For every expression e $\in$ Exp and values v_1, v_2 $\in$ Int
  - If e $\rightarrow v_1$ and e $\rightarrow$ v_2 then $v_1 = v_2$

Well Founded Induction

- Inductive arguments work since they cannot go forever
- Eventually we reach the basis

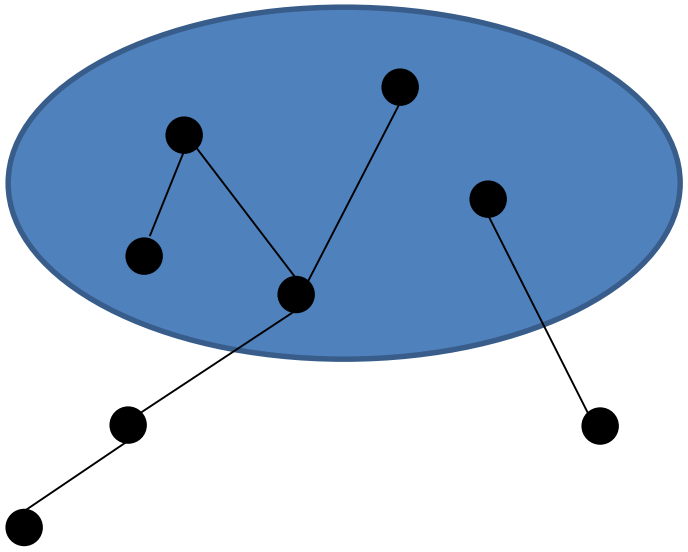
Well Founded Relations

- A binary relation $\prec$ on a set A is well founded if there are no infinite descending chains

  $\ldots \prec a_i \prec \ldots \prec a_1 \prec a_0$
- Examples
  - A=N and $x \prec y$ if $y = x+1$
  - A=N and $x \prec y$ if $x < y$
  - A is set of strings and $x \prec y$ if x is a prefix of y, i.e. there exists w such that x.w=y
  - A is set of strings and $x \prec y$ if y is a prefix of x, i.e. there exists w such that y.w=x

Minimal Elements

- A binary relation $\prec$ on a set A is well founded if and only if every non-empty subset Q of A has a minimal element m such that
  - $m \in Q \wedge \forall b.\ b \prec m \Rightarrow b \notin Q$



The Principle of Well-Founded Induction

- Let $\prec$ be a well-founded relation on a set A and P be a property then
  $\forall a.\ P(a)$
  iff
  $\forall c.\ [\forall b.\ (b \prec c \wedge P(b)) \Rightarrow P(c)]$
- Examples
  - For A=N and $x \prec y$ if we y = x+1 we get mathematical induction
  - For A=N and $x \prec y$ if we x <y we get Course of values Induction
  - For A = Derivation Trees and $x \prec y$ if x is a subtree of y we get structural induction

Further Reading

- Winskel: The formal semantics of programming languages: Chapter 3

Summary

- Induction is powerful
  - Mathematical
  - Course of values
  - Rule induction
  - Structural
  - Well Founded
- Coming up with induction hypothesis may be hard
- Induction is a key technique in programming languages