Concepts in Programming Languages – Recitation 5:

More (Untyped) Lambda Calculus

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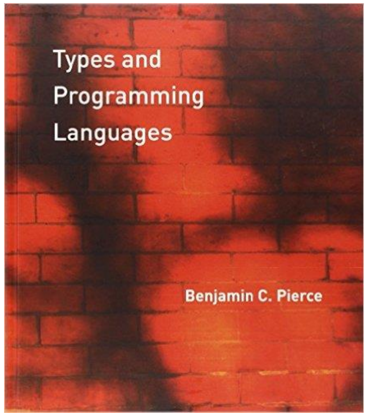
(original slides by Kathleen Fisher, John Mitchell, Shachar Itzhaky, S. Tanimoto)

Reference:
Types and Programming Languages
by Benjamin C. Pierce, Chapter 5



Untyped Lambda Calculus - Syntax

t ::=	terms
x	variable
λ x. t	abstraction
t t	application

- Terms can be represented as abstract syntax trees
- Syntactic Conventions:
  - Applications associates to left :
    $e_1\ e_2\ e_3 \equiv (e_1\ e_2)\ e_3$
  - The body of abstraction extends as far as possible:
    $\lambda x.\ \lambda y.\ x\ y\ x \equiv \lambda x.\ (\lambda y.\ (x\ y)\ x)$

Free vs. Bound Variables

- An occurrence of x in t is bound in λx. t
  - otherwise it is free
  - λx is a binder
- FV: t $\rightarrow$ P(Var) is the set free variables of t
  - FV(x) = {x}
  - FV(λ x. t) = FV(t) – {x}
  - FV (t_1 t_2) = FV(t_1) $\cup$ FV(t_2)
- Examples:
  - FV(x (y z)) =
  - FV(λx. λy. x (y z)) =
  - FV((λx. x)) =
  - FV((λx. x) x) =

Semantics: Substitution, β-reduction, α-conversion

- Substitution

  $[x \mapsto s]\ x = s$

  $[x \mapsto s]\ y = y$ if $y \neq x$

  $[x \mapsto s]\ (\lambda y.\ t_1) = \lambda y.\ [x \mapsto s]\ t_1$ if $y \neq x$ and $y \notin FV(s)$

  $[x \mapsto s]\ (t_1\ t_2) = ([x \mapsto s]\ t_1)\ ([x \mapsto s]\ t_2)$

- β-reduction

  $(\lambda\ x.\ t_1)\ t_2 \Rightarrow_\beta [x \mapsto t_2]\ t_1$

- α-conversion

  $(\lambda x.\ t) \Rightarrow_\alpha \lambda y.\ [x \mapsto y]\ t$ if $y \notin FV(t)$

Examples of β-reduction, α-conversion

$$\underline{(\lambda\ x.\ x)\ y} \Rightarrow_{\beta} \quad y$$

$$\underline{(\lambda\ x.\ x\ (\lambda\ x.\ x)\ )\ (u\ r)} \Rightarrow_{\text{”}\beta+\alpha\text{”}} \ u\ r\ (\lambda\ x.\ x)$$

$$\underline{(\lambda\ x\ (\lambda w.\ x\ w))\ (y\ z)} \Rightarrow_{\beta} \quad \lambda w.\ y\ z\ w$$

$$\underline{(\lambda x.\ (\lambda x.\ x))\ y} \ \Rightarrow_{\alpha} (\lambda x.\ (\lambda z.\ z))\ y \Rightarrow_{\beta} \ \lambda z.\ z$$

$$\underline{(\lambda x.\ (\lambda y.\ x))\ y} \ \Rightarrow_{\alpha} (\lambda x.\ (\lambda z.\ x))\ y \Rightarrow_{\beta} \ \lambda z.\ y$$

Currying – Multiple arguments

- Say we want to define a function with two arguments:
  - "f = λ(x, y). s"
- We do this by Currying:
  - f = λx. λy. s
  - f is now "a function of x that returns a function of y"
- Currying and β-reduction:

$$f\ v\ w \quad = (f\ v)\ w = \ ((\lambda x.\ \lambda y.\ s)\ v)\ w$$

$$\Rightarrow (\ \lambda y.[x \mapsto v]s\ )\ w \quad \Rightarrow [x \mapsto v]\ [y \mapsto w]\ s$$

- Conclusion:
  - "f = λ(x, y). s" → f = λx. λy. s
  - "f (v,w)" → f v w

Church Booleans

- Define: tru = $\lambda t.\ \lambda f.\ t$ fls = $\lambda t.\ \lambda f.\ f$ test = $\lambda l.\ \lambda m.\ \lambda n.\ l\ m\ n$
- test tru then else = $(\lambda l.\ \lambda m.\ \lambda n.\ l\ m\ n)\ (\lambda t.\ \lambda f.\ t)$ then else

  $\Rightarrow(\lambda m.\ \lambda n.\ (\lambda t.\ \lambda f.\ t)\ m\ n)$ then else

  $\Rightarrow(\lambda n.\ (\lambda t.\ \lambda f.\ t)$ then $n)$ else

  $\Rightarrow(\lambda t.\ \lambda f.\ t)$ then else

  $\Rightarrow(\lambda f.$ then$)$ else

  $\Rightarrow$then
- test fls then else = $(\lambda l.\ \lambda m.\ \lambda n.\ l\ m\ n)\ (\lambda t.\ \lambda f.\ f)$ then else

  $\Rightarrow(\lambda m.\ \lambda n.\ (\lambda t.\ \lambda f.\ f)\ m\ n)$ then else

  $\Rightarrow(\lambda n.\ (\lambda t.\ \lambda f.\ f)$ then $n)$ else

  $\Rightarrow(\lambda t.\ \lambda f.\ f)$ then else

  $\Rightarrow(\lambda f.\ f)$ else

  $\Rightarrow$else
- and = $\lambda b.\ \lambda c.\ b\ c$ fls
- or =
- not =

Church Numerals

- $c_0 = \lambda s.\ \lambda z.\ z$
- $c_1 = \lambda s.\ \lambda z.\ s\ z$
- $c_2 = \lambda s.\ \lambda z.\ s\ (s\ z)$
- $c_3 = \lambda s.\ \lambda z.\ s\ (s\ (s\ z))$
- …
- $\mathrm{scc} = \lambda n.\ \lambda s.\ \lambda z.\ s\ (n\ s\ z)$
- $\mathrm{plus} = \lambda m.\ \lambda n.\ \lambda s.\ \lambda z.\ m\ s\ (n\ s\ z)$
- $\mathrm{times} = \lambda m.\ \lambda n.\ m\ (\mathrm{plus}\ n)\ c_0$
- iszero =

Non-Deterministic Operational Semantics

(E-AppAbs) $(\lambda\ x.\ t_1)\ t_2 \Rightarrow [x \mapsto t_2]\ t_1$

$$\frac{t \Rightarrow t'}{\lambda\ x.\ t \Rightarrow \lambda\ x.\ t'}$$ (E-Abs)

(E-App$_1$) $$\frac{t_1 \Rightarrow t'_1}{t_1\ \ t_2 \Rightarrow t'_1\ t_2}$$

$$\frac{t_2 \Rightarrow t'_2}{t_1\ \ t_2 \Rightarrow t_1\ t'_2}$$ (E-App$_2$)

Why is this semantics non-deterministic?

Different Evaluation Orders

(E-AppAbs) $(\lambda\ x.\ t_1)\ t_2 \Rightarrow [x \mapsto t_2]\ t_1$

$$\frac{t \Rightarrow t'}{\lambda\ x.\ t \Rightarrow \lambda\ x.\ t'}$$ (E-Abs)

(E-App$_1$) $$\frac{t_1 \Rightarrow t'_1}{t_1\ \ t_2 \Rightarrow t'_1\ t_2}$$

$$\frac{t_2 \Rightarrow t'_2}{t_1\ \ t_2 \Rightarrow t_1\ t'_2}$$ (E-App$_2$)

$(\lambda x.\ (\text{add } x\ x))\ (\text{add } 2\ 3) \Rightarrow (\lambda x.\ (\text{add } x\ x))\ (5) \Rightarrow \text{add } 5\ 5 \Rightarrow 10$

$(\lambda x.\ (\text{add } x\ x))\ (\text{add } 2\ 3) \Rightarrow (\text{add } (\text{add } 2\ 3)\ (\text{add } 2\ 3)) \Rightarrow$

$(\text{add } 5\ (\text{add } 2\ 3)) \Rightarrow (\text{add } 5\ 5) \Rightarrow 10$

This example: same final result but lazy performs more computations

Different Evaluation Orders

(E-AppAbs) $(\lambda\ x.\ t_1)\ t_2 \Rightarrow [x \mapsto t_2]\ t_1$

$$\frac{t \Rightarrow t'}{\lambda\ x.\ t \Rightarrow \lambda\ x.\ t'}$$ (E-Abs)

(E-App$_1$) $$\frac{t_1 \Rightarrow t'_1}{t_1\ t_2 \Rightarrow t'_1\ t_2}$$

$$\frac{t_2 \Rightarrow t'_2}{t_1\ t_2 \Rightarrow t_1\ t'_2}$$ (E-App$_2$)

$(\lambda x.\ \lambda y.\ x)\ 3\ (\text{div}\ 5\ 0) \Rightarrow$ Exception: Division by zero

$(\lambda x.\ \lambda y.\ x)\ 3\ (\text{div}\ 5\ 0) \Rightarrow (\lambda y.\ 3)\ (\text{div}\ 5\ 0) \Rightarrow 3$

This example: lazy suppresses erroneous division and reduces to final result

Can also suppress non-terminating computation.

Many times we want this, for example:

```
if i < len(a) and a[i]==0: print “found zero”
```

Strict

(E-App$_1$)

$$\frac{t_1 \Rightarrow t'_1}{t_1\ t_2 \Rightarrow t'_1\ t_2}$$

precedence

(E-App$_2$)

$$\frac{t_2 \Rightarrow t'_2}{t_1\ t_2 \Rightarrow t_1\ t'_2}$$

precedence

(E-AppAbs)

$$(\lambda\ x.\ t_1)\ t_2 \Rightarrow [x \mapsto t_2]\ t_1$$

Lazy

(E-AppAbs)

$$(\lambda\ x.\ t_1)\ t_2 \Rightarrow [x \mapsto t_2]\ t_1$$

(E-App$_1$)

$$\frac{t_1 \Rightarrow t'_1}{t_1\ t_2 \Rightarrow t'_1\ t_2}$$

Normal Order

(E-AppAbs)

$$(\lambda\ x.\ t_1)\ t_2 \Rightarrow [x \mapsto t_2]\ t_1$$

precedence

(E-App$_1$)

$$\frac{t_1 \Rightarrow t'_1}{t_1\ t_2 \Rightarrow t'_1\ t_2}$$

precedence

(E-App$_2$)

$$\frac{t_2 \Rightarrow t'_2}{t_1\ t_2 \Rightarrow t_1\ t'_2}$$

(E-Abs)

$$\frac{t \Rightarrow t'}{\lambda\ x.\ t \Rightarrow \lambda\ x.\ t'}$$

Call-by-value Operations Semantics via Inductive Definition (no precedence)

$t ::=$	terms
x	variable
$\lambda\ x.\ t$	abstraction
$t\ t$	application

$v ::= \lambda\ x.\ t$ abstraction values

$$(\lambda\ x.\ t_1)\ v_2 \Rightarrow [x \mapsto v_2]\ t_1$$

(E-AppAbs)

$$\frac{t_1 \Rightarrow t'_1}{t_1\ t_2 \Rightarrow t'_1\ t_2}$$

(E-APPL1)

$$\frac{t_2 \Rightarrow t'_2}{v_1\ t_2 \Rightarrow v_1\ t'_2}$$

(E-APPL2)

Summary Order of Evaluation

- Full-beta-reduction
  - All possible orders
- Applicative order call by value (strict, eager)
  - Left to right
  - Fully evaluate arguments before function application
- Normal order
  - The leftmost, outermost redex is always reduced first
- Call by name (lazy)
  - Evaluate arguments as needed
- Call by need
  - Evaluate arguments as needed and store for subsequent usages
  - Implemented in Haskell

Church–Rosser Theorem

If:

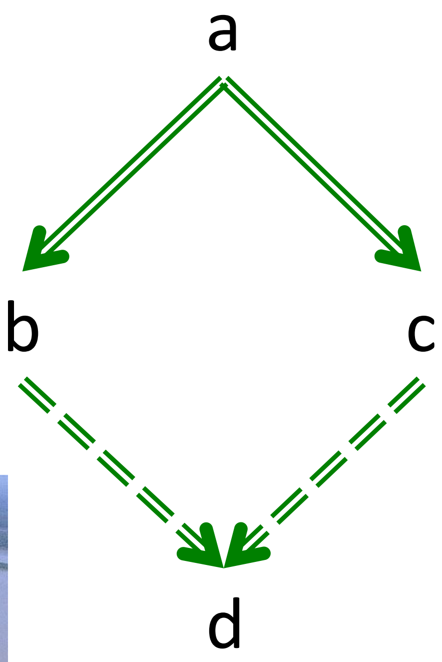
$a \Rightarrow^{*} b$,

$a \Rightarrow^{*} c$

then there exists d such that:

$b \Rightarrow^{*} d$, and

$c \Rightarrow^{*} d$



Normal Form & Halting Problem

- A term is in normal form if it is stuck in normal order semantics
- Under normal order every term either:
  - Reduces to normal form, or
  - Reduces infinitely
- For a given term, it is undecidable to decide which is the case

Combinators

- A combinator is a function in the Lambda Calculus having no free variables
- Examples
  - $\lambda x.\ x$ is a combinator
  - $\lambda x.\ \lambda y.\ (x\ y)$ is a combinator
  - $\lambda x.\ \lambda y.\ (x\ z)$ is not a combinator
- Combinators can serve nicely as modular building blocks for more complex expressions
- The Church numerals and simulated Booleans are examples of useful combinators

Iteration in Lambda Calculus

Williams + Hirakawa

- omega = $(\lambda x.\ x\ x)\ (\lambda x.\ x\ x)$
  - $(\lambda x.\ x\ x)\ (\lambda x.\ x\ x) \Rightarrow (\lambda x.\ x\ x)\ (\lambda x.\ x\ x)$

- $Y = \lambda f.\ (\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x))$
- $Z = \lambda f.\ (\lambda x.\ f\ (\lambda y.\ x\ x\ y))\ (\lambda x.\ f\ (\lambda y.\ x\ x\ y))$
- Recursion can be simulated
  - Y only works with call-by-name semantics
  - Z works with call-by-value semantics
- Defining factorial:
  - $g = \lambda f.\ \lambda n.$ if $n{=}{=}0$ then 1 else $(n * (f\ (n - 1)))$
  - fact = Y g (for call-by-name)
  - fact = Z g (for call-by-value)

Y-Combinator in action (lazy)

“g = λf. λn. if n==0 then 1 else (n * (f (n - 1)))”

Y = λf. (λx. f (x x)) (λx. f (x x))

Y g v = (λf. (λx. f (x x)) (λx. f (x x))) g v

$\Rightarrow \underline{((\lambda x.\ g\ (x\ x))\ (\lambda x.\ g\ (x\ x)))}\ v$

$\Rightarrow (g\ \underline{((\lambda x.\ g\ (x\ x))\ (\lambda x.\ g\ (x\ x)))}\ )\ v$

~ (g (Y g)) v

What happens to Y in strict semantics?

Z-Combinator in action (strict)

“g = λf. λn. if n==0 then 1 else (n * (f (n - 1)))”

Z = λf. (λx. f (λy. x x y)) (λx. f (λy. x x y))

Z g v = (λf. (λx. f (λy. x x y)) (λx. f (λy. x x y))) g v

$\Rightarrow \underline{((\lambda x.\ g\ (\lambda y.\ x\ x\ y))\ (\lambda x.\ g\ (\lambda y.\ x\ x\ y)))}\ v$

$\Rightarrow (g\ (\lambda y.\ \underline{(\lambda x.\ g\ (\lambda y.\ x\ x\ y))\ (\lambda x.\ g\ (\lambda y.\ x\ x\ y))}\ y))\ v$

$\sim\ \ (g\ \underline{(\lambda y.\ (Z\ g)\ y)}\ )\ v$

$\sim\ \ (g\ (Z\ g))\ v$

```
def f1(y):
    return f2(y)
```

Simulating laziness like Z-Combinator

```
def f(x):
    if ask_user("wanna see it?"):
        print x

def g(x, y, z):
    # very expensive computation without side effects

def main():
    # compute a, b, c with side effects
    f(g(a, b, c))
```

- In strict semantics, the above code computes g anyway
  - Lazy will avoid it
- How can achieve this in a strict programming language?

Simulating laziness like Z-Combinator

```
def f(x):
    if ask_user(“?"):
        print x

def g(x, y, z):
    # expensive

def main():
    # compute a, b, c
    f(g(a, b, c))
```

```
def f(x):
    if ask_user("?"):
        print x()

def g(x, y, z):
    # expensive

def main():
    # compute a, b, c
    f(lambda: g(a, b, c))
```

$$Z = \lambda f.\ (\lambda x.\ f\ (\lambda y.\ x\ x\ y))\ (\lambda x.\ f\ (\lambda y.\ x\ x\ y))$$

$$\xcancel{\begin{array}{c}\textbf{(E-Abs)}\\ \dfrac{t \Rightarrow t'}{\lambda\, x.\ t \Rightarrow \lambda\, x.\ t'}\end{array}}$$