Concepts in Programming Languages – Recitation 5:

More (Untyped) Lambda Calculus

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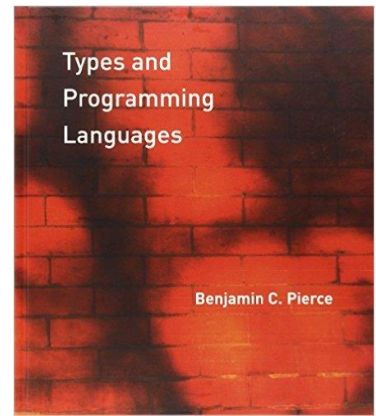
(original slides by Kathleen Fisher, John Mitchell, Shachar Itzhaky, S. Tanimoto)

Reference:
Types and Programming Languages
by Benjamin C. Pierce, Chapter 5



Untyped Lambda Calculus - Syntax

t ::=	terms
x	variable
λ x. t	abstraction
t t	application

- Terms can be represented as abstract syntax trees
- Syntactic Conventions:
  - Applications associates to left :
    $e_1\ e_2\ e_3 \equiv (e_1\ e_2)\ e_3$
  - The body of abstraction extends as far as possible:
    $\lambda x.\ \lambda y.\ x\ y\ x \equiv \lambda x.\ (\lambda y.\ (x\ y)\ x)$

Free vs. Bound Variables

- An occurrence of x in t is bound in λx. t
  - otherwise it is free
  - λx is a binder
- FV: t $\rightarrow$ P(Var) is the set free variables of t
  - $FV(x) = \{x\}$
  - $FV(\lambda x.\ t) = FV(t) - \{x\}$
  - $FV(t_1\ t_2) = FV(t_1) \cup FV(t_2)$
- Examples:
  - FV(x (y z)) =
  - FV(λx. λy. x (y z)) =
  - FV((λx. x)) =
  - FV((λx. x) x) =

Semantics: Substitution, β-reduction, α-conversion

- Substitution

$[x \mapsto s]\ x = s$

$[x \mapsto s]\ y = y$ if $y \neq x$

$[x \mapsto s]\ (\lambda y.\ t_1) = \lambda y.\ [x \mapsto s]\ t_1$ if $y \neq x$ and $y \notin FV(s)$

$[x \mapsto s]\ (t_1\ t_2) = ([x \mapsto s]\ t_1)\ ([x \mapsto s]\ t_2)$

- β-reduction

$(\lambda\ x.\ t_1)\ t_2 \Rightarrow_\beta [x \mapsto t_2]\ t_1$

- α-conversion

$(\lambda x.\ t) \Rightarrow_\alpha \lambda y.\ [x \mapsto y]\ t$ if $y \notin FV(t)$

Examples of β-reduction, α-conversion

$\underline{(\lambda\ x.\ x)\ y} \Rightarrow_{\beta}\ y$

$\underline{(\lambda\ x.\ x\ (\lambda\ x.\ x)\ )\ (u\ r)} \Rightarrow_{\text{"}\beta+\alpha\text{"}}\ u\ r\ (\lambda\ x.\ x)$

$\underline{(\lambda\ x\ (\lambda w.\ x\ w))\ (y\ z)} \Rightarrow_{\beta}\ \lambda w.\ y\ z\ w$

$\underline{(\lambda x.\ (\lambda x.\ x))\ y} \Rightarrow_{\alpha} (\lambda x.\ (\lambda z.\ z))\ y \Rightarrow_{\beta}\ \lambda z.\ z$

$\underline{(\lambda x.\ (\lambda y.\ x))\ y} \Rightarrow_{\alpha} (\lambda x.\ (\lambda z.\ x))\ y \Rightarrow_{\beta}\ \lambda z.\ y$

Non-Deterministic Operational Semantics

(E-AppAbs) $(\lambda\, x.\, t_1)\; t_2 \Rightarrow [x \mapsto t_2]\; t_1$

$$\frac{t \Rightarrow t'}{\lambda\, x.\, t \Rightarrow \lambda\, x.\, t'}$$ (E-Abs)

(E-App$_1$) $$\frac{t_1 \Rightarrow t'_1}{t_1\;\, t_2 \Rightarrow t'_1\; t_2}$$

$$\frac{t_2 \Rightarrow t'_2}{t_1\;\, t_2 \Rightarrow t_1\; t'_2}$$ (E-App$_2$)

Why is this semantics non-deterministic?

Strict

(E-App$_1$)

$$\frac{t_1 \Rightarrow t'_1}{t_1\ t_2 \Rightarrow t'_1\ t_2}$$

precedence

(E-App$_2$)

$$\frac{t_2 \Rightarrow t'_2}{t_1\ t_2 \Rightarrow t_1\ t'_2}$$

precedence

(E-AppAbs)

$$(\lambda\ x.\ t_1)\ t_2 \Rightarrow [x \mapsto t_2]\ t_1$$

Lazy

(E-AppAbs)

$$(\lambda\ x.\ t_1)\ t_2 \Rightarrow [x \mapsto t_2]\ t_1$$

(E-App$_1$)

$$\frac{t_1 \Rightarrow t'_1}{t_1\ t_2 \Rightarrow t'_1\ t_2}$$

Normal Order

(E-AppAbs)

$$(\lambda\ x.\ t_1)\ t_2 \Rightarrow [x \mapsto t_2]\ t_1$$

precedence

(E-App$_1$)

$$\frac{t_1 \Rightarrow t'_1}{t_1\ t_2 \Rightarrow t'_1\ t_2}$$

precedence

(E-App$_2$)

$$\frac{t_2 \Rightarrow t'_2}{t_1\ t_2 \Rightarrow t_1\ t'_2}$$

(E-Abs)

$$\frac{t \Rightarrow t'}{\lambda\ x.\ t \Rightarrow \lambda\ x.\ t'}$$

Call-by-value Operations Semantics via Inductive Definition (no precedence)

t ::=	terms
x	variable
λ x. t	abstraction
t t	application

v ::= λ x. t   abstraction values

$$(\lambda\ x.\ t_1)\ v_2 \Rightarrow [x \mapsto v_2]\ t_1$$ (E-AppAbs)

$$\frac{t_1 \Rightarrow t'_1}{t_1\ \ t_2\ \Rightarrow\ t'_1\ t_2}$$ (E-APPL1)

$$\frac{t_2 \Rightarrow t'_2}{v_1\ \ t_2\ \Rightarrow\ v_1\ t'_2}$$ (E-APPL2)

Summary Order of Evaluation

- Full-beta-reduction
  - All possible orders
- Applicative order call by value (strict, eager)
  - Left to right
  - Fully evaluate arguments before function application
- Normal order
  - The leftmost, outermost redex is always reduced first
- Call by name (lazy)
  - Evaluate arguments as needed
- Call by need
  - Evaluate arguments as needed and store for subsequent usages
  - Implemented in Haskell

Church–Rosser Theorem

If:

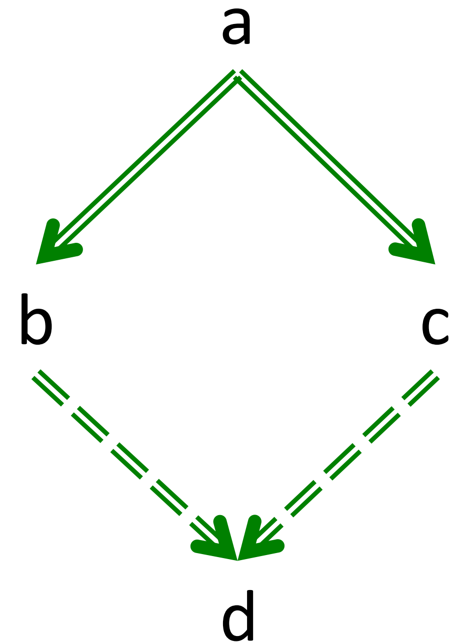
$a \Rightarrow^{*} b$,

$a \Rightarrow^{*} c$

then there exists d such that:

$b \Rightarrow^{*} d$, and

$c \Rightarrow^{*} d$



Normal Form & Halting Problem

- A term is in normal form if it is stuck in normal order semantics
- Under normal order every term either:
  - Reduces to normal form, or
  - Reduces infinitely
- For a given term, it is undecidable to decide which is the case

Iteration in Lambda Calculus

- omega = $(\lambda x.\ x\ x)\ (\lambda x.\ x\ x)$
  - $(\lambda x.\ x\ x)\ (\lambda x.\ x\ x) \Rightarrow (\lambda x.\ x\ x)\ (\lambda x.\ x\ x)$

- Y = $\lambda f.\ (\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x))$
- Z = $\lambda f.\ (\lambda x.\ f\ (\lambda y.\ x\ x\ y))\ (\lambda x.\ f\ (\lambda y.\ x\ x\ y))$
- Recursion can be simulated
  - Y only works with call-by-name semantics
  - Z works with call-by-value semantics
- Defining factorial:
  - g = $\lambda f.\ \lambda n.$ if n==0 then 1 else (n * (f (n - 1)))
  - fact = Y g (for call-by-name)
  - fact = Z g (for call-by-value)

Y-Combinator in action (lazy)

$Y = \lambda f.\ (\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x))$

$Y\ g\ v = (\lambda f.\ (\lambda x.\ f\ (x\ x))\ (\lambda x.\ f\ (x\ x)))\ g\ v$

$\Rightarrow ((\lambda x.\ g\ (x\ x))\ (\lambda x.\ g\ (x\ x)))\ v$

$\Rightarrow (g\ ((\lambda x.\ g\ (x\ x))\ (\lambda x.\ g\ (x\ x)))\ )\ v$

$\sim\ (g\ (Y\ g))\ v$

Z-Combinator in action (strict)

$Z = \lambda f.\ (\lambda x.\ f\ (\lambda y.\ x\ x\ y))\ (\lambda x.\ f\ (\lambda y.\ x\ x\ y))$

$Z\ g\ v = (\lambda f.\ (\lambda x.\ f\ (\lambda y.\ x\ x\ y))\ (\lambda x.\ f\ (\lambda y.\ x\ x\ y)))\ g\ v$

$\Rightarrow ((\lambda x.\ g\ (\lambda y.\ x\ x\ y))\ (\lambda x.\ g\ (\lambda y.\ x\ x\ y)))\ v$

$\Rightarrow (g\ (\lambda y.\ (\lambda x.\ g\ (\lambda y.\ x\ x\ y))\ (\lambda x.\ g\ (\lambda y.\ x\ x\ y))\ y))\ v$

$\sim\ (g\ (\lambda y.\ (Z\ g)\ y)\ )\ v$

$\sim\ (g\ (Z\ g))\ v$

```
def f1(y):
    return f2(y)
```

Simulating laziness like Z-Combinator

```
def f(x):
    if ask_user("wanna see it?"):
        print x

def g(x, y, z):
    # very expensive computation without side effects

def main():
    # compute a, b, c with side effects
    f(g(a, b, c))
```

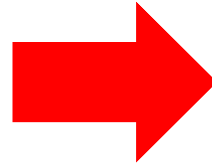
- In strict semantics, the above code computes g anyway
  - Lazy will avoid it
- How can achieve this in a strict programming language?

Simulating laziness like Z-Combinator

```
def f(x):
    if ask_user(“?"):
        print x

def g(x, y, z):
    # expensive

def main():
    # compute a, b, c
    f(g(a, b, c))
```



```
def f(x):
    if ask_user("?"):
        print x()

def g(x, y, z):
    # expensive

def main():
    # compute a, b, c
    f(lambda: g(a, b, c))
```

$Z = \lambda f.\ (\lambda x.\ f\ (\lambda y.\ x\ x\ y))\ (\lambda x.\ f\ (\lambda y.\ x\ x\ y))$