Concepts in Programming Languages – Recitation 5:

Untyped Lambda Calculus

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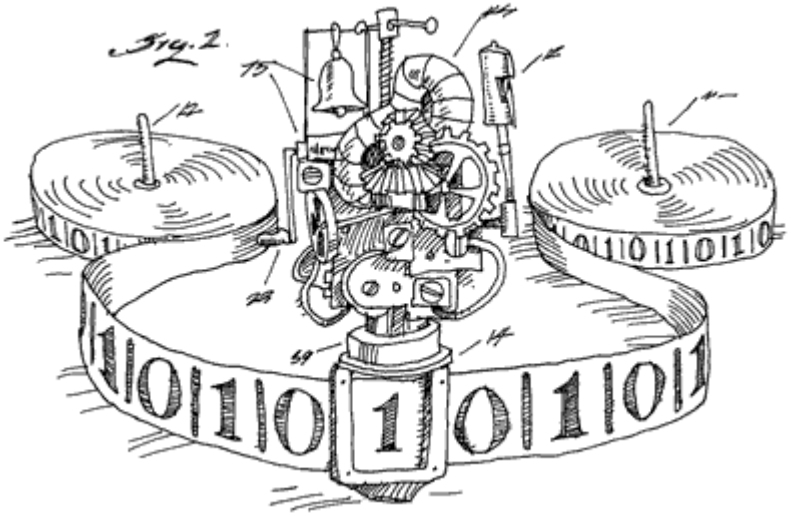
(original slides by Kathleen Fisher, John Mitchell, Shachar Itzhaky, S. Tanimoto)

Reference:

Types and Programming Languages by Benjamin C. Pierce, Chapter 5

Computation Models

- Turing Machines
- Wang Machines
- Counter Programs
- Lambda Calculus



Historical Context

Like Alan Turing, another mathematician, Alonzo Church, was very interested, during the 1930s, in the question “What is a computable function?”

He developed a formal system known as the pure lambda calculus, in order to describe programs in a simple and precise way.

Today the Lambda Calculus serves as a mathematical foundation for the study of functional programming languages, and especially for the study of “denotational semantics.”

Reference: http://en.wikipedia.org/wiki/Lambda_calculus

Untyped Lambda Calculus - Syntax

t ::=	terms
x	variable
λ x. t	abstraction
t t	application

Terms can be represented as abstract syntax trees

Syntactic Conventions

- Applications associates to left

  $e_1\ e_2\ e_3 \equiv (e_1\ e_2)\ e_3$

- The body of abstraction extends as far as possible
  - $\lambda x.\ \lambda y.\ x\ y\ x \equiv \lambda x.\ (\lambda y.\ (x\ y)\ x)$

Lambda Calculus Syntax vs. Python Syntax

$(\lambda\ x.\ x)\ y$ (lambda x: x) (y)

Free vs. Bound Variables

- An occurrence of x is bound in t if it occurs in $\lambda x.\ t$
  - otherwise it is free
  - λx is a binder
- Examples
  - Id= $\lambda x.\ x$
  - $\lambda y.\ x\ (y\ z)$
  - $\lambda z.\ \lambda x.\ \lambda y.\ x\ (y\ z)$
  - $(\lambda x.\ x)\ x$

FV: t $\rightarrow$ P(Var) is the set free variables of t

$FV(x) = \{x\}$

$FV(\lambda x.\ t) = FV(t) - \{x\}$

$FV\ (t_1\ t_2) = FV(t_1) \cup FV(t_2)$

Semantics: Beta-Reduction

$(\lambda\, x.\, t_1)\; t_2 \Rightarrow_\beta [x \mapsto t_2]\; t_1$ (β-reduction)

$[x \mapsto s]\; x = s$

$[x \mapsto s]\; y = y$ if $y \neq x$

$[x \mapsto s]\; (\lambda y.\, t_1) = \lambda y.\, [x \mapsto s]\; t_1$ if $y \neq x$ and $y \notin FV(s)$

$[x \mapsto s]\; (t_1\; t_2) = ([x \mapsto s]\; t_1)\; ([x \mapsto s]\; t_2)$

Beta-Reduction: Examples

$(\lambda\ x.\ t_1)\ t_2 \Rightarrow_\beta [x \mapsto t_2]\ t_1$ (β-reduction)

redex

$(\lambda\ x.\ x)\ y \Rightarrow_\beta\ y$

$(\lambda\ x.\ x\ (\lambda\ x.\ x)\ )\ (u\ r) \Rightarrow_\beta\ u\ r\ (\lambda\ x.\ x)$

$(\lambda\ x\ (\lambda w.\ x\ w))\ (y\ z) \Rightarrow_\beta\ \lambda w.\ y\ z\ w$

Substitution Subtleties

$(\lambda\, x.\, t_1)\; t_2 \Rightarrow_\beta [x \mapsto t_2]\; t_1$ (β-reduction)

$[x \mapsto s]\; x = s$

$[x \mapsto s]\; y = y$ if $y \neq x$

$[x \mapsto s]\; (\lambda y.\, t_1) = \lambda y.\, [x \mapsto s]\; t_1$ **if $y \neq x$ and $y \notin FV(s)$**

$[x \mapsto s]\; (t_1\; t_2) = ([x \mapsto s]\; t_1)\; ([x \mapsto s]\; t_2)$

$(\lambda x.\, (\lambda x.\, x))\; y \Rightarrow_\beta [x \mapsto y]\; (\lambda x.\, x) = \lambda x.\, x$

$(\lambda x.\, (\lambda y.\, x))\; y \Rightarrow_\beta [x \mapsto y]\; (\lambda y.\, x) = \lambda y.\, y?$

$(\lambda x.\, (\lambda y.\, x))\; y$ is stuck! it has no β-reduction

Alpha – Conversion

Alpha conversion:
Renaming of a bound variable and its bound occurrences

$(\lambda x.\ t) \Rightarrow_\alpha \lambda y.\ [x \mapsto y]\ t \quad$ if $y \notin FV(t)$

$(\lambda x.\ (\lambda y.\ x))\ y \Rightarrow_\alpha (\lambda x.\ (\lambda z.\ x))\ y \Rightarrow_\beta [x \mapsto y]\ (\lambda z.\ x) = \lambda z.\ y \neq \lambda y.\ y$

Non-Deterministic Operational Semantics

$$(\lambda\, x.\, t_1)\; t_2 \Rightarrow [x \mapsto t_2]\; t_1$$

$$\frac{t \Rightarrow t'}{\lambda\, x.\, t \Rightarrow \lambda\, x.\, t'}$$

$$\frac{t_1 \Rightarrow t'_1}{t_1\; t_2 \Rightarrow t'_1\; t_2}$$

$$\frac{t_2 \Rightarrow t'_2}{t_1\; t_2 \Rightarrow t_1\; t'_2}$$

Why is this semantics non-deterministic?

Different Evaluation Orders

$$(\lambda\, x.\, t_1)\; t_2 \Rightarrow [x \mapsto t_2]\; t_1$$

$$\frac{t \Rightarrow t'}{\lambda\, x.\, t \Rightarrow \lambda\, x.\, t'}$$

$$\frac{t_1 \Rightarrow t'_1}{t_1\; t_2 \Rightarrow t'_1\; t_2}$$

$$\frac{t_2 \Rightarrow t'_2}{t_1\; t_2 \Rightarrow t_1\; t'_2}$$

$(\lambda x.\ (\text{add } x\ x))\ (\text{add } 2\ 3) \Rightarrow (\lambda x.\ (\text{add } x\ x))\ (5) \Rightarrow \text{add } 5\ 5 \Rightarrow 10$

$(\lambda x.\ (\text{add } x\ x))\ (\text{add } 2\ 3) \Rightarrow (\text{add } (\text{add } 2\ 3)\ (\text{add } 2\ 3)) \Rightarrow$

$(\text{add } 5\ (\text{add } 2\ 3)) \Rightarrow \quad (\text{add } 5\ 5) \Rightarrow \quad 10$

This example: same final result but lazy performs more computations

Different Evaluation Orders

$$(\lambda\, x.\, t_1)\; t_2 \Rightarrow [x \mapsto t_2]\; t_1$$

$$\frac{t \Rightarrow t'}{\lambda\, x.\, t \Rightarrow \lambda\, x.\, t'}$$

$$\frac{t_1 \Rightarrow t'_1}{t_1\; t_2 \Rightarrow t'_1\; t_2}$$

$$\frac{t_2 \Rightarrow t'_2}{t_1\; t_2 \Rightarrow t_1\; t'_2}$$

$(\lambda x.\, \lambda y.\, x)\; 3\; (\text{div } 5\; 0) \Rightarrow$ Exception: Division by zero

$(\lambda x.\, \lambda y.\, x)\; 3\; (\text{div } 5\; 0) \Rightarrow (\lambda y.\, 3)\; (\text{div } 5\; 0) \Rightarrow 3$

This example: lazy suppresses erroneous division and reduces to final result

Can also suppress non-terminating computation.

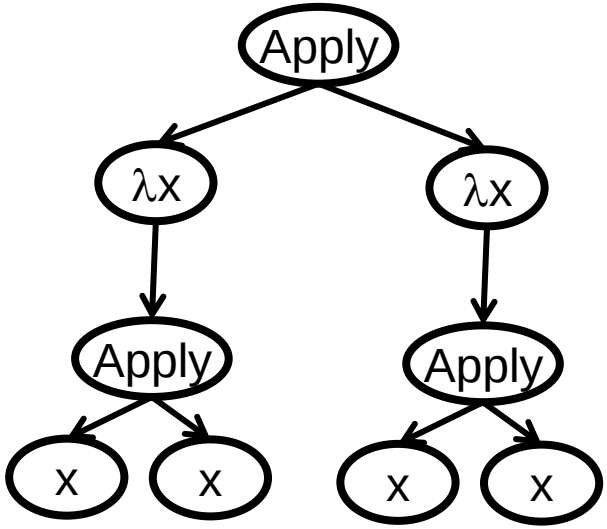
Many times we want this, for example:

```
if i < len(a) and a[i]==0: print “found zero”
```

Divergence

$(\lambda\ x.\ t_1)\ t_2 \Rightarrow_\beta [x \mapsto t_2]\ t_1$ (β-reduction)

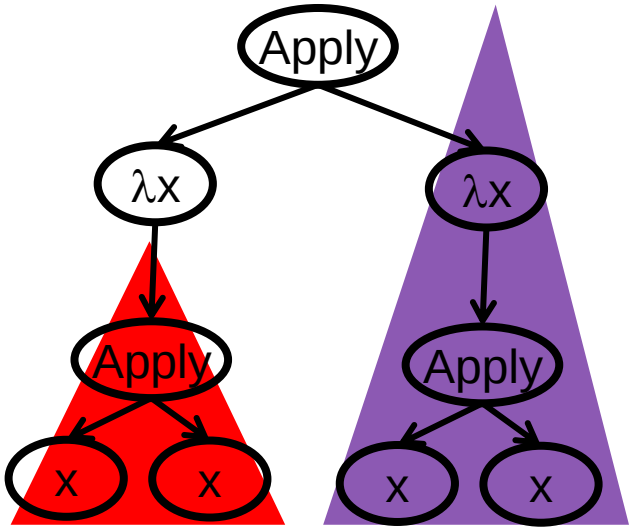
$(\lambda\ x.(x\ \ x))\ (\lambda\ x.(x\ \ x))$



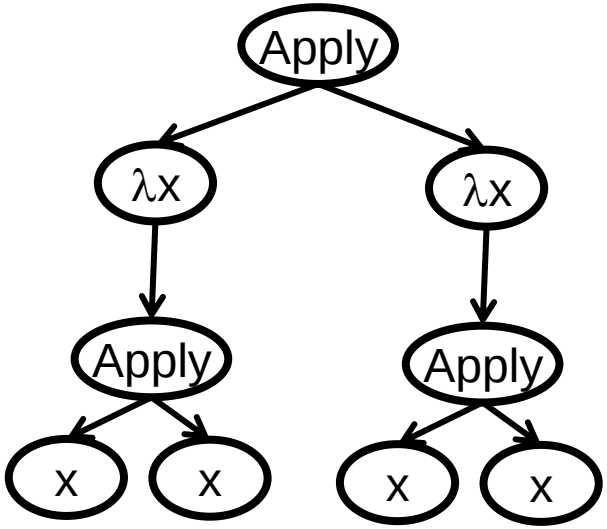
Divergence

$(\lambda\ x.\ t_1)\ t_2 \Rightarrow_\beta [x \mapsto t_2]\ t_1$ (β-reduction)

$(\lambda\ x.(x\ \ x))\ (\lambda\ x.(x\ \ x))$



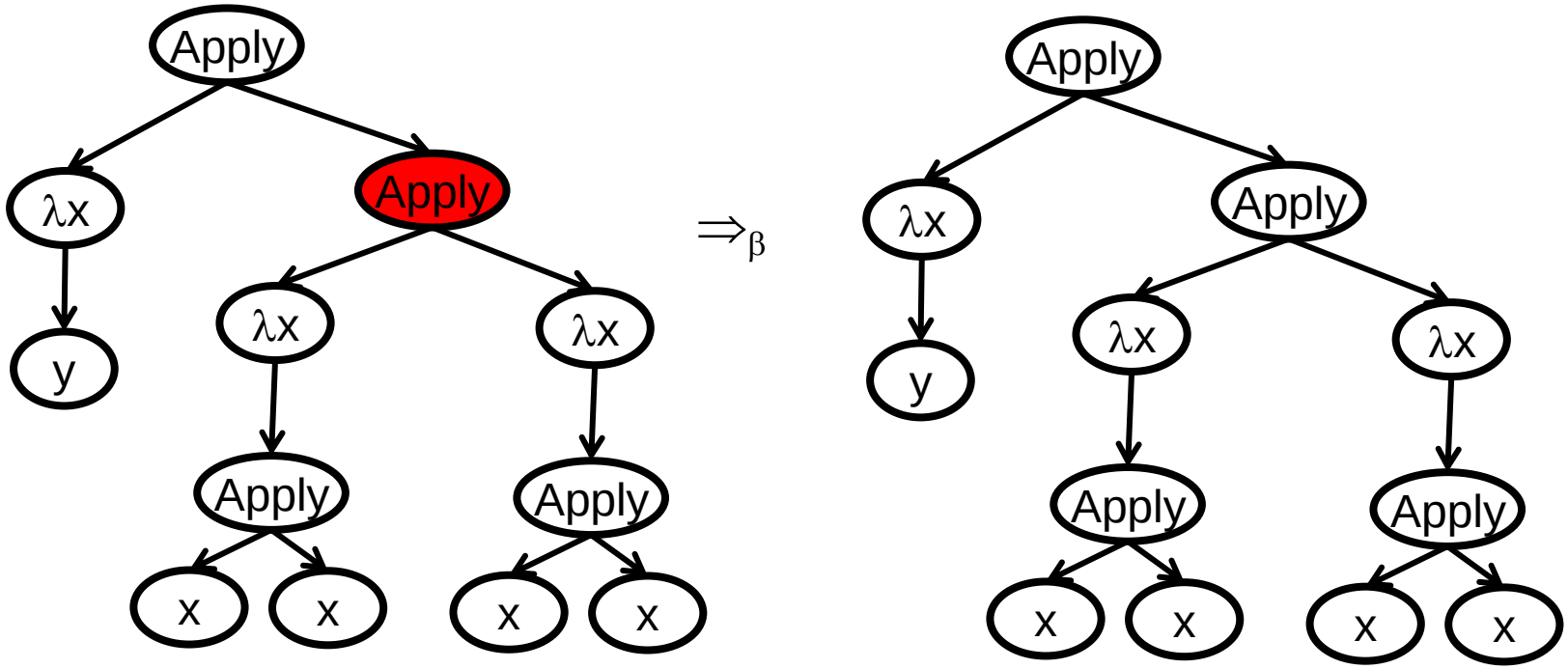
$\Rightarrow_\beta$



Different Evaluation Orders

$(\lambda\ x.\ t_1)\ t_2 \Rightarrow_\beta [x \mapsto t_2]\ t_1$ (β-reduction)

$(\lambda\ x.y)\ ((\lambda\ x.(x\ \ x))\ (\lambda\ x.(x\ \ x)))$



Different Evaluation Orders

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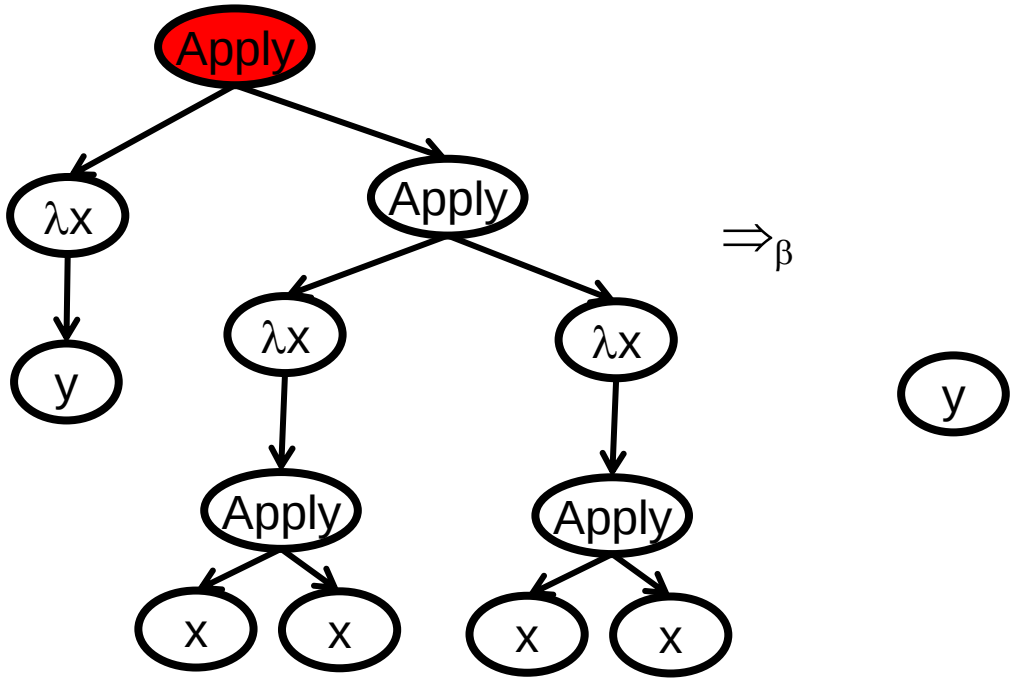

Different Evaluation Orders

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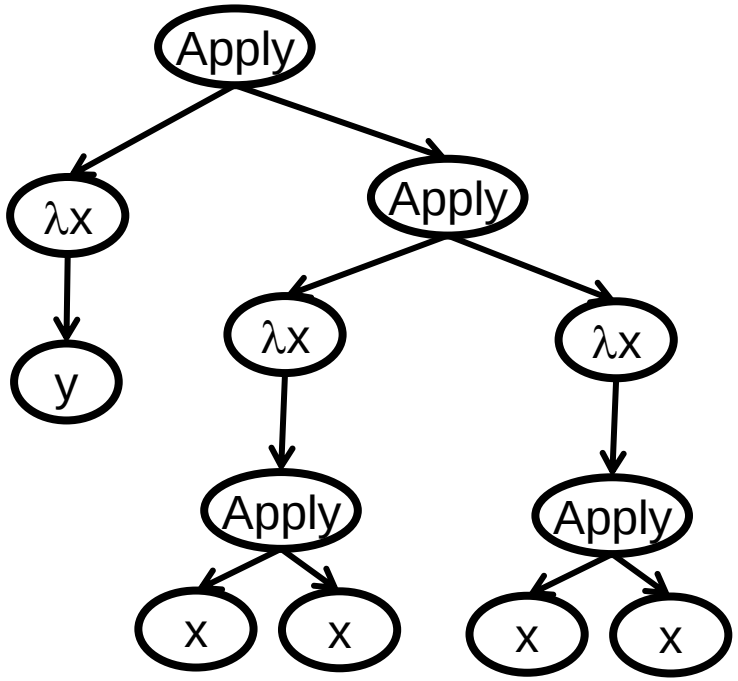
$(\lambda\ x.y)\ ((\lambda\ x.(x\ \ x))\ (\lambda\ x.(x\ \ x)))$




```
def f():
    while True: pass

def g(x):
    return 2

print g(f())
```

Summary Order of Evaluation

- Full-beta-reduction
  - All possible orders
- Applicative order call by value
  - Left to right
  - Fully evaluate arguments before function
- Normal order
  - The leftmost, outermost redex is always reduced first
- Call by name
  - Evaluate arguments as needed
- Call by need
  - Evaluate arguments as needed and store for subsequent usages
  - Implemented in Haskell

Call-by-value Operational Semantics

t ::=	terms
x	variable
λ x. t	abstraction
t t	application

v ::=	values
λ x. t	abstraction values
other values	

$$(\lambda\, x.\, t_1)\; v_2 \Rightarrow [x \mapsto v_2]\; t_1 \qquad \text{(E-AppAbs)}$$

$$\frac{t_1 \Rightarrow t'_1}{t_1\ t_2 \Rightarrow t'_1\ t_2} \qquad \text{(E-APPL1)}$$

$$\frac{t_2 \Rightarrow t'_2}{v_1\ t_2 \Rightarrow v_1\ t'_2} \qquad \text{(E-APPL2)}$$

Currying – Multiple arguments

$f = \lambda(x, y).\ s$

-> Currying

$f = \lambda x.\ \lambda y.\ s$

$f\ v\ w =$

$(f\ v)\ w =$

$(\lambda x.\ \lambda y.s\ v)\ w \Rightarrow$

$\lambda y.[x \mapsto v]s)\ w) \Rightarrow$

$[x \mapsto v]\ [y \mapsto w]\ s$

Church Booleans

- tru = λt. λf. t
- fls = λt. λf. f
- test = λl. λm. λn. l m n
- test tru then else = (λl. λm. λn. l m n) (λt. λf. t)
- test fls then else = (λl. λm. λn. l m n) (λt. λf. f)
- and = λb. λc. b c fls
- or =
- not =

Church Numerals

- $c_0 = \lambda s.\ \lambda z.\ z$
- $c_1 = \lambda s.\ \lambda z.\ s\ z$
- $c_2 = \lambda s.\ \lambda z.\ s\ (s\ z)$
- $c_3 = \lambda s.\ \lambda z.\ s\ (s\ (s\ z))$
- $succ = \lambda n.\ \lambda s.\ \lambda z.\ s\ (n\ s\ z)$
- $plus = \lambda m.\ \lambda n.\ \lambda s.\ \lambda z.\ m\ s\ (n\ s\ z)$
- $times = \lambda m.\ \lambda n.\ m\ (plus\ n)\ c_0$
- iszero =

Combinators

- A combinator is a function in the Lambda Calculus having no free variables
- Examples
  - $\lambda x.\ x$ is a combinator
  - $\lambda x.\ \lambda y.\ (x\ y)$ is a combinator
  - $\lambda x.\ \lambda y.\ (x\ z)$ is not a combinator
- Combinators can serve nicely as modular building blocks for more complex expressions
- The Church numerals and simulated booleans are examples of useful combinators

Iteration in Lambda Calculus

- omega = (λx. x x) (λx. x x)
  - (λx. x x) (λx. x x) $\Rightarrow$ (λx. x x) (λx. x x)

- Y = λf. (λx. f (x x)) (λx. f (x x))
- Z = λf. (λx. f (λy. x x y)) (λx. f (λy. x x y))
- Recursion can be simulated
  - Y only works with call-by-name semantics
  - Z works with call-by-value semantics
- Defining factorial:
  - g = λf. λn. if n==0 then 1 else (n * (f (n - 1)))
  - fact = Z g