Iterative Program Analysis
Abstract Interpretation

Mooly Sagiv
http://www.cs.tau.ac.il/~msagiv/courses/pa11.html
Tel Aviv University
640-6706

Textbook: Principles of Program Analysis

Chapter 4

CC79, CC92

Outline

- Reminder Chaotic Iterations
- The abstract interpretation technique
  - Relating Concrete and Abstract Interpretation
  - More examples
  - Precision
- Later
  - Backward analysis
  - Complexity
  - Widening and Narrowing
  - Shape Analysis

Specialized Chaotic Iterations System of Equations

$S =$

$$\left\{ \begin{array}{l} df_{entry}[s] = \iota \\ df_{entry}[v] = \sqcup \{ f(u, v)\, (df_{entry}[u]) \mid (u, v) \in E \} \end{array} \right\}$$

$F_S : L^n \rightarrow L^n$

$$F_S\,(X)[s] = \iota$$

$$F_S(X)[v] = \sqcup \{ f(u, v)(X[u]) \mid (u, v) \in E \}$$

$$lfp(S) = lfp(F_S)$$

Specialized Chaotic Iterations

Chaotic(G(V, E): Graph, s: Node, L: Lattice, ι: L, f: E →(L →L)){

for each v in V to n do $df_{entry}[v] := \bot$

df[s] = ι

WL = {s}

while (WL ≠ Ø) do

select and remove an element u ∈ WL

for each v, such that. (u, v) ∈E do

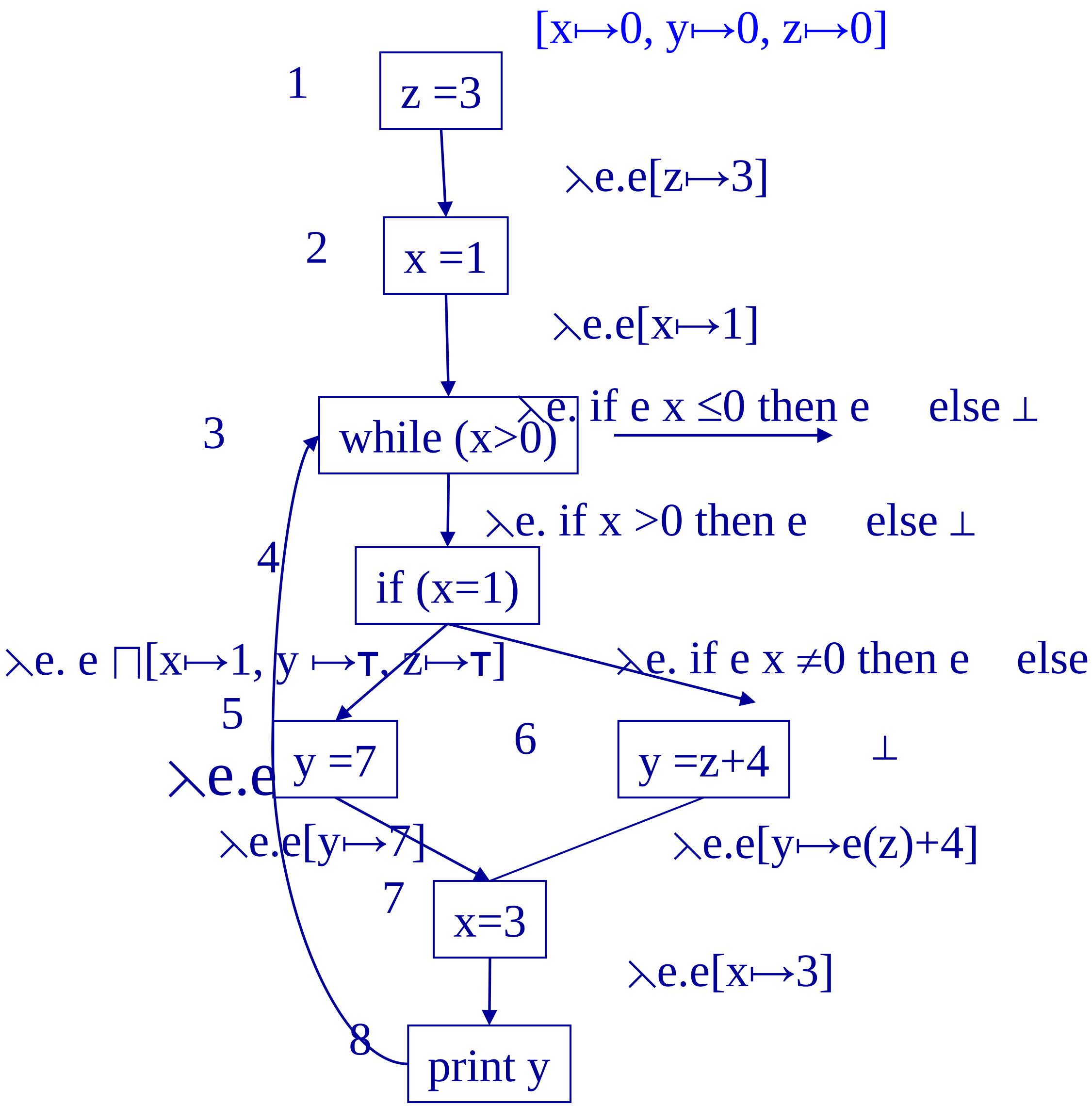
temp = $f(e)(df_{entry}[u])$

new := $df_{entry}(v) \sqcup$ temp

if (new ≠ $df_{entry}[v]$) then

$df_{entry}[v]$:= new;

WL := WL ∪{v}



WL	$df_{entry}[v]$
{1}	
{2}	df[2]:=[x↦0, y↦0, z↦3]
{3}	df[3]:=[x↦1, y↦0, z↦3]
{4}	df[4]:=[x↦1, y↦0, z↦3]
{5}	df[5]:=[x↦1, y↦0, z↦3]
{7}	df[7]:=[x↦1, y↦7, z↦3]
{8}	df[8]:=[x↦3, y↦7, z↦3]
{3}	df[3]:=[x↦T, y↦T, z↦3]
{4}	df[4]:=[x↦T, y↦T, z↦3]
{5,6}	df[5]:=[x↦1, y↦T, z↦3]
{6,7}	df[6]:=[x↦T, y↦T, z↦3]
{7}	df[7]:=[x↦T, y↦7, z↦3]

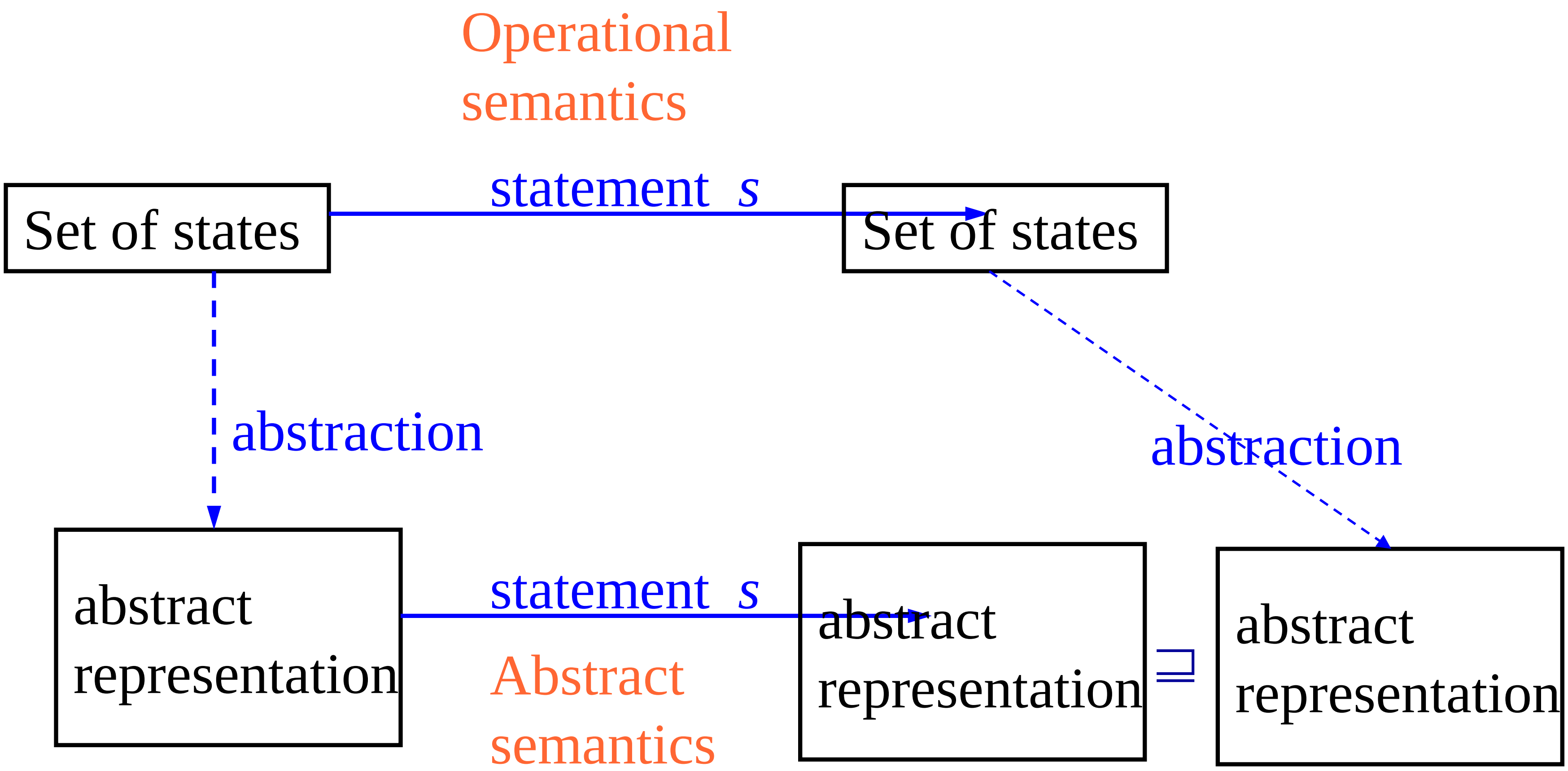
Complexity of Chaotic Iterations

- Parameters:
  - n the number of CFG nodes
  - k is the maximum outdegree of edges
  - A lattice of height h
  - c is the maximum cost of
    - applying $f_{(e)}$
    - $\sqcup$
    - L comparisons
- Complexity
  O(n * h * c * k)

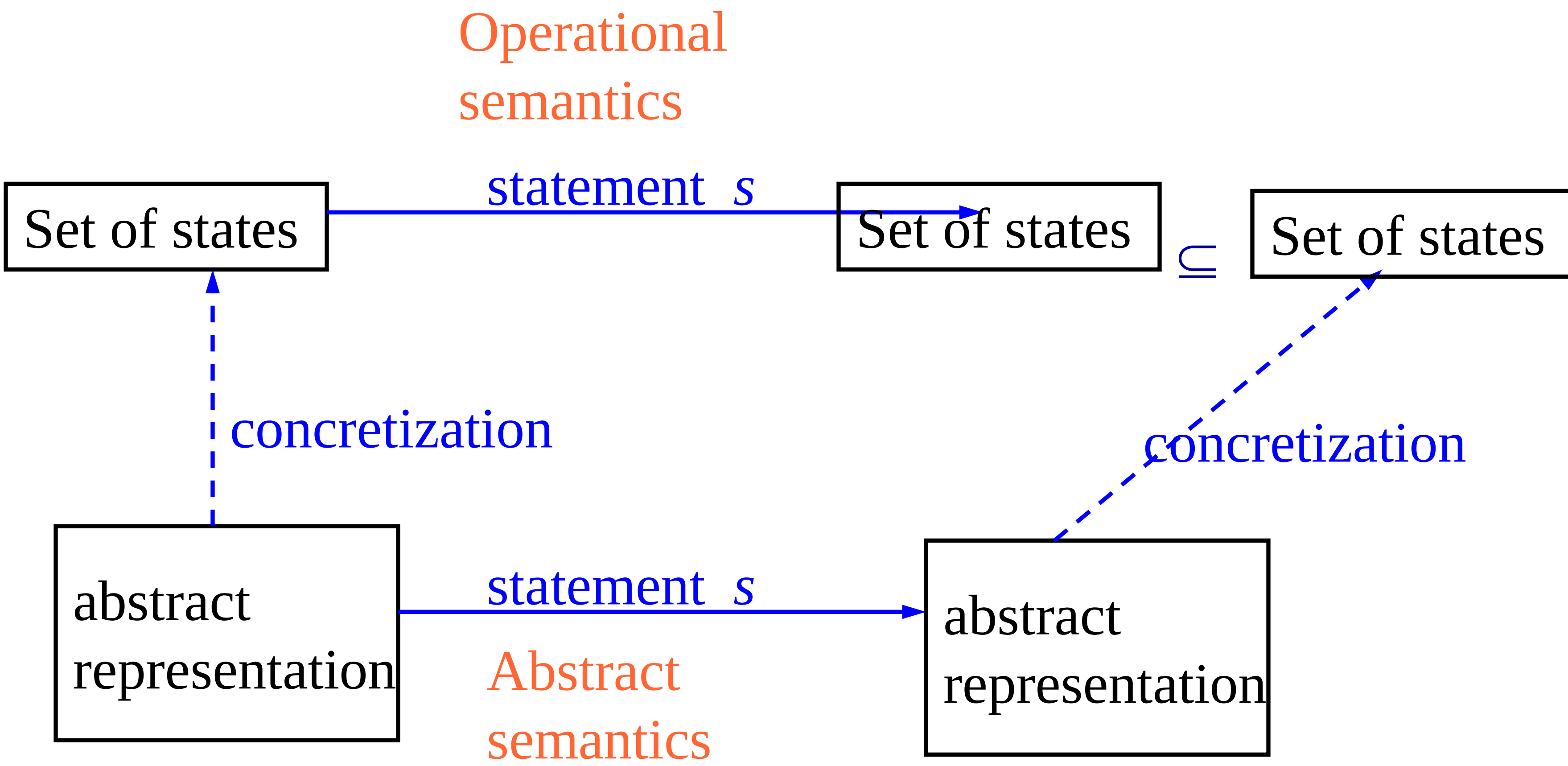
The Abstract Interpretation Technique (Cousot & Cousot)

- The foundation of program analysis
- Defines the meaning of the information computed by static tools
- A mathematical framework
- Allows proving that an analysis is sound in a local way
- Identify design bugs
- Understand where precision is lost
- New analysis from old
- Not limited to certain programming style

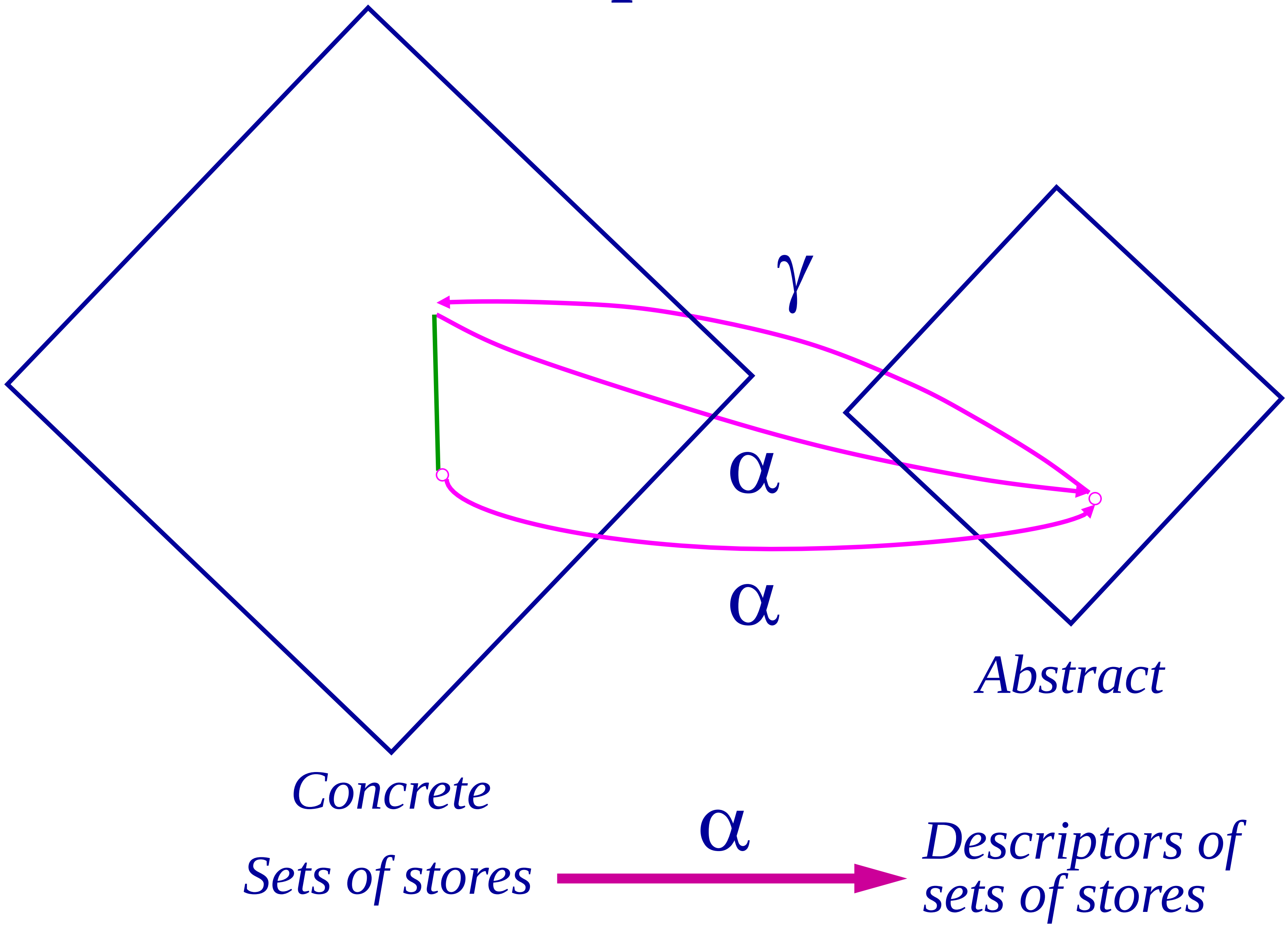
Abstract (Conservative) interpretation



Abstract (Conservative) interpretation



Abstract Interpretation



Galois Connections

- Lattices C and A and functions $\alpha: C \to A$ and $\gamma: A \to C$
- The pair of functions (α, γ) form Galois connection if
  - α and γ are monotone
  - $\forall a \in A$
    - » $\alpha(\gamma(a)) \sqsubseteq a$
  - $\forall c \in C$
    - » $c \sqsubseteq \gamma(\alpha(C))$
- Alternatively if:

  $\forall c \in C$

  $\forall a \in A$

  $\alpha(c) \sqsubseteq a$ iff $c \sqsubseteq \gamma(a)$

- α and γ uniquely determine each other

The Abstraction Function (CP)

- Map collecting states into constants
- The abstraction of an individual state

  $\beta_{CP}:[Var_* \rightarrow Z] \rightarrow [Var_* \rightarrow Z \cup \{\bot, \top\}]$

  $\beta_{CP}(\sigma) = \sigma$
- The abstraction of set of states

  $\alpha_{CP}:P([Var_* \rightarrow Z]) \rightarrow [Var_* \rightarrow Z \cup \{\bot, \top\}]$

  $\alpha_{CP}(CS) = \sqcup \{ \beta_{CP}(\sigma) \mid \sigma \in CS\} = \sqcup\{\sigma \mid \sigma \in CS\}$
- Soundness

  $\alpha_{CP}(Reach(v)) \sqsubseteq df(v)$
- Completeness

The Concretization Function

- Map constants into collecting states
- The formal meaning of constants
- The concretization

  γ_{CP}: $[Var_* \rightarrow Z \cup \{\perp, \top\}] \rightarrow P([Var_* \rightarrow Z])$

  $\gamma_{CP}(df) = \{\sigma \mid \beta_{CP}(\sigma) \sqsubseteq df\} = \{\sigma \mid \sigma \sqsubseteq df\}$
- Soundness

  $Reach(v) \subseteq \gamma_{CP}(df(v))$
- Completeness

Galois Connection Constant Propagation

- α_{CP} is monotone
- γ_{CP} is monotone
- $\forall$ df $\in$ [$Var_* \rightarrow Z \cup \{\bot, \top\}$]
  - $\alpha_{CP}(\gamma_{CP}(df)) \sqsubseteq df$
- $\forall$ c $\in$ P([$Var_* \rightarrow Z$])
  - $c_{CP} \sqsubseteq \gamma_{CP}(\alpha_{CP}(C))$

Upper Closure (CP)

Proof of Soundness

- Define an “appropriate” operational semantics
- Define “collecting” structural operational semantics
- Establish a Galois connection between collecting states and abstract states
- (Local correctness) Show that the abstract interpretation of every atomic statement is sound
  w.r.t. the collecting semantics
- (Global correctness) Conclude that the analysis is sound

Collecting Semantics

- The input state is not known at compile-time
- “Collect” all the states for all possible inputs to the program
- No lost of precision

A Simple Example Program

$\{[x \mapsto 0, y \mapsto 0, z \mapsto 0]\}$

z = 3 $\{[x \mapsto 0, y \mapsto 0, z \mapsto 3]\}$

x = 1 $\{[x \mapsto 1, y \mapsto 0, z \mapsto 3]\}$

while (x > 0) ($\{[x \mapsto 1, y \mapsto 0, z \mapsto 3], [x \mapsto 3, y \mapsto 0, z \mapsto 3],\}$

if (x = 1) then y = 7 $\{[x \mapsto 1, y \mapsto 7, z \mapsto 3], [x \mapsto 3, y \mapsto 7, z \mapsto 3]\}$

else y = z + 4

$\{[x \mapsto 1, y \mapsto 7, z \mapsto 3], [x \mapsto 3, y \mapsto 7, z \mapsto 3]\}$

x = 3

$\{[x \mapsto 3, y \mapsto 7, z \mapsto 3]\}$

print y

$\{[x \mapsto 3, y \mapsto 7, z \mapsto 3]\}$

)

Another Example

```
x= 0
while (true) do
   x = x +1
```

An "Iterative" Definition

- Generate a system of monotone equations
- The least solution is well-defined
- The least solution is the collecting interpretation
- But may not be computable

Equations Generated for Collecting Interpretation

- Equations for elementary statements
  - [skip]

    $CS_{exit}(1) = CS_{entry}(l)$
  - [b]
    $CS_{exit}(1) = \{\sigma: \sigma \in CS_{entry}(l), [\![b]\!]\sigma = tt\}$
  - [x := a]
    $CS_{exit}(1) = \{(s[x \mapsto \mathbf{A}[\![a]\!]s]) \mid s \in CS_{entry}(l)\}$
- Equations for control flow constructs
  $CS_{entry}(l) = \cup CS_{exit}(l') \; l'$ immediately precedes l in the control flow graph
- An equation for the entry
  $CS_{entry}(1) = \{\sigma \mid \sigma \in Var_{*} \to Z\}$

Specialized Chaotic Iterations System of Equations (Collecting Semantics)

S =

$$
\left\{
\begin{array}{l}
CS_{entry}[s] = \{\sigma_0\} \\
CS_{entry}[v] = \cup\{f(e)(CS_{entry}[u]) \mid (u, v) \in E\} \\
\text{where } f(e) = \lambda X. \{[\![st(e)]\!]\, \sigma \mid \sigma \in X\} \text{ for atomic statements} \\
\qquad f(e) = \lambda X. \{\sigma \mid [\![b(e)]\!]\, \sigma = tt\}
\end{array}
\right\}
$$

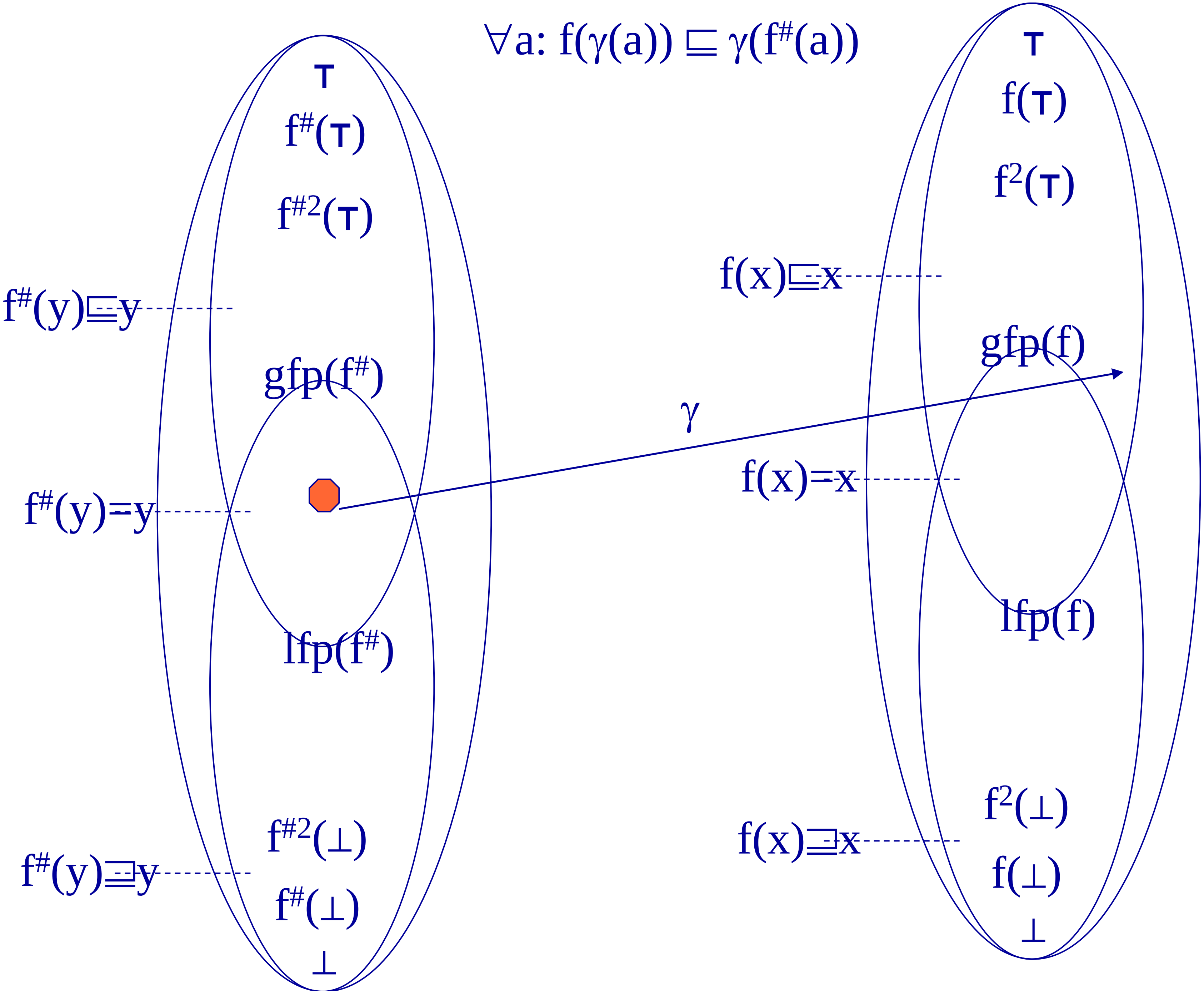
$$F_S : L^n \rightarrow L^n$$

$$F_s(X)[v] = \cup\{f(e)[u] \mid (u, v) \in E\}$$

$$lfp(S) = lfp(F_S)$$

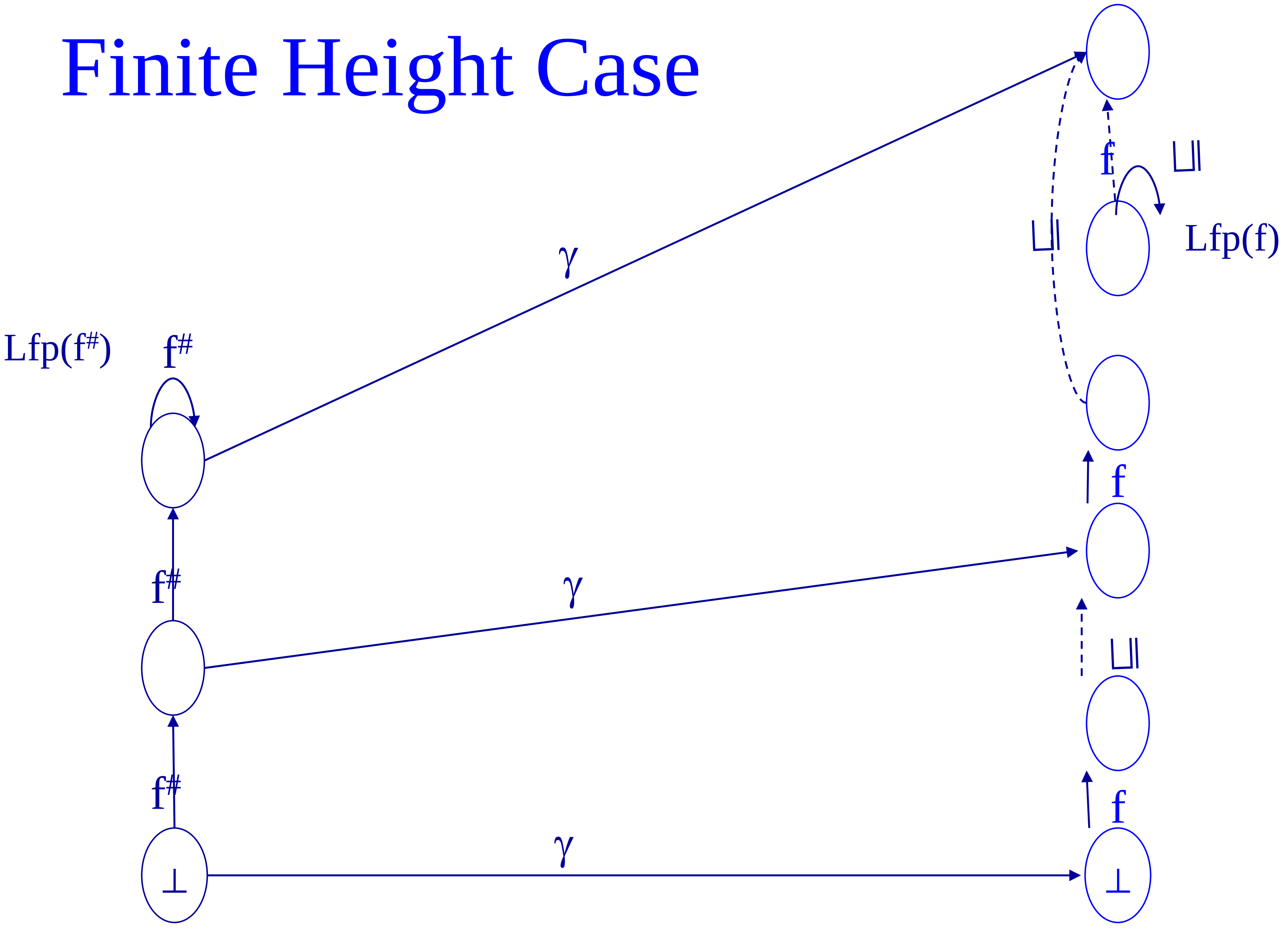
The Least Solution

- 2n sets of equations
  $CS_{entry}(1), \ldots, CS_{entry}(n), CS_{exit}(1), \ldots, CS_{exit}(n)$
- Can be written in vectorial form
  $$\overrightarrow{CS} = F_{cs}(\overrightarrow{CS})$$
- The least solution $lfp(F_{cs})$ is well-defined
- Every component is minimal
- Since F_{cs} is monotone such a solution always exists
- $CS_{entry}(v) = \{s \mid \exists s_0 \mid \langle P, s_0 \rangle \Rightarrow^* (S', s)), \; init(S')=v\}$
- Simplify the soundness criteria



∀a: f(γ(a)) ⊑ γ(f#(a))
⊤
f#(⊤)
f#2(⊤)
f#(y)⊑y
gfp(f#)
f#(y)=y
lfp(f#)
f#2(⊥)
f#(⊥)
f#(y)⊒y
⊥
γ
⊤
f(⊤)
f2(⊤)
f(x)⊑x
gfp(f)
f(x)=x
lfp(f)
f2(⊥)
f(x)⊒x
f(⊥)
⊥

Finite Height Case



Soundness Theorem(1)

1. Let (α, γ) form Galois connection from C to A
2. $f: C \rightarrow C$ be a monotone function
3. $f^{\#}: A \rightarrow A$ be a monotone function
4. $\forall a \in A: f(\gamma(a)) \sqsubseteq \gamma(f^{\#}(a))$

$$lfp(f) \sqsubseteq \gamma(lfp(f^{\#}))$$

$$\alpha(lfp(f)) \sqsubseteq lfp(f^{\#})$$

Soundness Theorem(2)

1. Let (α, γ) form Galois connection from C to A
2. $f: C \rightarrow C$ be a monotone function
3. $f^{\#}: A \rightarrow A$ be a monotone function
4. $\forall c \in C: \alpha(f(c)) \sqsubseteq f^{\#}(\alpha(c))$

$$\alpha(lfp(f)) \sqsubseteq lfp(f^{\#})$$

$$lfp(f) \sqsubseteq \gamma(lfp(f^{\#}))$$

Soundness Theorem(3)

1. Let (α, γ) form Galois connection from C to A
2. $f: C \rightarrow C$ be a monotone function
3. $f^{\#}: A \rightarrow A$ be a monotone function
4. $\forall a \in A: \alpha(f(\gamma(a))) \sqsubseteq f^{\#}(a)$

$$\alpha(lfp(f)) \sqsubseteq lfp(f^{\#})$$

$$lfp(f) \sqsubseteq \gamma(lfp(f^{\#}))$$

Proof of Soundness (Summary)

- Define an "appropriate" structural operational semantics
- Define "collecting" structural operational semantics
- Establish a Galois connection between collecting states and reaching definitions
- (Local correctness) Show that the abstract interpretation of every atomic statement is sound w.r.t. the collecting semantics
- (Global correctness) Conclude that the analysis is sound

Example Dataflow Problem

- Formal available expression analysis
- Find out which expressions are available at a given program point
- Example program

```
x = y + t
z = y + r
while (...) {
      t = t + (y + r)
          }
```

- Lattice
- Galois connection
- Basic statements
- Soundness

Example: May-Be-Garbage

- A variable x may-be-garbage at a program point v if there exists a execution path leading to v in which x's value is unpredictable:
  - Was not assigned
  - Was assigned using an unpredictable expression
- Lattice
- Galois connection
- Basic statements
- Soundness

The **PWhile** Programming Language
Abstract Syntax

$a := x \mid *x \mid \&x \mid n \mid a_1\ op_a\ a_2$

$b :=$ true | false | not b | $b_1\ op_b\ b_2 \mid a_1\ op_r\ a_2$

$S := x := a \mid *x := a \mid$ skip | S_1 ; S_2 |

if b **then** S_1 **else** S_2 | **while** b do S

Concrete Semantics for PWhile

$\text{State1} = [\text{Loc} \to \text{Loc} \cup Z]$

For every atomic statement S

$$[\![S]\!] : \text{States1} \to \text{States1}$$

$$[\![x := a]\!](\sigma) = \sigma[\text{loc}(x) \mapsto \mathbf{A}[\![a]\!]\,\sigma]$$

$$[\![x := \&y]\!](\sigma)$$

$$[\![x := *y]\!](\sigma)$$

$$[\![x := y]\!](\sigma)$$

$$[\![*x := y]\!](\sigma)$$

Points-To Analysis

- Lattice L_{pt} =
- Galois connection

```
t := &a;
y := &b;

z := &c;

 if x> 0;
        then p:= &y;
        else p:= &z;

 *p := t;
```

```
/* ∅ */ t := &a; /* {(t, a)}*/
 /* {(t, a)}*/  y := &b; /* {(t, a), (y, b) }*/

/* {(t, a), (y, b)}*/ z := &c; /* {(t, a), (y, b), (z, c) }*/

 if x> 0;
        then p:= &y; /* {(t, a),  (y, b), (z, c), (p, y)}*/


        else p:= &z; /* {(t, a),  (y, b), (z, c), (p, z)}*/
/* {(t, a),  (y, b), (z, c), (p, y), (p, z)}*/

 *p := t;

/* {(t, a), (y, b), (y, c), (p, y), (p, z), (y, a), (z, a)}*/
```

Abstract Semantics for PWhile

State# = $P(Var^* \times Var^*)$

For every atomic statement S

$[\![x := a]\!](\sigma)$

$[\![x := \&y]\!](\sigma)$

$[\![x := *y]\!](\sigma)$

$[\![x := y]\!](\sigma)$

$[\![*x := y]\!](\sigma)$

```
/*  ∅ */ t := &a; /* {(t, a)}*/
 /* {(t, a)}*/  y := &b; /* {(t, a), (y, b) }*/

/* {(t, a), (y, b)}*/ z := &c; /* {(t, a), (y, b), (z, c) }*/

 if x> 0;
         then p:= &y; /* {(t, a),  (y, b), (z, c), (p, y)}*/


         else p:= &z; /* {(t, a),  (y, b), (z, c), (p, z)}*/
/* {(t, a),  (y, b), (z, c), (p, y), (p, z)}*/

 *p := t;

/* {(t, a), (y, b), (y, c), (p, y), (p, z), (y, a), (z, a)}*/
```

Flow insensitive points-to-analysis Steengard 1996

- Ignore control flow
- One set of points-to per program
- Can be represented as a directed graph
- Conservative approximation
  - Accumulate pointers
- Can be computed in almost linear time
  - Union find

```
t := &a;
y := &b;

z := &c;

 if x> 0;
        then p:= &y;
        else p:= &z;

 *p := t;
```

Precision

- We cannot usually have
  - $\alpha(CS) = DF$
    on all programs
- But can we say something about precision in all programs?

The Join-Over-All-Paths (JOP)

- Let paths(v) denote the potentially infinite set paths from start to v (written as sequences of labels)
- For a sequence of edges $[e_1, e_2, \ldots, e_n]$ define $f[e_1, e_2, \ldots, e_n]: L \rightarrow L$ by composing the effects of basic blocks
  $f[e_1, e_2, \ldots, e_n](l) = f(e_n)(\ldots(f(e_2)(f(e_1)(l))\ldots)$
- JOP[v] $= \sqcup\{f[e_1, e_2, \ldots, e_n](\iota) \mid [e_1, e_2, \ldots, e_n] \in \text{paths}(v)\}$

JOP vs. Least Solution

- The DF solution obtained by Chaotic iteration satisfies for every *l*:
  - $JOP[v] \sqsubseteq DF_{entry}(v)$
- A function f is additive (distributive) if
  - $f(\sqcup\{x| x \in X\}) = \sqcup\{f(x) | \in X\}$
- If every f_l is additive (distributive) for all the nodes v
  - $JOP[v] = DF_{entry}(v)$
- Examples
  - Maybe garbage
  - Available expressions
  - Constant Propagation
  - Points-to

Conclusion

- Chaotic iterations is a powerful technique
- Easy to implement
- Rather precise
- But expensive
  - More efficient methods exist for structured programs
- Abstract interpretation relates runtime semantics and static information
- The concrete semantics serves as a tool in designing abstractions
  - More intuition will be given in the sequel