Deductive Verification

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First-Order Logic

- A formal notation for mathematics, with expressions involving
  - Propositional symbols
  - Predicates
  - Functions and constant symbols
  - Quantifiers
- In contrast, propositional (Boolean) logic only involves propositional symbols and operators

First-Order Logic: Syntax

- As with propositional logic, expressions in first-order logic are made up of sequences of symbols
- Symbols are divided into *logical symbols* and *non-logical symbols* or *parameters*
- Example:

$$(x = y) \wedge (y = z) \wedge (f(z) \rightarrow f(x)+1)$$

First-Order Logic: Syntax

- Logical Symbols
  - Propositional connectives: $\wedge$, $\vee$, $\neg$, $\rightarrow$,…
  - Variables: V1, V2, . . .
  - Quantifiers: $\exists$, $\forall$
- Non-logical symbols/Parameters
  - Equality: =
  - Functions: +, -, %, bit-wise &, f(), concat, …
  - Predicates: $\leq$, is_substring, …
  - Constant symbols: 0, 1.0, null, …

Example

$\forall X{:}S.\ \ p(f(X),X) \Rightarrow \exists Y{:}S\ .\ p(f(g(X,Y)),g(X,Y)$

Quantifier-free Subset

- We will largely restrict ourselves to formulas without quantifiers ($\forall$, $\exists$)
- This is called the quantifier-free subset/fragment of first-order logic with the relevant theory

Logical Theory

- Defines a set of parameters (non-logical symbols) and their meanings
- This definition is called a *signature*
- Example of a signature:

Theory of linear arithmetic over integers

Signature is (0,1,+,-,·) interpreted over Z

Presbruger Arithmetic

- Signature (0, 1, +, =) interpreted over Z
- Axioms
  - $\forall X: Z. \neg ((X+1) = 0)$
  - $\forall X, Y: Z. (X+1) = (Y+1) \Rightarrow X + Y$
  - $\forall X: Z. X+0 = X$
  - $\forall X, Y: Z. X+(Y+1) = (X+ Y)+1$
  - Let P(X) be a first order formula over X
    - $(P(0) \wedge \forall X: Z. P(X) \Rightarrow P(X+1)) \Rightarrow \forall Y: Z. P(Y)$

Many Sorted First Order Vocabulary

- A finite set of sorts S
- A finite set of function symbols F each with a fixed signature $S^* \to S$
  - Zero arity functions are constant symbols
- A finite set of relation symbols R each with a fixed arity S^*

An Interpretation ι

- A domain D_s for every $s \in S$
  - $D = \cup_{s \in S:} Ds$
- For every function symbol $f \in F$, an interpretation $\iota[f]: D_{s1} \times D_{s2} \times \ldots \times D_{sn} \rightarrow Ds$
- For every relation symbol $r \in R$, an interpretation
  $\iota[r] \subseteq D_{s1} \times D_{s2} \times \ldots \times D_{sm}$

Many-Sorted First Oder Formulas

- Logical Variables
  - Begin with Capital variables
- Typed Terms

  <term> ::= <variable> | f [(<term>, … <term>)]
- Formulas

  <form> ::= <term> = <term> | r(<term>, … <term>) // atomic

  <form> $\vee$ <form> | <form> $\wedge$<form> | $\neg$ <form> // Boolean

  $\exists$X: s <form> | $\forall$ X : s. <form> // Quantifications

Free Variables

- FV: <term>, <formula> $\rightarrow$ 2^{Var}
- Terms
  - $FV(X) = \{X\}$
  - $FV(f(<t_1, t_2, \ldots, t_n)) = \cup_{i=1..n:} FV(t_i)$
- Formulas
  - $FV(t1 = t2) = FV(t1) \cup FV(t2)$
  - $FV(r(<t_1, t_2, \ldots, t_n)) = \cup_{i \in 1..n} FV(t_i)$
  - $FV(f1 \vee, \wedge f2) = FV(f_1) \cup FV(f_2)$
  - $FV(\neg f2) = FV(f)$
  - $FV(\exists X{:}s.f) = FV(f) - \{X\}$
  - $FV(\forall X{:}s.\ f) = FV(f) - \{X\}$

Assignments and Models

- Assignment A: Var → D
- Extended to terms
  - $A(f(t_1, t_2, \ldots, t_n) = \iota[f](A(t_1), A(t_2), \ldots, A(t_n))$
- An assignment A models a formula f under interpretation ι (denoted by A, $\iota \models f$) if f is true in A (Tarsky's semantics)
- $A, \iota \models t_1 = t_2$ if $A(t_1) = A(t_2)$
- $A, \iota \models r(t_1, t_2, \ldots, t_n)$ if $\langle A(t_1), A(t_2), \ldots, A(t_n)\rangle \in \iota[r]$
- $A, \iota \models f_1 \vee f_2$ if $A, \iota \models f_1$ or $A, \iota \models f_2$
- $A, \iota \models \neg f$ if not $A, \iota \models f$
- $A, \iota \models \exists X\!: t.\ f$ if there exists $d \in D_t$ such that $A[X \mapsto d]$ if $A, \iota \models f$

A T-Interpretation

- A domain D_s for every $s \in S$
  - $D = \cup_{s \in S:} Ds$
- For every function symbol $f \in F$, an interpretation $\iota[f]: D_{s1} \times D_{s2} \times \ldots \times D_{sn} \rightarrow Ds$
- For every relation symbol $r \in R$, an interpretation $\iota[r] \subseteq D_{s1} \times D_{s2} \times \ldots \times D_{sm}$
- The domain and the interpretations satisfy the theory requirements(axioms)

Example Linear Arithmetic

- $S = \{\text{int}\}$, $F = \{\mathbf{0}^0, \mathbf{1}^1, +^2\}$, $r = \{\leq^2\}$
- Domain
  - $D_{int} = Z$
- Functions
  - $[\![\mathbf{0}]\!] = 0$
  - $[\![\mathbf{1}]\!] = 1$
  - $[\![+]\!] = \lambda x, y\text{: int. } x + y$
- Relations
  - $[\![\leq]\!] = \lambda x, y\text{: int. } x \leq y$

Assignments and T-Models

- Assignment A: Var → D
- Extended to terms
  - $A(f(t_1, t_2, \ldots, t_n) = \iota[f](A(t_1), A(t_2), \ldots, A(t_n))$
- An assignment A which models a theory T, T-models a formula f under interpretation ι (denoted by $A, \iota \models_T f$) if f is true in A (Tarsky's semantics)
- $A, \iota \models_T t_1 = t_2$ if $A(t_1) = A(t_2)$
- $A, \iota \models_T r(t_1, t_2, \ldots, t_n)$ if $\langle A(t_1), A(t_2), \ldots, A(t_n)\rangle \in \iota[r]$
- $A, \iota \models_T f_1 \vee f_2$ if $A, \iota \models_T f_1$ or $A, \iota \models_T f_2$
- $A, \iota \models_T \neg f$ if not $A, \iota \models_T f$
- $A, \iota \models_T \exists X{:}\ t.\ f$ if there exists $d \in D_t$ such that $A[X \mapsto d]$ if $A, \iota \models_T f$

The SMT decision problem

- Input: A quantifier-free formula f over a theory T
- Does there exist an T-interpretation ι and an assignment A:FV(f) → D such that A $\models_T$ f
- The complexity depends on the complexity of the theory solvers
  - NPC-Undecidable

Summary of Decidability Results

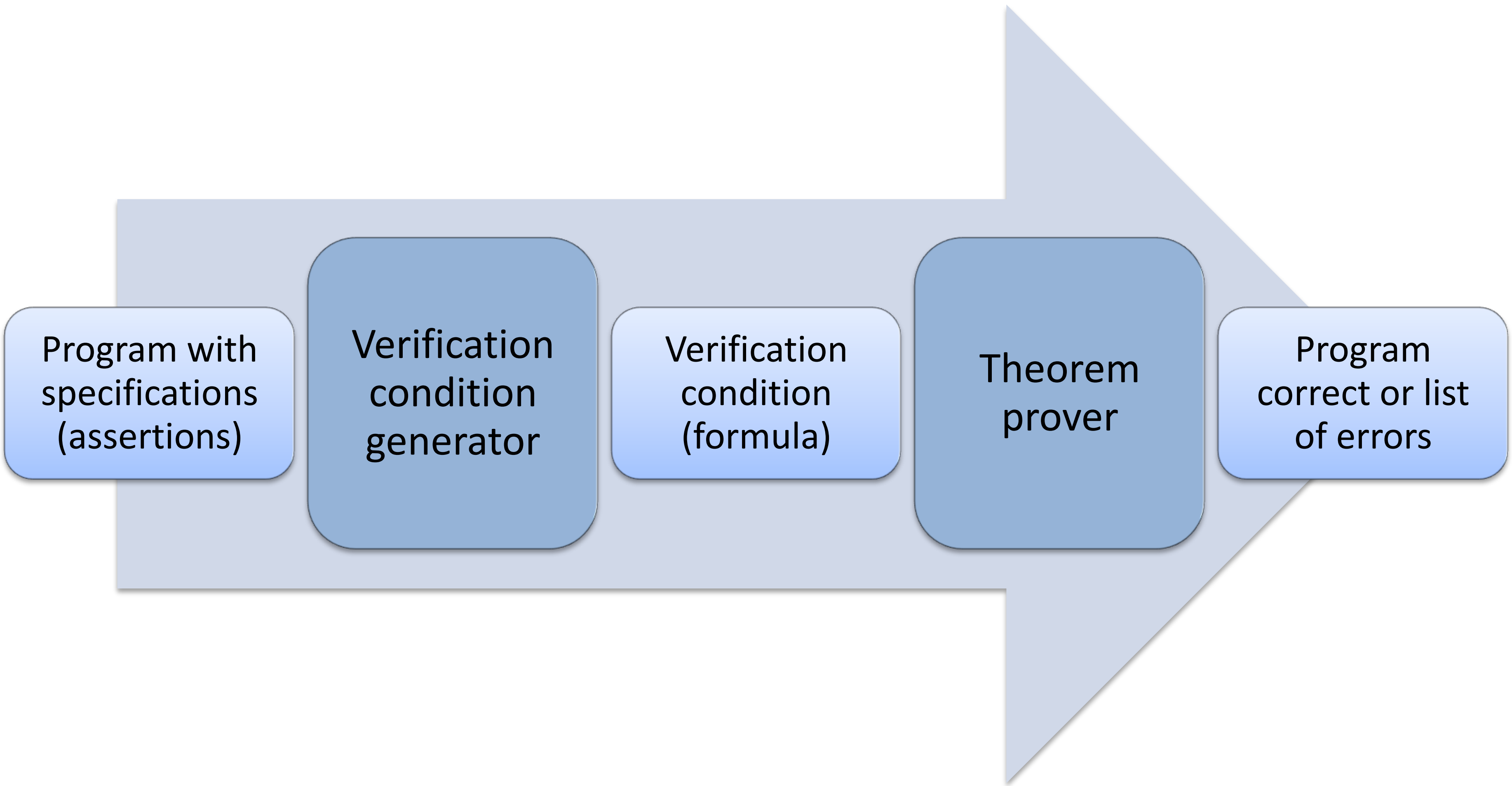
Theory		Quantifiers Decidable	QFF Decidable
T_E	Equality	NO	YES
T_{PA}	Peano Arithmetic	NO	NO
$T_{\mathbb{N}}$	Presburger Arithmetic	YES	YES
$T_{\mathbb{Z}}$	Linear Integer Arithmetic	YES	YES
$T_{\mathbb{R}}$	Real Arithmetic	YES	YES
$T_{\mathbb{Q}}$	Linear Rationals	YES	YES
T_A	Arrays	NO	YES

Summary of Complexity Results

Theory		Quantifiers	QF Conjunctive
PL	Propositional Logic	NP-complete	O(n)
T_E	Equality	–	O(n log n)
$T_{\mathbb{N}}$	Presburger Arithmetic	O(2^2^2^(kn))	NP-complete
$T_{\mathbb{Z}}$	Linear Integer Arithmetic	O(2^2^2^(kn))	NP-complete
$T_{\mathbb{R}}$	Real Arithmetic	O(2^2^(kn))	O(2^2^(kn))
$T_{\mathbb{Q}}$	Linear Rationals	O(2^2^(kn))	PTIME
T_A	Arrays	–	NP-complete

n – input formula size; k – some positive integer

Basic Verifier Architecture



Verification Condition Generator

- Creates verification conditions (mathematical logic formulas) from program's source code
  - If VC is valid – program is correct
  - If VC is invalid – possible error in program
- Based on the theory of Hoare triples
  - Formalization of software semantics for verification
- Verification conditions computed automatically using *weakest preconditions* (wp)

Simple Command Language

x := E

havoc x

assert P

assume P

S ; T [sequential composition]

S $\square$ T [choice statement]

Program States

- Program state s
  - Assignment of values (of proper type) to all program variables
  - Sometimes includes program counter variable pc
    - Holds current program location
- Example

  $s : (x \mapsto -1, y \mapsto 1)$

  $s : (pc \mapsto \mathrm{L}, a \mapsto 0, i \mapsto 3)$
- Reachable state is a state that can be reached during some computation

Program States cont.

- A set of program states can be described using a FOL formula
- Example

  Set of states:

  s : { $(x \mapsto 1)$, $(x \mapsto 2)$, $(x \mapsto 3)$ }

  FOL formulas defining s:

  $x = 1$ Ç $x = 2$ Ç $x = 3$

  $0 < x$ Æ $x < 4$ [if x is integer]

Hoare Triple

- Used for reasoning about (program) executions

$$\{ P \} \; S \; \{ Q \}$$

- S is a command
- P is a precondition – formula about program state before S executes
- Q is a postcondition – formula about program state after S executes

Hoare Triple Definition

$$\{ P \} \ S \ \{ Q \}$$

- When a state *s* satisfies precondition P, every terminating execution of command S starting in *s*
  - does not go wrong, and
  - establishes postcondition Q

Hoare Triple Examples

- $\{a = 2\}$ b := a + 3; $\{b > 0\}$
- $\{a = 2\}$ b := a + 3; $\{b = 5\}$
- $\{a > 3\}$ b := a + 3; $\{a > 0\}$
- $\{a = 2\}$ b := a * a; $\{b > 0\}$

Weakest Precondition [Dijkstra]

- The most general (i.e., weakest) P that satisfies

  $$\{ P \} S \{ Q \}$$

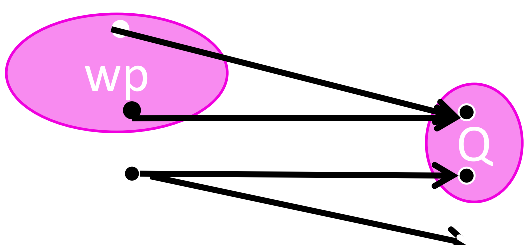
  is called the weakest precondition of S with respect to Q, written:

  $$wp(S, Q)$$

- To check $\{ P \} S \{ Q \}$ prove $P \rightarrow wp(S, Q)$

Weakest Precondition

- wp: Stm $\rightarrow$(Ass$\rightarrow$Ass)
- wp $[\![S]\!]$(Q) – the weakest condition such that every terminating computation of S results in a state satisfying Q
- $\sigma \models \text{wp}\, [\![S]\!](Q) \leftrightarrow \;\; \forall \sigma' : \sigma\, [\![S]\!]\, \sigma' \rightarrow \sigma' \models Q$



Weakest Precondition [Dijkstra]

- The most general (i.e., weakest) P that satisfies

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$$wp(S, Q)$$

- To check { P } S { Q } prove $P \rightarrow wp(S, Q)$
- Example

$$\{?P?\}\ \ b := a + 3;\ \{b > 0\}$$

$$\{a + 3 > 0\}\ \ b := a + 3;\ \{b > 0\}$$

$$wp(b := a + 3,\ b > 0) = a + 3 > 0$$

Strongest Postcondition

- The strongest Q that satisfies

  $$\{ P \} S \{ Q \}$$

  is called the strongest postcondition of S with respect to P, written:

  $$sp(S, P)$$

- To check $\{ P \} S \{ Q \}$ prove $sp(S, P) \rightarrow Q$
- Strongest postcondition is (almost) a dual of weakest precondition

Weakest Preconditions Cookbook

- wp(x := E, Q) =   Q[E / x]
- wp(havoc x, Q) =   (∀ x . Q)
- wp(assert P, Q) =   P ∧ Q
- wp(assume P, Q) =   P → Q
- wp(S ; T, Q) =   wp(S, wp(T, Q))
- wp(S □ T, Q) =   wp(S, Q) ∧ wp(T, Q)

Checking Correctness with wp

{true}

```
x := 1;

y := x + 2;

assert y = 3;
```

{true}

Checking Correctness with wp cont.

{true}

wp(x := 1, x + 2 = 3) = 1 + 2 = 3 $\wedge$ true

```
x := 1;
```

wp(y := x + 2, y = 3) = x + 2 = 3 $\wedge$ true

```
y := x + 2;
```

wp(assert y = 3, true) = y = 3 $\wedge$ true

```
assert y = 3;
```

{true}

Check: true $\rightarrow$ 1 + 2 = 3 $\wedge$ true

Example II

{x > 1}

```
y := x + 2;

assert y > 3;
```

{true}

Example II cont.

$\{x > 1\}$

$wp(y := x + 2, y > 3) = x + 2 > 3$

```
y := x + 2;
```

$wp(\text{assert } y > 3, \text{true}) = y > 3 \wedge \text{true} = y > 3$

```
assert y > 3;
```

$\{\text{true}\}$

Check: $x > 1 \rightarrow (x + 2 > 3)$

Example III

{true}

```
assume x > 1;

y := x * 2;

z := x + 2;

assert y > z;
```

{true}

Example III cont.

{true}

wp(assume $x > 1$, $x * 2 > x + 2$) = $x>1 \rightarrow x*2 > x+2$

```
assume x > 1;
```

wp($y := x * 2$, $y > x + 2$) = $x * 2 > x + 2$

```
y := x * 2;
```

wp($z := x + 2$, $y > z$) = $y > x + 2$

```
z := x + 2;
```

wp(assert $y > z$, true) = $y > z \wedge \text{true}$ = $y > z$

```
assert y > z;
```

{true}

Structured if Statement

- Just a "syntactic sugar":

  if E then S else T

  gets desugared into

  (assume E ; S) □ (assume :E ; T)

Absolute Value Example

```
if (x >= 0) {
   abs_x := x;
} else {
   abs_x := -x;
}
assert abs_x >= 0;
```

While Loop

loop condition

while E

loop body

do

S

end

- Loop body S executed as long as loop condition E holds

Desugar While Loop by Unrolling N Times

```
while E do S end =
if E {
   S;
   if E {
      S;
      if E {
         S;
         if E {assume false;} // blocks execution
      }
   }
}
```

Example

```
i := 0;
while i < 2 do i := i + 1 end

i := 0;
if i < 2 {
   i := i + 1;
   if i < 2 {
      i := i + 1;
      if i < 2 {
         i := i + 1;
         if i < 2 {assume false;} // blocks execution
      }
   }
}
```

First Issue with Unrolling

```
i := 0;
while i < 4 do i := i + 1 end

i := 0;
if i < 4 {
  i := i + 1;
  if i < 4 {
    i := i + 1;
    if i < 4 {
      i := i + 1;
      if i < 4 {assume false;} // blocks execution
    }
  }
}
```

Second Issue with Unrolling

```
i := 0;
while i < n do i := i + 1 end

i := 0;
if i < n {
  i := i + 1;
  if i < n {
    i := i + 1;
    if i < n {
      i := i + 1;
      if i < n {assume false;} // blocks execution
    }
  }
}
```

While Loop with Invariant

```
while E
    invariant J
do
    S
end
```

loop condition

loop invariant

loop body

- Loop body S executed as long as loop condition E holds
- Loop invariant J must hold on every iteration
  - J must hold initially and is evaluated before E
  - J must hold even on final iteration when E is false
  - J must be inductive
  - Provided by a user or inferred automatically

Desugaring While Loop Using Invariant

- while E invariant J do S end

check that the loop invariant holds initially

assert J;

jump to an arbitrary iteration of the loop

havoc x; assume J;

where x denotes the assignment targets of S

(

assume E; S; assert J; assume false

□

check that the loop invariant is maintained by the loop body

assume ¬E

)

exit the loop

Weakest Precondition of While

- wp(while E invariant J do S end, Q) =

Dafny

- Simple “verifying compiler”
  - Proves procedure contracts statically for all possible inputs
  - Uses theory of weakest preconditions
- Input
  - Annotated program written in simple imperative language
    - Preconditions
    - Postconditions
    - Loop invariants
- Output
  - Correct or list of failed annotations

Dafny Architecture