Satisfiability of Propositional Formulas

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Based on a presentation by Sharad Malik & Ohad Shacham
The SAT Problem

• Given a propositional formula (Boolean function)
  – \( \varphi = (a \lor b) \land (\neg a \lor \neg b \lor c) \)

• Determine if \( \varphi \) is valid

• Determine if \( \varphi \) is satisfiable
  – Find a satisfying assignment or report that such does not exist

• For \( n \) variables, there are \( 2^n \) possible truth assignments to be checked
Why Bother?

- Core computational engine for major applications
  - Artificial Intelligence
    - Knowledge base deduction
    - Automatic theorem proving
  - Electronic Design Automaton
    - Testing and Verification
    - Logic synthesis
    - FPGA routing
    - Path delay analysis
    - And more…
Problem Representation

- Represent the formulas in Conjunctive Normal Form (CNF)
- Conversion to CNF is straightforward
  - \( a \lor (b \land \neg (c \lor \neg d)) \equiv (a \lor (b \land \neg c \land \neg \neg d)) \equiv (a \lor (b \land \neg c \land \neg d)) \equiv (a \lor b) \land (a \lor \neg c) \land (a \lor d) \)
  - **May need to add variables**
- Notations
  - **Literals**
    - Variable or its negation
  - **Clauses**
    - Disjunction of literals
  - \( \varphi = (a \lor b) \land (\neg a \lor \neg b \lor c) \equiv (a + b)(a' + b' + c) \)
- Advantages of CNF
  - Simple data structure
  - All the clauses need to be satisfied
Complexity Results

• First established NP-Complete problem
  – Even when at most 3 literals per clause (3-SAT)
    – No polynomial algorithm for all instances unless P = NP

• Becomes polynomial when
  – At most two literals per clause (2-SAT)
  – At most one positive literal in every clause (Horn)
Goals

• Develop algorithms which solve all SAT instances

• Exponential worst case complexity

• But works well on many instances
  – Interesting Heuristics
  – Annual SAT conferences
  – SAT competitions
    • Randomly, Handmade, Industrial, AI
  – 10 Millions variables!
Resolution

- Resolution of a pair of clauses with exactly ONE incompatible variable

\[ a + b + c' + f \]

\[ g + h' + c + f \]

\[ a + b + g + h' + f \]

- What if more than one incompatible variables?

- Iteratively select a variable for resolution till no more variables are left
- Report UNSAT when the empty clause occurs
- Can discard resolved clauses after each iteration

Potential memory explosion problem!
Can we avoid using exponential space?
DLL Algorithm

• Davis, Logemann and Loveland


• Basic framework for many modern SAT solvers

• Also known as DPLL for historical reasons
Basic DLL Procedure - DFS

(a' + b + c)
(a + c + d)
(a + c + d')
(a' + c' + d)
(a + c' + d')
(b' + c' + d)
(a' + b + c')
(a' + b' + c)
Basic DLL Procedure - DFS

(a' + b + c)
(a + c + d)
(a + c + d')
(a + c' + d)
(a + c' + d')
(b' + c' + d)
(a' + b + c')
(a' + b' + c)
Basic DLL Procedure - DFS

\[(a' + b + c)\]
\[(a + c + d)\]
\[(a + c + d')\]
\[(a' + c + d)\]
\[(a' + c + d')\]
\[(b' + c' + d)\]
\[(a' + b + c')\]
\[(a' + b' + c)\]
Basic DLL Procedure - DFS

(a' + b + c)
(a + c + d)
(a + c + d')
(a + c' + d)
(a + c' + d')
(b' + c' + d)
(a' + b + c')
(a' + b' + c)

\[ a + c + d \]
\[ a + c + d' \]
\[ a + c' + d \]
\[ a + c' + d' \]
\[ a' + b + c \]
\[ b' + c' + d \]
\[ a' + b + c' \]
\[ a' + b' + c \]
Basic DLL Procedure - DFS

(a' + b + c)
(a + c + d)
(a + c + d')
(a + c' + d)
(a + c' + d')
(b' + c' + d)
(a' + b + c')
(a' + b' + c)

⇐ Decision
Basic DLL Procedure - DFS

(a' + b + c)
(a + c + d)
(a + c + d')
(a + c' + d)
(a + c' + d')
(b' + c' + d)
(a' + b + c')
(a' + b' + c)

d=1
c=0
a=0
b
0
0
c
(Conflict!)
(a + c + d)
d=1
(a + c + d')
d=0
c=0
(a + c + d')
(a + c + d)

Implication Graph
Basic DLL Procedure - DFS

\[(a' + b + c)\]
\[(a + c + d)\]
\[(a + c + d')\]
\[(a + c' + d)\]
\[(a + c' + d')\]
\[(a' + b + c)\]
\[(a' + b' + c)\]

\[c=0\]
\[d=0\]
\[d=1\]
\[a=0\]

Conflict!
Basic DLL Procedure - DFS

(a' + b + c)
(a + c + d)
(a + c + d')
(a + c' + d)
(a + c' + d')
(b' + c' + d)
(a' + b + c')
(a' + b' + c)

← Backtrack
Basic DLL Procedure - DFS

\[(a' + b + c)\]
\[(a + c + d)\]
\[(a + c + d')\]
\[(a + c' + d)\]
\[(a + c' + d')\]
\[(a' + b + c)\]
\[(b' + c' + d)\]
\[(a' + b + c')\]
\[(a' + b' + c)\]

Conflict!

Forced Decision
Basic DLL Procedure - DFS

(a’ + b + c)
(a + c + d)
(a’ + c + d’)
(a + c’ + d)
(a + c’ + d’)
(b’ + c’ + d)
(a’ + b + c’)
(a’ + b’ + c)

(a + c + d)

⇐ Backtrack
Basic DLL Procedure - DFS

\[(a' + b + c)\]
\[(a + c + d)\]
\[(a + c + d')\]
\[(a + c' + d)\]
\[(a + c' + d')\]
\[(b' + c' + d)\]
\[(a' + b + c')\]
\[(a' + b' + c)\]
Basic DLL Procedure - DFS

\[(a' + b + c)\]
\[(a + c + d)\]
\[(a + c + d')\]
\[(a + c' + d)\]
\[(a + c' + d')\]
\[(a' + b + c)\]
\[(a' + b' + c)\]

\[c = 0\]
\[d = 1\]
\[c = 0\]
\[d = 0\]

Conflict!
Basic DLL Procedure - DFS

(a' + b + c)
(a + c + d)
(a + c + d')
(a + c' + d)
(a + c' + d')
(b' + c' + d)
(a' + b + c')
(a' + b' + c)

⇐ Backtrack
Basic DLL Procedure - DFS

\[
\begin{align*}
(a' + b + c) \\
(a + c + d) \\
(a + c + d') \\
(a + c' + d) \\
(a + c' + d') \\
(a' + b + c) \\
(b' + c' + d) \\
(a' + b + c') \\
(a' + b' + c)
\end{align*}
\]

Conflict!

\[\Rightarrow\text{Forced Decision}\]

Conflict!
Basic DLL Procedure - DFS

(a' + b + c)
(a + c + d)
(a + c + d')
(a + c' + d)
(a + c' + d')
(b' + c' + d)
(a' + b + c')
(a' + b' + c)

⇐ Backtrack
Basic DLL Procedure - DFS

(a' + b + c)
(a + c + d)
(a + c + d')
(a + c' + d)
(a + c' + d')
(b' + c' + d)
(a' + b + c')
(a' + b' + c)

Forced Decision
Basic DLL Procedure - DFS

(a' + b + c)
(a + c + d)
(a + c + d')
(a + c' + d)
(a + c' + d')
(b' + c' + d)
(a' + b + c')
(a' + b' + c)

Decision
Basic DLL Procedure - DFS

\[(a' + b + c)\]
\[(a + c + d)\]
\[(a + c + d')\]
\[(a + c' + d)\]
\[(a + c' + d')\]
\[(b' + c' + d)\]
\[(a' + b + c')\]
\[(a' + b' + c)\]

Conflict!
Basic DLL Procedure - DFS

(a' + b + c)
(a + c + d)
(a + c + d')
(a + c' + d)
(a + c' + d')
(b' + c' + d)
(a' + b + c')
(a' + b' + c)
Basic DLL Procedure - DFS

\[(a' + b + c)\]  
\[(a + c + d)\]  
\[(a + c + d')\]  
\[(a + c' + d)\]  
\[(a + c' + d')\]  
\[(a' + b + c')\]  
\[(a' + b' + c)\]

Forced Decision

\(a=1\)  
\(b=1\)  
\(c=1\)
Basic DLL Procedure - DFS

(a' + b + c)
(a + c + d)
(a + c + d')
(a + c' + d)
(a + c' + d')
(b' + c' + d)
(a' + b + c')
(a' + b' + c)

(a' + b' + c)
(b' + c' + d)

a=1
b=1
c=1
d=1
Basic DLL Procedure - DFS

(a' + b + c)
(a + c + d)
(a + c + d')
(a + c' + d)
(a + c' + d')
(b' + c' + d)
(a' + b + c')
(a' + b' + c)

\[ \text{SAT} \]
Features of DLL

- Eliminates the exponential memory requirements of DP
- Exponential time is still a problem
- Limited practical applicability – largest use seen in automatic theorem proving
- Very limited size of problems are allowed
  - 32K word memory
  - Problem size limited by total size of clauses (1300 clauses)
Implications and Boolean Constraint Propagation

• Implication
  – A variable is forced to be assigned to be True or False based on previous assignments

• Unit clause rule (rule for elimination of one literal clauses)
  – An unsatisfied clause is a unit clause if it has exactly one unassigned literal
  - The unassigned literal is implied because of the unit clause

• Boolean Constraint Propagation (BCP)
  – Iteratively apply the unit clause rule until there is no unit clause available.

• Workhorse of DLL based algorithms

\[(a + b' + c)(b + c')(a' + c')\]

\[a = T, b = T, c \text{ is unassigned}\]
GRASP

- Marques-Silva and Sakallah [SS96, SS99]

- Incorporates conflict driven learning and non-chronological backtracking

- Practical SAT instances can be solved in reasonable time

- Bayardo and Schrag’s RelSAT also proposed conflict driven learning [BS97]
Conflict Driven Learning and Non-chronological Backtracking

\( x_1 + x_4 \)
\( x_1 + x_3' + x_8' \)
\( x_1 + x_8 + x_{12} \)
\( x_2 + x_{11} \)
\( x_{7'} + x_3' + x_9 \)
\( x_{7'} + x_8 + x_9' \)
\( x_7 + x_8 + x_{10'} \)
\( x_7 + x_{10} + x_{12'} \)
Conflict Driven Learning and Non-chronological Backtracking

\[
x_1 + x_4
\]
\[
x_1 + x_3' + x_8'
\]
\[
x_1 + x_8 + x_{12}
\]
\[
x_2 + x_{11}
\]
\[
x_7' + x_3' + x_9
\]
\[
x_7' + x_8 + x_9'
\]
\[
x_7 + x_8 + x_{10'}
\]
\[
x_7 + x_{10} + x_{12'}
\]

\[
x_1 = 0
\]
Conflict Driven Learning and Non-chronological Backtracking

\[
\begin{align*}
x_1 + x_4 \\
x_1 + x_3' + x_8' \\
x_1 + x_8 + x_{12} \\
x_2 + x_{11} \\
x_7' + x_3' + x_9 \\
x_7' + x_8 + x_9' \\
x_7 + x_8 + x_{10'} \\
x_7 + x_{10} + x_{12'}
\end{align*}
\]
Conflict Driven Learning and Non-chronological Backtracking

\[ x_1 + x_4 \]
\[ x_1 + x_3' + x_8' \]
\[ x_1 + x_8 + x_{12} \]
\[ x_2 + x_{11} \]
\[ x_7' + x_3' + x_9 \]
\[ x_7' + x_8 + x_9' \]
\[ x_7 + x_8 + x_{10'} \]
\[ x_7 + x_{10} + x_{12'} \]

- \( x_4 = 1 \)
- \( x_1 = 0 \)
- \( x_3 = 1 \)

- \( x_1 = 0, x_4 = 1 \)
- \( x_3 = 1 \)
Conflict Driven Learning and Non-chronological Backtracking

\[
x_1 + x_4
x_1 + x_3' + x_8'
\]

\[
x_1 + x_8 + x_{12}
\]
\[
x_2 + x_{11}
\]
\[
x_7' + x_3' + x_9
\]
\[
x_7' + x_8 + x_9'
\]
\[
x_7 + x_8 + x_{10'}
\]
\[
x_7 + x_{10} + x_{12'}
\]

\[
x_1 = 0, x_4 = 1
\]

\[
x_3 = 1, x_8 = 0
\]
Conflict Driven Learning and Non-chronological Backtracking

\[ x_1 + x_4 \]
\[ x_1 + x_3' + x_8' \]
\[ x_1 + x_8 + x_{12} \]
\[ x_2 + x_{11} \]
\[ x_7' + x_3' + x_9 \]
\[ x_7' + x_8 + x_9' \]
\[ x_7 + x_8 + x_{10'} \]
\[ x_7 + x_{10} + x_{12'} \]
Conflict Driven Learning and Non-chronological Backtracking

\[ x_1 + x_4 \]
\[ x_1 + x_3' + x_8' \]
\[ x_1 + x_8 + x_{12} \]
\[ x_2 + x_{11} \]
\[ x_7' + x_3' + x_9 \]
\[ x_7' + x_8 + x_9' \]
\[ x_7 + x_8 + x_{10'} \]
\[ x_7 + x_{10} + x_{12'} \]
Conflicts Driven Learning and
Non-chronological Backtracking

\[
x_1 + x_4 \\
x_1 + x_3' + x_8' \\
x_1 + x_8 + x_{12} \\
x_2 + x_{11} \\
x_7' + x_3' + x_9 \\
x_7' + x_8 + x_9' \\
x_7 + x_8 + x_{10'} \\
x_7 + x_{10} + x_{12'}
\]
Conflict Driven Learning and Non-chronological Backtracking

\[ x_1 + x_4 \]
\[ x_1 + x_3' + x_8' \]
\[ x_1 + x_8 + x_{12} \]
\[ x_2 + x_{11} \]
\[ x_7' + x_3' + x_9 \]
\[ x_7' + x_8 + x_9' \]
\[ x_7 + x_8 + x_{10'} \]
\[ x_7 + x_{10} + x_{12'} \]

\[ x_1 = 0, \ x_4 = 1 \]
\[ x_3 = 1, \ x_8 = 0, \ x_{12} = 1 \]
\[ x_2 = 0, \ x_{11} = 1 \]
\[ x_7 = 1 \]
Conflict Driven Learning and Non-chronological Backtracking

\[ x_1 + x_4 \]
\[ x_1 + x_3' + x_8' \]
\[ x_1 + x_8 + x_{12} \]
\[ x_2 + x_{11} \]
\[ x_7' + x_3' + x_9 \]
\[ x_7' + x_8 + x_9' \]
\[ x_7 + x_8 + x_{10}' \]
\[ x_7 + x_{10} + x_{12}' \]
Conflict Driven Learning and Non-chronological Backtracking

\[ x_1 + x_4 \]
\[ x_1 + x_3' + x_8' \]
\[ x_1 + x_8 + x_{12} \]
\[ x_2 + x_{11} \]
\[ x_7' + x_3' + x_9 \]
\[ x_7' + x_8 + x_9' \]
\[ x_7 + x_8 + x_10' \]
\[ x_7 + x_{10} + x_{12'} \]

\[ x_1 = 0, \ x_4 = 1 \]
\[ x_3 = 1, \ x_8 = 0, \ x_{12} = 1 \]
\[ x_2 = 0, \ x_{11} = 1 \]
\[ x_7 = 1, \ x_9 = 1 \]

\[ x_3 = 1 \land x_7 = 1 \land x_8 = 0 \rightarrow \text{conflict} \]
Conflict Driven Learning and Non-chronological Backtracking

\[ x_1 + x_4 \]
\[ x_1 + x_3' + x_8' \]
\[ x_1 + x_8 + x_{12} \]
\[ x_2 + x_{11} \]
\[ x_7' + x_3' + x_9 \]
\[ x_7' + x_8 + x_9' \]
\[ x_7 + x_8 + x_{10'} \]
\[ x_7 + x_{10} + x_{12'} \]

Add conflict clause: \( x_3' + x_7' + x_8 \)

| x1=0, x4=1 |
| x3=1, x8=0, x12=1 |
| x2=0, x11=1 |
| x7=1, x9=1 |

\[ x_3=1 \wedge x_7=1 \wedge x_8=0 \rightarrow \text{conflict} \]

Add conflict clause: \( x_3' + x_7' + x_8 \)
Conflict Driven Learning and Non-chronological Backtracking

\[ x_1 + x_4 \]
\[ x_1 + x_3' + x_8' \]
\[ x_1 + x_8 + x_{12} \]
\[ x_2 + x_{11} \]
\[ x_{7'} + x_3' + x_9 \]
\[ x_{7'} + x_8 + x_9' \]
\[ x_7 + x_8 + x_{10'} \]
\[ x_7 + x_{10} + x_{12'} \]

Add conflict clause: \( x_3' + x_7' + x_8 \)

Conflict clause: \( x_3 = 1 \land x_7 = 1 \land x_8 = 0 \rightarrow \text{conflict} \)

Add conflict clause: \( x_3' + x_7' + x_8 \)
Conflict Driven Learning and Non-chronological Backtracking

\[ x_1 + x_4 \]
\[ x_1 + x_3' + x_8' \]
\[ x_1 + x_8 + x_{12} \]
\[ x_2 + x_{11} \]
\[ x_7' + x_3' + x_9 \]
\[ x_7' + x_8 + x_9' \]
\[ x_7 + x_8 + x_{10}' \]
\[ x_7 + x_{10} + x_{12}' \]
\[ x_3' + x_8 + x_7' \]

Backtrack to the decision level of \( x_3 = 1 \)
\( x_7 = 0 \)
Conflict Driven Learning and Non-chronological Backtracking

\[
x_1 + x_4 \\
x_1 + x_3' + x_8' \\
x_1 + x_8 + x_{12} \\
x_2 + x_{11} \\
x_7' + x_3' + x_9 \\
x_7' + x_8 + x_9' \\
x_7 + x_8 + x_{10'} \\
x_7 + x_{10} + x_{12'} \\
x_3' + x_8 + x_7'
\]
What's the big deal?

Significantly prune the search space – learned clause is useful forever!

Useful in generating future conflict clauses.

Conflict clause: $x_1' + x_3 + x_5'$
Restart

- Abandon the current search tree and reconstruct a new one
- The clauses learned prior to the restart are still there after the restart and can help pruning the search space
- Adds to robustness in the solver

Conflict clause: $x_1' + x_3 + x_5'$
SAT becomes practical!

- Conflict driven learning greatly increases the capacity of SAT solvers (several thousand variables) for structured problems
- Realistic applications become feasible
  - Usually thousands and even millions of variables
  - Typical EDA applications that can make use of SAT
    - circuit verification
    - FPGA routing
    - many other applications...
- Research direction changes towards more efficient implementations
Large Example: Tough

• Industrial Processor Verification
  – Bounded Model Checking, 14 cycle behavior

• Statistics
  – 1 million variables
  – 10 million literals initially
    • 200 million literals including added clauses
    • 30 million literals finally
  – 4 million clauses (initially)
    • 200K clauses added
  – 1.5 million decisions
  – 3 hours run time
Chaff

- One to two orders of magnitude faster than other solvers…

- Widely Used:
  - BlackBox – AI Planning
    - Henry Kautz (UW)
  - NuSMV – Symbolic Verification toolset
  - GrAnDe – Automatic theorem prover
  - Several industrial licenses
Chaff Philosophy

• Make the core operations fast
  – profiling driven, most time-consuming parts:
    • Boolean Constraint Propagation (BCP) and Decision

• Emphasis on coding efficiency and elegance

• Emphasis on optimizing data cache behavior

• As always, good search space pruning (i.e. conflict resolution and learning) is important
Motivating Metrics: Decisions, Instructions, Cache Performance and Run Time

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<thead>
<tr>
<th></th>
<th>1dlx_c_mc_ex_bp_f</th>
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<td>Num Variables</td>
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</tr>
<tr>
<td>Num Clauses</td>
<td>3725</td>
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<tr>
<td>Num Literals</td>
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<table>
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<th>GRASP</th>
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<td># Decisions</td>
<td>3166</td>
<td>1795</td>
</tr>
<tr>
<td># Instructions</td>
<td>86.6M</td>
<td>1415.9M</td>
</tr>
<tr>
<td># L1/L2 accesses</td>
<td>24M / 1.7M</td>
<td>416M / 153M</td>
</tr>
<tr>
<td>% L1/L2 misses</td>
<td>4.8% / 4.6%</td>
<td>32.9% / 50.3%</td>
</tr>
<tr>
<td># Seconds</td>
<td>0.22</td>
<td>11.78</td>
</tr>
</tbody>
</table>
BCP Algorithm

• What “causes” an implication? When can it occur?
  – All literals in a clause but one are assigned to F
    • \((v_1 + v_2 + v_3)\): implied cases: \((0 + 0 + v_3)\) or \((0 + v_2 + 0)\) or \((v_1 + 0 + 0)\)
  – For an N-literal clause, this can only occur after N-1 of the literals have been assigned to F
  – So, (theoretically) we could completely ignore the first N-2 assignments to this clause
  – In reality, we pick two literals in each clause to “watch” and thus can ignore any assignments to the other literals in the clause.
    • Example: \((v_1 + v_2 + v_3 + v_4 + v_5)\)
    • \((v_1=X + v_2=X + v_3=? \text{ i.e. X or 0 or 1} + v_4=? + v_5=? )\)
BCP Algorithm

• Big Invariants
  – Each clause has two watched literals
  – If a clause can become newly implied via any sequence of assignments, then this sequence will include an assignment of one of the watched literals to F.
    • Example again: \((v_1 + v_2 + v_3 + v_4 + v_5)\)
    • \((v_1=X + v_2=X + v_3=? + v_4=? + v_5=? )\)

• BCP consists of identifying implied clauses (and the associated implications) while maintaining the “Big Invariants”

• No actions on backtracking
BCP Algorithm

• Let’s illustrate this with an example:

\[v_2 + v_3 + v_1 + v_4 + v_5\]
\[v_1 + v_2 + v_3'\]
\[v_1 + v_2'\]
\[v_1' + v_4\]
\[v_1'\]
Let’s illustrate this with an example:

- Initially, we identify any two literals in each clause as the watched ones.
- Clauses of size one are a special case.
BCP Algorithm

- We begin by processing the assignment $v_1 = F$ (which is implied by the size one clause)

State: $(v_1 = F)$

Pending:

- $v_2 + v_3 + v_1 + v_4 + v_5$
- $v_1 + v_2 + v_3'$
- $v_1 + v_2'$
- $v_1' + v_4$
BCP Algorithm

- We begin by processing the assignment \( v_1 = F \) (which is implied by the size one clause)

State: \((v_1=F)\)

Pending:

\[
\begin{align*}
\text{v2} + \text{v3} + \text{v1} + \text{v4} + \text{v5} \\
\text{v1} + \text{v2} + \text{v3}' \\
\text{v1} + \text{v2}' \\
\text{v1}' + \text{v4}
\end{align*}
\]

To maintain our invariants, we must examine each clause where the assignment being processed has set a watched literal to F.
BCP Algorithm

• We begin by processing the assignment $v_1 = F$ (which is implied by the size one clause)

State: $(v_1 = F)$
Pending:

$\neg v_2 + v_3 + v_1 + v_4 + v_5$
$v_1 + v_2 + v_3'$
$v_1 + v_2'$
$v_1' + v_4$

- To maintain our invariants, we must examine each clause where the assignment being processed has set a watched literal to $F$.
- We need not process clauses where a watched literal has been set to $T$, because the clause is now satisfied and so can not become implied.
BCP Algorithm

- We begin by processing the assignment $v_1 = F$ (which is implied by the size one clause).

To maintain our invariants, we must examine each clause where the assignment being processed has set a watched literal to $F$.

- We need not process clauses where a watched literal has been set to $T$, because the clause is now satisfied and so can not become implied.

- We *certainly* need not process any clauses where neither watched literal changes state (in this example, where $v_1$ is not watched).
BCP Algorithm

- Now let's actually process the second and third clauses:

\[
\begin{align*}
  v_2 + v_3 + v_1 + v_4 + v_5 \\
  v_1 + v_2 + v_3' \\
  v_1 + v_2' \\
  v_1' + v_4
\end{align*}
\]

State: \((v_1=F)\)
Pending:
Now let’s actually process the second and third clauses:

- For the second clause, we replace $v_1$ with $v_3'$ as a new watched literal. Since $v_3'$ is not assigned to $F$, this maintains our invariants.
BCP Algorithm

Now let’s actually process the second and third clauses:

- For the second clause, we replace $v_1$ with $v_3'$ as a new watched literal. Since $v_3'$ is not assigned to $F$, this maintains our invariants.

- The third clause is implied. We record the new implication of $v_2'$, and add it to the queue of assignments to process. Since the clause cannot again become newly implied, our invariants are maintained.
Next, we process v2’. We only examine the first 2 clauses.

For the first clause, we replace v2 with v4 as a new watched literal. Since v4 is not assigned to F, this maintains our invariants.

The second clause is implied. We record the new implication of v3’, and add it to the queue of assignments to process. Since the clause cannot again become newly implied, our invariants are maintained.
Next, we process $v_3'$. We only examine the first clause.

For the first clause, we replace $v_3$ with $v_5$ as a new watched literal. Since $v_5$ is not assigned to $F$, this maintains our invariants.

Since there are no pending assignments, and no conflict, BCP terminates and we make a decision. Both $v_4$ and $v_5$ are unassigned. Let’s say we decide to assign $v_4=T$ and proceed.
Next, we process v4. We do nothing at all.

Since there are no pending assignments, and no conflict, BCP terminates and we make a decision. Only v5 is unassigned. Let’s say we decide to assign v5=F and proceed.
BCP Algorithm

• Next, we process v5=F. We examine the first clause.

\[
\begin{align*}
\text{State:}(v_1=\text{F}, v_2=\text{F}, v_3=\text{F}, v_4=\text{T}, v_5=\text{F})
\end{align*}
\]

The first clause is implied. However, the implication is v4=T, which is a duplicate (since v4=T already) so we ignore it.

Since there are no pending assignments, and no conflict, BCP terminates and we make a decision. No variables are unassigned, so the problem is sat, and we are done.
BCP Algorithm Summary

• During forward progress: Decisions and Implications
  – Only need to examine clauses where watched literal is set to F
    • Can ignore any assignments of literals to T
    • Can ignore any assignments to non-watched literals

• During backtrack: Unwind Assignment Stack
  – Any sequence of chronological unassignments will maintain our invariants
    • So no action is required at all to unassign variables

• Overall
  – Minimize clause access
  – Better memory locality
Decision Heuristics – Conventional Wisdom

• DLIS is a relatively simple dynamic decision heuristic
  – Simple and intuitive: At each decision simply choose the assignment that satisfies the most unsatisfied clauses
  – However, considerable work is required to maintain the statistics necessary for this heuristic – for one implementation:
    • Must touch *every* clause that contains a literal that has been set to true. Often restricted to initial (not learned) clauses
    • Maintain “sat” counters for each clause
    • When counters transition 0→1, update rankings
    • Need to reverse the process for unassignment
  – The total effort required for this and similar decision heuristics is *much more* than for our BCP algorithm.

• Look ahead algorithms even more compute intensive
  C. Li, Anbulagan, “Look-ahead versus look-back for satisfiability problems”
Chaff Decision Heuristic - VSIDS

- Variable State Independent Decaying Sum
  - Rank variables by literal count in the initial clause database
  - Periodically, divide all counts by a constant
  - Only increment counts as new clauses are added

- Quasi-static:
  - Static because it doesn’t depend on var state
  - Not static because it gradually changes as new clauses are added
    - Decay causes bias toward *recent* conflicts.

- Use heap to find unassigned var with the highest ranking
  - Even single linear pass though variables on each decision would dominate run-time!

- Seems to work fairly well in terms of # decisions
  - hard to compare with other heuristics because they have too much overhead
Interplay of BCP and Decision

• This is only an intuitive description …
  – Reality depends heavily on specific instance

• Take some variable ranking (from the decision engine)
  – Assume several decisions are made
    • Say v2=T, v7=F, v9=T, v1=T (and any implications thereof)
  – Then a conflict is encountered that forces v2=F
    • The next decisions may still be v7=F, v9=T, v1=T !
    • But the BCP engine has recently processed these assignments … so these variables are unlikely to still be watched.
    • Thus, the BCP engine *inherently does a differential update.*
  – And the Decision heuristic makes differential changes more likely to occur in practice.

• In a more general sense, the more “active” a variable is, the more likely it is to *not* be watched.
Missing

- Post Chaff SAT solvers
  - BerkMin
  - Seige
  - miniSat
  - HaifaSAT
  - JeruSAT (Alex Nadel)

- The Stålmarck’s algorithm
- Hyperresolution
- Local Search
Local Search (GSAT, WSAT)


- Incomplete SAT solvers
  - Geared towards satisfiable instances, cannot prove unsatisfiability
- Hill climbing algorithm for local search
- Make short local moves
- Probabilistically accept moves that worsen the cost function to enable exits from local minima

![Solution Space Diagram](Image)
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Open Question

• Is there a subset of propositional logic beyond Horn clauses which:
  – Allows polynomial SAT
  – Includes many of the practical instances
Summary

- Rich history of emphasis on practical efficiency
- Need to account for computation cost in search space pruning
- Need to match algorithms with underlying processing system architectures
- Specific problem classes can benefit from specialized algorithms
  - Identification of problem classes?
  - Dynamically adapting heuristics?
- We barely understand the tip of the iceberg here
  - much room to learn and improve