Incidences between points and lines in $\mathbb{R}^4$ *

[Extended Abstract]

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ABSTRACT

We show that the number of incidences between m distinct points and n distinct lines in $\mathbb{R}^4$ is at most

$$A_\varepsilon\left(m^{2/5+\varepsilon}n^{4/5}+m^{1/2+\varepsilon}n^{2/3}q^{1/12}+m^{2/3+\varepsilon}n^{4/9}s^{2/9}\right)+A(m+n),$$

for any $\varepsilon>0$, where A_ε is a constant that depends on ε and A is an absolute constant, provided that no 2-plane contains more than s input lines and no 3-dimensional algebraic variety of degree at most c_ε, for a suitable constant c_ε that depends on ε, contains more than q lines.

Categories and Subject Descriptors

F.2.2 [**Analysis of Algorithms and Problem Complexity**]: Nonnumerical algorithms and problems—*Geometrical problems and computations*; G.2.1 [**Discrete Mathematics**]: Combinatorics—*Counting problems*

General Terms

Theory

Keywords

Combinatorial geometry, incidences, the polynomial method, algebraic geometry

1. INTRODUCTION

Let P be a set of m distinct points in $\mathbb{R}^4$ and let L be a set of n distinct lines in $\mathbb{R}^4$. Let $I(P,L)$ denote the number of incidences between the points of P and the lines of L; that is, the number of pairs (p,ℓ) with $p\in P$, $\ell\in L$, and $p\in\ell$. If all the points of P and all the lines of L lie in a common plane, then the classical Szemerédi–Trotter theorem [21] yields the worst-case tight bound

$$I(P,L)=O\left(m^{2/3}n^{2/3}+m+n\right). \tag{1}$$

This bound clearly also holds in three, four, or any higher dimensions, by projecting the given lines and points onto some generic plane. Moreover, the bound will continue to be worst-case tight by placing all the points and lines in a common plane, in a configuration that yields the planar lower bound.

In the recent groundbreaking paper of Guth and Katz [7], an improved bound has been derived for $I(P,L)$, for a set P of m points and a set L of n lines in $\mathbb{R}^3$, provided that not too many lines of L lie in a common plane or regulus. Specifically, they showed:

Theorem 1.1. *Let P be a set of m distinct points and L a set of n distinct lines in $\mathbb{R}^3$, and let $s\le n$ be a parameter, such that no plane contains more than s lines of L. Then*

$$I(P,L)=O\left(m^{1/2}n^{3/4}+m^{2/3}n^{1/3}s^{1/3}+m+n\right).$$

This bound is tight in the worst case.

In this paper, we establish the following analogous and sharper result in four dimensions.

Theorem 1.2. *For each $\varepsilon>0$, there exists an integer c_ε, so that the following holds. Let P be a set of m distinct points and L a set of n distinct lines in $\mathbb{R}^4$, and let $q,s\le n$ be parameters, such that (i) for any polynomial $f\in\mathbb{R}[x,y,z,w]$ of degree $\le c_\varepsilon$, its zero set $Z(f)$ does not contain more than q lines of L, and (ii) no 2-plane contains more than s lines of L. Then,*

$$I(P,L)\le A_\varepsilon\left(m^{2/5+\varepsilon}n^{4/5}+m^{1/2+\varepsilon}n^{2/3}q^{1/12}+m^{2/3+\varepsilon}n^{4/9}s^{2/9}\right)+A(m+n), \tag{2}$$

where A_ε depends on ε, and A is an absolute constant. This bound is nearly tight in the worst case, in a sense made precise in Section 3.

By a standard argument, the theorem implies, and is in fact equivalent to, the following corollary.

Corollary 1.3. *Let L be a set of n lines in $\mathbb{R}^4$, satisfying the assumptions (i)–(ii) in Theorem 1.2, for given*

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parameters ε, q and s. Then the number $m_{\geq k}$ of points incident to at least $k \geq k_0$ lines of L, where k_0 is a suitable sufficiently large absolute constant, satisfies

$$m_{\geq k} = O\left(\frac{n^{4/3+\varepsilon}}{k^{5/3}} + \frac{n^{4/3+\varepsilon}q^{1/6}}{k^2} + \frac{n^{4/3+\varepsilon}s^{2/3}}{k^3}\right),$$

for any $\varepsilon > 0$, where the constant of proportionality depends on ε.

Remark. It is instructive to compare Corollary 1.3 with the analysis of *joints* in a set L of n lines. In $\mathbb{R}^d$, a joint of L is a point incident to at least d lines of L, not all in a common hyperplane. As shown in [11, 14], the maximum number of joints of such a set is $O(n^{d/(d-1)})$, and this bound is worst-case tight. In four dimensions, this bound is $O(n^{4/3})$, which is a cleaner version (but restricted only to joints) of the bound given in Corollary 1.3.

As a matter of fact, the quantity $n^{4/3}$ plays an important role in the bound in Theorem 1.2, which is qualitatively different for $m < n^{4/3}$ and for $m > n^{4/3}$. Ignoring the terms that depend on q and s, the bound in the former case is of the form $O^*(m^{2/5}n^{4/5} + n)$ (where, as in the abstract, the $O^*(\cdot)$ notation hides subpolynomial, ε-dependent factors), and in the latter case it is simply $O^*(m)$. Moreover, as is easily checked, the terms that depend on q and s are subsumed by the other terms when q and s are not too large, specifically when $q, s = O((n^4/m^3)^{2/5})$ for $m \leq n^{4/3}$, and when $q = O((m^3/n^4)^2)$ and $s = O((m^3/n^4)^{1/2})$ for $m > n^{4/3}$.

Background.

Incidence problems have been a major topic in combinatorial and computational geometry for the past thirty years, starting with the Szemerédi-Trotter bound [21] back in 1983. Several techniques, interesting in their own right, have been developed, or adapted, for the analysis of incidences, including the crossing-lemma technique of Székely [20], and the use of cuttings as a divide-and-conquer mechanism (e.g., see [2]). Connections with range searching and related problems in computational geometry have also been noted. See Pach and Sharir [13] for a comprehensive survey of the topic.

The study of incidence geometry has dramatically changed in the past five years, due to the infusion, in two ground-breaking papers by Guth and Katz [6, 7], of new tools and techniques drawn from algebraic geometry. Although their two direct goals have been to obtain a tight upper bound on the number of joints in a set of lines in three dimensions [6], and a lower bound for the classical distinct distances problem of Erdős [7], the new tools have quickly been recognized as useful for incidence bounds. See [5, 9, 10, 18, 19, 23, 24] for a sample of recent works on incidence problems that use the new algebraic machinery.

The simplest instances of incidence problems involve points and lines. Szemerédi and Trotter solved completely this special case in the plane [21]. Guth and Katz's second paper [7] provides a worst-case tight bound in three dimensions, under the assumption that no plane contains too many lines; see Theorem 1.1. Under this assumption, the bound in three dimensions is significantly smaller than the planar bound (unless one of m, n is significantly smaller than the other), and the intuition is that this phenomenon also shows up as we move to higher dimensions. Unfortunately, the analysis becomes more involved in higher dimensions, and requires the development or adaptation of progressively more complex tools from algebraic geometry. Most of these tools are still unavailable, and appear to be interesting (new) open problems in the area.

The present paper is a first step in this direction, by considering the four-dimensional case. It does indeed derive a sharper bound, under assumptions that effectively require the configuration of points and lines to be "truly four-dimensional", in the more precise sense spelled out in Theorem 1.2.

We also note that studying incidence problems in four (or higher) dimensions is already taking place in several contemporary works, such as in Solymosi and Tao [19] and Zahl [24]. These works, though, consider incidences with higher-dimensional varieties, and the study of incidences involving lines, presented in this paper, is new.

Besides being an interesting problem in its own right, our study of point-line incidences in four dimensions has lead to a "shopping list" of tools from algebraic geometry that seem to be needed for tackling the problem successfully. As a matter of fact, it is the present lack (or inadequacy) of some of these tools that has forced us to use partitioning polynomials of constant degree (see [7] and below for more details concerning these polynomials), making our bound slightly worse (involving the m^ε factors) than what could have been obtained with polynomials of non-constant degree. Moreover, the assumption concerning the threshold degree c_ε is also a consequence of the use of constant-degree polynomials. We will further discuss these issues towards the end of the paper, and list the relevant open problems (or rather problems that we are unaware of any adequate solution thereof) in algebraic geometry that they raise. It is our hope that the final version of this paper will be able to overcome these issues, and (a) to derive the slightly sharper bound without the m^ε factors, and (b) to weaken requirement (i) in the theorem.

Overview of the proof.

In this overview we assume some familiarity of the reader with the new "polynomial method" of Guth and Katz. Otherwise, the overview can be skipped on first reading.

The analysis follows the general approach of Guth and Katz [7], with certain adaptations and modifications; see also Elekes et al. [5]. Specifically, the proof proceeds inductively. We apply the polynomial partitioning technique of [7], but we use a partitioning polynomial f of *constant* degree (as done, e.g., in [18, 19]). This yields a partition of $\mathbb{R}^4$ into cells, and we apply the induction hypothesis within each cell, to the points in the cell and the lines that cross it. The remaining, and major part of the analysis involves points and lines that lie on the zero set $Z(f)$ of f. To handle them, we further partition $Z(f)$ using a second partitioning polynomial g (as done, e.g., in [9]). Incidences involving points in $Z(f) \setminus Z(g)$ and lines not fully contained in $Z(g)$ are handled inductively, similarly to the treatment of points and lines in the cells of the first partitioning, and we are left with points and lines fully contained in the two-dimensional variety $V = Z(f, g)$ (the common zero set of f and g). Here our strategy is to project V onto some generic (three-dimensional) hyperplane, and carry out the analysis within the resulting projected variety (a two-dimensional surface in 3-space). This part involves certain algebraic subtleties, since, over the reals, the projection of an algebraic variety is not always algebraic. Nevertheless, these subtleties can be handled using resultants and similar basic tools in algebraic geometry.

As mentioned, the use of constant-degree partitioning polynomials is not new; see, e.g., Solymosi and Tao [19]. It makes certain parts of the analysis considerably simpler (or doable at all), at the cost of making the bounds slightly worse, involving the extra m^ε factors. Moreover, the requirement (condition (i) in the theorem) that no three-dimensional variety of degree at most c_ε contain more than q lines is also dictated by the use of constant-degree polynomials. We are currently working on extending the analysis so that it can use partitioning polynomials of high non-constant degree, and we hope that the full version of the paper will reflect this refined approach, with its slightly improved bound, and with the replacement of condition (i) by the assumption that no *hyperplane or quadric* contains more than q lines of L (see the discussion at the end of the paper for details).

2. PROOF OF THEOREM 1.2

The proof establishes the bound in (2) by induction on m, keeping ε fixed, where the base case is when $m \le n^{1/2}$ or when $m \le r$, for some sufficiently large constant r that will be set later. Using the trivial well-known bound $I(P, L) = O(m^2+n)$, we get that in this range of m, we have $I(P, L) \le An$, for a suitable absolute constant A. That is, (2) holds in this case.

Suppose that (2) holds for all sets P', L', with $|P'| < m$, and consider the case where the sets P, L are of respective sizes m, n, and $m > n^{1/2}$ and $m > r$.

Applying the polynomial partitioning technique (first partitioning step).

We fix a sufficiently large constant parameter $r \le m$ (the one just mentioned above), whose choice will be specified later, and apply the polynomial partition theorem (see [7] and [10, Theorem 2.6]) to obtain an *r-partitioning* 4-variate polynomial f of degree $D = O(r^{1/4})$. That is, every (open) connected component of $\mathbb{R}^4 \setminus Z(f)$ contains at most m/r points of P, where $Z(f)$ denotes the zero set of f. By Warren's theorem [22] (see also [10]), the number of components of $\mathbb{R}^4 \setminus Z(f)$ is $O(D^4) \le ar$, for a suitable absolute constant a.

Set $P_0 := P \cap Z(f)$ and $P' := P \setminus P_0$. Each line $\ell \in L$ is either fully contained in $Z(f)$ or intersects it in at most $D = O(r^{1/4})$ points. (This follows since the restriction of f to ℓ is a non-zero univariate polynomial of degree at most D.) Let L_0 denote the subset of lines of L that are fully contained in $Z(f)$ and put $L' = L \setminus L_0$. We have

$$I(P, L) = I(P_0, L_0) + I(P_0, L') + I(P', L'). \qquad (3)$$

As can be expected, the harder part of the analysis is the estimation of $I(P_0, L_0)$. Indeed, it might happen that f is of low degree (e.g., $Z(f)$ is a hyperplane, or a surface of degree at most c_ε). In this case, the best upper bound we can offer is the bound specified by Theorem 1.1. This bound is immediate when $Z(f)$ is a hyperplane, and otherwise it is obtained by projecting $Z(f)$ onto some generic hyperplane. It might also happen that $Z(f)$ contains some 2-plane, in which case we are back in the planar scenario, for which the best (and worst-case tight) bound we can offer is the Szemerédi–Trotter bound (1). Of course, the assumptions (i) and (ii) of the theorem come to the rescue, and we will see below how exactly they are used.

We first bound the second and third terms of (3). As already observed, we have

$$I(P_0, L') \le |L'| \cdot D = O(n).$$

To estimate $I(P', L')$, let us denote the (open) connected components of $\mathbb{R}^4 \setminus Z(f)$ as $K_1, \ldots, K_t$, for $t = O(r)$. For $i = 1, \ldots, t$, put $P_i = P \cap K_i$, and let L_i denote the set of the lines of L' that cross K_i; put $m_i = |P_i| \le m/r$, and $n_i = |L_i|$. Since every line $\ell \in L'$ crosses at most $1 + D$ components of $\mathbb{R}^4 \setminus Z(f)$, we have $\sum_i n_i \le n(1 + D) \le bnr^{1/4}$, for a suitable absolute constant b. Clearly, we have $I(P', L') = \sum_{i=1}^{t} I(P_i, L_i)$, so we apply the induction hypothesis within each cell K_i, and get

$$I(P_i, L_i) \le A_\varepsilon \left(m_i^{2/5+\varepsilon} n_i^{4/5} + m_i^{1/2+\varepsilon} n_i^{2/3} q^{1/12} + m_i^{2/3+\varepsilon} n_i^{4/9} s^{2/9} \right) + A(m_i + n_i).$$

Summing these bounds, and using Hölder's inequality, we get

$$\begin{aligned} I(P', L') &= \sum_{i=1}^{t} I(P_i, L_i) \\ &\le A_\varepsilon \Big((m/r)^{1/5+\varepsilon} m^{1/5} (bnr^{1/4})^{4/5} \\ &\quad + (m/r)^{1/6+\varepsilon} m^{1/3} (bnr^{1/4})^{2/3} q^{1/12} \\ &\quad + (m/r)^{1/9+\varepsilon} m^{5/9} (bnr^{1/4})^{4/9} s^{2/9} \Big) + A(m' + bnr^{1/4}) \\ &= A_\varepsilon \Big(\frac{b^{4/5}}{r^\varepsilon} m^{2/5+\varepsilon} n^{4/5} + \frac{b^{2/3}}{r^\varepsilon} m^{1/2+\varepsilon} n^{2/3} q^{1/12} \\ &\quad + \frac{b^{4/9}}{r^\varepsilon} m^{2/3+\varepsilon} n^{4/9} s^{2/9} \Big) + A(m' + bnr^{1/4}), \end{aligned}$$

where $m' = |P'|$. Recall that we assume that $m > n^{1/2}$; in this case we have $n < m^{2/5} n^{4/5}$, so the last term is at most $Abr^{1/4} m^{2/5} n^{4/5}$. Now, choosing r sufficiently large, in terms of b and ε, and then choosing A_ε sufficiently large, the whole preceding expression is upper bounded by

$$\frac{1}{3} A_\varepsilon \left(m^{2/5+\varepsilon} n^{4/5} + m^{1/2+\varepsilon} n^{2/3} q^{1/12} + m^{2/3+\varepsilon} n^{4/9} s^{2/9} \right) + Am'. \qquad (4)$$

Note that, using again the inequality $n < m^{2/5} n^{4/5}$ that holds for the range under consideration, we may assume, with appropriate choice of parameters, that (4) is also an upper bound for $I(P_0, L') + I(P', L')$.

Estimating $I(P_0, L_0)$.

We next bound the number of incidences between points and lines that are contained in $Z(f)$; as already noted, this is the hardest part of the analysis.

By the nature of its construction, f is in general reducible (see [7]). However, to apply successfully the forthcoming analysis we will need to assume that f is irreducible, so we will apply the analysis separately to each irreducible factor of f, and then sum up the resulting bounds.

Write the irreducible factors of f, in an arbitrary order, as $f_1, \ldots, f_k$. The points of P_0 are partitioned among the zero sets of these factors, by assigning each point $p \in P_0$

to the first factor in this order whose zero set contains p. A line $\ell \in L_0$ is similarly assigned to the first factor whose zero set fully contains ℓ. Then $I(P_0, L_0)$ is the sum, over $i = 1, \ldots, k$, of the number of incidences between the points and the lines that are assigned to the ith factor, plus the number of incidences between points and lines assigned to different factors. The latter kind of incidences is easy to handle, as will be discussed later. For the former kind, we assume in what follows that we have a single irreducible polynomial f, and denote by m, for short, the number of points assigned to f, and by n the number of lines assigned to f (and thus lying fully in $Z(f)$).

Second partitioning step.

In what follows, the actual degree of f does matter, and not the upper bound D chosen above (as dictated by the choice of r). Therefore, from now on, we denote by D_1 the actual degree of f.

We note that D_1 may be small, but we may assume that $D_1 > c_\varepsilon$. Otherwise, by assumption (i), L_0 consists of at most q lines. In that case, we simply project P_0 and L_0 onto some generic 3-space, so that the property that no plane contains more than s lines of (the projected) L_0 continues to hold, and no pair of lines of L_0 project to the same line. We can then apply the upper bound of Guth and Katz [7] (as provided in Theorem 1.1) to the projected scenario, and conclude that

$$\begin{aligned} I(P_0, L_0) &= O(m^{1/2}q^{3/4} + m^{2/3}q^{1/3}s^{1/3} + m + q) \\ &= O(m^{1/2}q^{3/4} + m^{2/3}n^{1/3}s^{1/3} + m + n), \qquad (5) \end{aligned}$$

which is certainly subsumed by the bound we are after. More concretely, with an appropriate choice of parameters, the sum of the bounds (5), over all relevant factors (with degrees smaller than c_ε) will be at most the bound in (4). Hence, in what follows, we assume that $D_1 > c_\varepsilon$.

As in [9], we construct a second partitioning polynomial[1] g, of degree at most $D_2 \geq D_1$ (but proportional to D_1), such that g does not vanish identically on $Z(f)$, and partitions $Z(f)$ (actually, the entire 4-space) into $s = \Theta(D_1 D_2^3)$ cells, each containing at most $m/(D_1 D_2^3)$ points of P_0. The bounds on the number of cells and on the size of each subset follow from an appropriate generalization to four dimensions of the analysis in [9].

Incidences with lines not contained in $Z(f, g)$.

A line $\ell \in L$ either crosses at most $D_2 + 1$ cells, or is fully contained in $Z(g)$. We let L_0' denote the subset of lines of L_0 that are not fully contained in $Z(g)$. Enumerate the cells of the new partition as $K_1', \ldots, K_s'$, and for each $i = 1, \ldots, s$ put $P_{0,i} = P_0 \cap K_i'$, and $m_i = |P_{0,i}| \leq m/(D_1 D_2^3)$. Let $L_{0,i}$ denote the set of the lines of L_0' that cross K_i', and put $n_i = |L_{0,i}|$. We have

$$\sum_{i=1}^{s} n_i \leq (D_2 + 1)|L_0| = O(D_2 n).$$

Applying the induction hypothesis for each i, we have, using Hölder's inequality (where b is another suitable absolute constant),

$$I(P_0, L_0') = \sum_{i=1}^{s} I(P_{0,i}, L_{0,i})$$

$$\leq A_\varepsilon \left(\left(\frac{m}{D_1 D_2^3}\right)^{2/5+\varepsilon} \sum_{i=1}^{s} n_i^{4/5} + \left(\frac{m}{D_1 D_2^3}\right)^{1/2+\varepsilon} \sum_{i=1}^{s} n_i^{2/3} q^{1/12} \right.$$

$$\left. + \left(\frac{m}{D_1 D_2^3}\right)^{2/3+\varepsilon} \sum_{i=1}^{s} n_i^{4/9} s^{2/9} \right) + A \left(\sum_{i=1}^{s} (m_i + n_i) \right)$$

$$\leq bA_\varepsilon \left(\frac{m^{2/5+\varepsilon}}{(D_1 D_2^3)^{2/5+\varepsilon}} (D_2 n)^{4/5} (D_1 D_2^3)^{1/5} \right.$$

$$+ \frac{m^{1/2+\varepsilon}}{(D_1 D_2^3)^{1/2+\varepsilon}} (D_2 n)^{2/3} (D_1 D_2^3)^{1/3} q^{1/12}$$

$$\left. + \frac{m^{2/3+\varepsilon}}{(D_1 D_2^3)^{2/3+\varepsilon}} (D_2 n)^{4/9} (D_1 D_2^3)^{5/9} s^{2/9} \right) + A(m + D_2 n)$$

$$= bA_\varepsilon \left(\frac{D_2^{1/5-3\varepsilon}}{D_1^{1/5+\varepsilon}} m^{2/5+\varepsilon} n^{4/5} + \frac{D_2^{1/6-3\varepsilon}}{D_1^{1/6+\varepsilon}} m^{1/2+\varepsilon} n^{2/3} q^{1/12} \right.$$

$$\left. + \frac{D_2^{1/9-3\varepsilon}}{D_1^{1/9+\varepsilon}} m^{2/3+\varepsilon} n^{4/9} s^{2/9} \right) + A(m_0' + D_2 n).$$

As already mentioned, we choose, as we may (see [9] for details) D_2 to be (greater than and) proportional to D_1, say $D_2 \leq \beta D_1$, for a suitable absolute constant β. Then we get

$$\begin{aligned} I(P_0, L_0') \leq \frac{b_1 A_\varepsilon}{D_1^{4\varepsilon}} \Big(& m^{2/5+\varepsilon} n^{4/5} + m^{1/2+\varepsilon} n^{2/3} q^{1/12} \\ & + m^{2/3+\varepsilon} n^{4/9} s^{2/9} \Big) + A(m + \beta D_1 n), \end{aligned}$$

for yet another suitable absolute constant b_1. As before, if we assume that c_ε is sufficiently large (recall that we assume that $D_1 > c_\varepsilon$), and assume that A_ε is also sufficiently large (so as to "swallow" the factor $\beta A D_1$), then the linear term $\beta A D_1 n$ will be subsumed by the first term, and the whole expression will be at most

$$\begin{aligned} \frac{1}{3} A_\varepsilon \Big(& m^{2/5+\varepsilon} n^{4/5} + m^{1/2+\varepsilon} n^{2/3} q^{1/12} \\ & + m^{2/3+\varepsilon} n^{4/9} s^{2/9} \Big) + Am. \qquad (6) \end{aligned}$$

Before we continue, we note that summing (6) over all the irreducible facors of the original polynomial f, we obtain the same upper bound, now with $m = |P_0|$ and $n = |L_0|$.

Another observation concerns the number of incidences between points and lines assigned to different factors. This is easily done, by noting that each such incidence, between a point p assigned to some factor f_i and a line ℓ assigned to some factor f_j (necessarily with $j > i$) can be interpreted as an intersection of ℓ with $Z(f_i$. The overall number of such intersections is at most nD, a bound that is subsumed by, say, the bound in (6).

Incidences with lines contained in $Z(f, g)$.

We now continue the analysis with a single factor, still denoted as f, and with its second partitioning polynomial g.

[1] It is in this construction that we need f to be irreducible; see [9] for details.

We need to bound the number of incidences between points and lines that (are assigned to f and) are fully contained in $Z(f,g)$.

With a slight recycling of notation, we have a set P_0 of m points and a set L_0 of n lines in $\mathbb{R}^4$, so that all the points and lines are contained in the two-dimensional algebraic variety $V = Z(f,g) \subset \mathbb{R}^4$, where $f,g \in \mathbb{R}[x,y,z,w]$ are polynomials of constant degrees D_1 and D_2, respectively, so that $D_2 = \Theta(D_1)$. We want to project V onto some generic hyperplane, and then analyze the number of incidences within the resulting projected surface. Indeed, notice that distinct lines of L are projected to distinct lines, when the projection is *generic*. This implies that the number of incidences does not decrease in a generic projection. However, some care is needed here. In the complex projective case, the projection of an algebraic variety is always algebraic (e.g., see [1, Theorem 4.103]). In the real (affine) case, the projection of a (real) algebraic variety is always semi-algebraic (e.g., see [1, Theorem 2.62]), but does not have to be algebraic. (For example, the projection of a circle in the plane onto a line is a closed segment, and similar examples exist in higher dimensions.) We therefore use the following notion of an *algebraic projection* in the real case, specialized to the scenario at hand.

The algebraic projection.

We delegate the technical details concerning algebraic projections to an appendix, and mention them here very briefly. In a nutshell, we assume that the generic projection is onto the xyz-hyperplane $w = 0$ (which can be made without loss of generality), and denote this hyperplane as h. We construct the *resultant* $Res(f,g,w)$ of f and g (see [3]), viewed as polynomials in w (with coefficients in $\mathbb{R}[x,y,z]$). This is a trivariate polynomial, in $\mathbb{R}^3$, of degree $d := D_1D_2$, and its zero set (within h) contains the standard projection of V onto h (the two projections coincide in the complex projective case, but can differ over $\mathbb{R}$. We call this zero set $Z(Res(f,g,w))$ the *algebraic projection* of V, and denote it as S. Fuller details concerning this step are given in the appendix.

Consider the prime factorization $Res(f,g,w) = \prod_{i=1}^{v} q_i^{t_i}$ of the resultant (as a real trivariate polynomial), with the factors q_i all distinct, and put $p = \prod_{i=1}^{v} q_i$. Clearly, $S = Z(Res(f,g,w)) = Z(p)$, and p is square-free, of degree at most d.

Critical and flat points and lines.

We introduce some notations and propositions from Guth and Katz [6] and from Elekes et al. [5]. A point a is *critical* (or *singular*) for a trivariate polynomial p if $p(a) = 0$ and $\nabla p(a) = 0$; any other point a in $Z(p)$ is called *regular*. A line ℓ is critical for p if $\ell \subseteq Z(p)$ and all its points are critical. By [6, Proposition 3.1] or [5, Proposition 3], there are at most $d(d-1)$ critical lines for p (for this argument one needs p to be square-free, which is why we replaced $Res(f,g,w)$ by p).

Clearly (e.g., see [5, Proposition 4]), if p vanishes identically on three lines passing through some regular point a, then these lines must be coplanar (as they are contained in the tangent plane to $Z(p)$ at a).

Following the notation in [6] (see also [5]), call a regular point a of a trivariate real polynomial p *linearly flat* if it is incident to three distinct (necessarily coplanar) lines on which p vanishes identically. The condition for a point a to be linearly flat can be expressed by the vanishing of another polynomial $\Pi(p)$, of degree $3d-4$, which is constructed from p and from its first- and second-order partial derivatives, and is related to the *second fundamental form* of p; see [5, 6] for more details.

A line ℓ is *flat* for p, if $\ell \subseteq Z(p)$ and all of its points are linearly flat (with the possible exception of a finite number of critical points). By [6, Corollary 3.4] and [5, Proposition 8], if p has no linear factors, there are at most $d(3d-4)$ flat lines for p.

Incidences within the projection S.

Returning to our scenario, we have a set $P^* \subset h$ of at most m points and a set L^* of at most n distinct lines in h, which are the respective projections of the points of P_0 and the lines of L_0 onto h. (For these simple objects, the standard projection and the algebraic projection always coincide.) Clearly, the original incidences in $\mathbb{R}^4$ are preserved in this projection. We may assume that (i) each point of P^* is incident to at least three lines, because the number of incidences involving points incident to at most two lines is at most $2m$, and that (ii) each line of L^* is incident to at least $5d$ points of P^*, because the other lines contribute at most $5dn = O(n)$ incidences. Let p be the reduced square-free form of the resultant $Res(f,g,w)$, as defined above. Clearly, p can have at most d linear factors; i.e., p can vanish on at most d planes $\pi_1, \ldots, \pi_k$, for some $k \le d$. We factor out all the linear factors from p, and let $\tilde{p}$ denote the resulting polynomial, which is a square-free polynomial, without any linear factors, of degree at most d.

Let L_1^* (resp., P_1^*) denote the set of lines of L^* (resp., points of P^*) that are contained in some of the planes $\pi_i, i = 1, \ldots, k$, and put $L_2^* = L^* \setminus L_1^*$, $P_2^* = P^* \setminus P_1^*$. For each line $\ell \in L_2^*$, $\tilde{p}$ vanishes identically on it, and at most d points of $P \cap \ell$ can lie on the planes π_i. Thus, ℓ contains at least $4d$ points of P_2^*. Each of these points is either critical or linearly flat for $\tilde{p}$. (A line passing through such a point and contained in $Z(p)$ must also be contained in $Z(\tilde{p})$ because it cannot be fully contained in any of the planes π_i.) Hence, either at least d of these points are critical points of $\tilde{p}$, and then ℓ is a critical line for $\tilde{p}$, or at least $3d$ of these points are linearly flat points of $\tilde{p}$, and then ℓ is a flat line for $\tilde{p}$ (because the polynomial $\Pi(\tilde{p})$, of degree at most $3d-4$, vanishes in at least $3d$ points on ℓ). Applying, e.g., [5, Propositions 3 and 8], the overall number of lines in L_2^* is therefore at most

$$d(d-1) + d(3d-4) < 4d^2,$$

and the number of incidences involving lines in L_2^* is therefore at most

$$4md^2 = O(m). \tag{7}$$

It therefore remains to bound the number of incidences involving the lines in L_1^*. Note that these lines can be incident only to the points of P_1^*. In complete analogy to the way we handled the different irreducible factors of f, we assign each point of P_1^* to the first plane π_j (the one with the smallest index) that contains it, and similarly assign each line in L_1^* to the first plane π_j that fully contains it. Clearly, if a point $u \in P_1^*$ is incident to a line $\ell \in L_1^*$, then the plane π_i to which u is assigned and the plane π_j to which ℓ is assigned satisfy $i \le j$. For a line $\ell \in L_1^*$, assigned to some π_j, the number of incidences of ℓ with points that are assigned to preceding planes is at most $j - 1 \le k - 1 < d$. Hence, the number of incidences between lines and points which are not assigned to the same plane is at most $dn = O(n)$.

Finally, we bound the number of incidences inside each plane π_i, for $i = 1, \ldots, k$; that is, between points and lines assigned to the same π_i. The intuition is that the pre-image in V of each π_i should also be a 2-flat, in which case we know, by assumption (ii) of the theorem, that it contains at most s lines of L, so π_i contains at most s projected lines. We need some care though. When we let the projection h vary, the identity and number of the resulting planes π_i might change.

Let U denote some irreducible component of V. In a suffieicntly small neighborhood of some smooth point, U looks like a plane. For a (projected) plane π_i^h, let U_i^h denote its pre-image in U under h. By definition, the projection of U_i^h is the intersection of the projection of U with π_i^h. This intersection is either the entire π_i^h or a 1-dimensional curve. If, for some projection h, we have $\dim(U_i^h) = 1$ for all indices i of the resulting planes, we can bound the number of incidences involving U as we did earlier, because then U projects into the nonlinear part $Z(\tilde{p})$ of S. Therefore, it suffices to consider only irreducible components U of V that are always generically projected onto some plane π_i. This situation is handled by the following simple lemma.

LEMMA 2.1. *Let X be a two-dimensional real algebraic variety in $\mathbb{R}^4$, such that a generic (real) algebraic projection of V on $\mathbb{R}^3$ is a plane. Then X is a 2-flat in $\mathbb{R}^4$.*

Proof. Assume to the contrary that X is not a 2-flat in $\mathbb{R}^4$. This implies that, for a generic hyperplane h, the curve $C_h = X \cap h$ is not a straight line. For any 2-flat $g \subset h$, let $C_{h,g}$ denote the projection of C_h onto the plane g. This implies that for a generic choice of a plane g and a hyperplane h satisfying $g \subset h \subset \mathbb{R}^4$, the curve $C_{h,g}$ is not a straight line in the plane g. Next, by taking a suitable rotation of the coordinate frame, we may assume that g is the xy-plane, and h is the xyz-hyperplane. In these coordinates, it is easy to verify that $C_{h,g}$ can be obtained by first projecting X onto the xyw-hyperplane, and then cutting it with the xy-plane. But the projection of X onto the xyw-hyperplane (which is actually a generic hyperplane) is a plane by assumption, and then cutting it with any 2-flat yields a line, a contradiction that completes the proof. □

We thus have a collection of k 2-flats $U_1, \ldots, U_k$ in $\mathbb{R}^4$, and we wish to bound the number of incidences within each U_i. For $i = 1, \ldots, k$, let $L_{1,i}$ (resp., $P_{1,i}$) denote the set of lines of L_0 (resp., points of P_0), which are contained in U_i, and put $n_i = |L_{1,i}|$, $m_i = |P_{1,i}|$. By assumption (ii) of the theorem, $n_i \le s$ for each $i = 1, \ldots, k$, so the classical Szemerédi–Trotter bound (1) yields

$$I(P_{1,i}, L_{1,i}) = O\left(m_i^{2/3} s^{2/3} + m_i + s\right), \qquad i = 1, \ldots, k.$$

Summing over $i = 1, \ldots, k$, we get a total of $O(m^{2/3} d^{1/3} s^{2/3} + m + s)$ incidences, which, combined with (7), yields the bound

$$I(P_0, L_0) = O\left(m^{2/3} s^{2/3} + m + s\right), \tag{8}$$

which is subsumed in the bound in (2), for a sufficiently large constant A_ε.

Recall that so far we have assumed in this part of the analysis that f is irreducible. In general, we apply the preceding analysis to each irreducible factor of f separately, and sum up the resulting bounds (8). It is easily checked that this sum is still dominated by the bound in (6) (the term s is dominated by $m^{2/3} n^{4/9} s^{2/9}$ when $m > n^{1/2}$, as is easily checked). Finally, putting together the bounds in (4) and (6) (now also including the bound (8) for points and lines within $Z(f, g)$), and ensuring that A_ε is chosen sufficiently large, we establish the induction step and thus complete the proof of the theorem. □

3. THE LOWER BOUND

We use the following generalization to four dimensions of a construction due to Elekes; see [4].

We fix two integer parameters k and ℓ, with concrete values that will be set later, and take P to be the set of vertices of the integer grid

$$\{(x, y, z, w) \mid 1 \le x \le k,\ 1 \le y, z, w \le 2k\ell\}.$$

We have $|P| = 8k^4\ell^3$.

We then take L to be the set of all lines of the form

$$y = ax + b, \qquad z = cx + d, \qquad w = ex + f, \tag{9}$$

where $1 \le a, c, e \le \ell$ and $1 \le b, d, f \le k\ell$. We have $|L| = k^3\ell^6$. Note that each line in L has k incidences with the points of P, one for each $x = 1, 2, \ldots, k$, so

$$I(P, L) = k^4\ell^6 = \Theta(|P|^{2/5}|L|^{4/5}),$$

as is easily checked. Note that $|L|^{1/2} \le |P| \le 8|L|^{4/3}$, which is (asymptotically) the range of interest for this bound to be significant (see the discussion in the introduction). Moreover, for any pair of integers m, n, with $n^{1/2} \le m \le n^{4/3}$, we can find k and ℓ for which $|P| = \Theta(m)$ and $|L| = \Theta(n)$.

To complete the construction, we show that no hyperplane, no constant-degree algebraic surface, and no plane can contain more than $q = s := O\left(|L|^{8/5}/|P|^{6/5}\right) = O(\ell^6)$ lines of L, under certain constraints on $|P|$ and $|L|$. As discussed in the introduction, this means that, for this configuration, the terms that depend on q and s do not dominate the bound in Theorem 1.2.

Let h be an arbitrary hyperplane. If h is orthogonal to the x-axis then it does not contain any line of L, as is easily checked, so we may assume that h intersects any hyperplane of the form $x = i$ in a 2-plane π_i. The intersection of P with $x = i$ is a $2k\ell \times 2k\ell \times 2k\ell$ lattice, that we denote as Q_i. Every line $\lambda \in L$ in h meets π_i at a single point (as noted, it cannot be fully contained in π_i), which is necessarily a point in $\pi_i \cap Q_i$ (every line of L contains a point of every Q_i). The size of $\pi_i \cap Q_i$ is easily seen to be $O((k\ell)^2)$, and each point is incident to at most ℓ^2 lines that lie in h. To see this latter property, substitute the equations (9) of a line of L into the linear equation defining h, say $Ax + By + Cz + Dw - 1 = 0$. This yields a linear equation in x, whose x-coefficient has to vanish. This in turn yields a linear equation in a, c, and e, which can have at most ℓ^2 solutions over $[1, \ldots, \ell]^3$ (it is easily checked that the x-coefficient cannot be identically zero for all choices of a, c, e). The number of lines of L in h is thus $O(\ell^2(k\ell)^2) = O(k^2\ell^4)$.

This analysis easily extends to show that no constant-degree algebraic surface contains more than $O(k^2\ell^4)$ lines of L; we omit the routine details.

Finally, let π be a 2-plane, where again we may assume that π is not orthogonal to the x-axis. Then π meets a hyperplane $x = i$ in a line μ, and $\mu \cap Q_i$ contains at most $k\ell$ points. Every line λ in π meets μ at one of these points and, arguing as above, each such point can be incident to at most ℓ lines that lie in π (now instead of one linear equation in a, c, e, we get two). Hence, π contains at most $k\ell^2$ lines of L.

All these bounds are $O(\ell^6)$ when $k = O(l)$, which translates to $|P| = O(|L|^{7/9})$. Hence, we have shown that in this range of cardinalities, the bound in Theorem 1.2 is nearly tight in the worst case, and is $\Theta(|P|^{2/5}|L|^{4/5})$.

4. DISCUSSION

The main challenge that we face is to refine the analysis so that it uses a partitioning polynomial of high non-constant degree. If the various technical difficulties that such a choice might create can be overcome, this will eliminate the need for induction, and will have two positive effects on the result: (a) The ε's will be removed from the exponents. (b) Condition (i) in Theorem 1.2 will be replaced by the weaker and more natural requirement that no *hyperplane* or *quadric* contains more than q lines of L (see below for the intuition why we think that quadrics should also be included).

The difficulties in using a high-degree partitioning polynomial are in analyzing $I(P_0, L_0)$, involving lines that are fully contained in $Z(f)$. Here the use of a second partitioning polynomial will not yield the desired bound when the actual degree of the first polynomial is too low, say a constant. Informally, in such a case the configuration of points and lines is "essentially three-dimensional" (in a sense that we do not elaborate here), and the analysis will get stuck at the weaker bound of Theorem 1.1.

The intuition is that when the algebraic surface $Z(f)$ contains many lines, it should be *ruled*, meaning that each point on $Z(f)$ is incident to a line that is fully contained in $Z(f)$. Such a special structure could be exploited in the analysis. For example, in three dimensions a surface of degree d that fully contains more than $d(11d - 24)$ lines must be ruled (this follows from the Cayley-Salmon theorem [16]). Ruled surfaces in $\mathbb{R}^3$ are either planes, or quadrics (hyperbolic paraboloids or hyperboloids of one sheet), or are *singly ruled*, meaning that each generic point on the surface is incident to only one line fully contained in the surface. As follows from the arguments in Guth and Katz [7], points that are "accidentally" incident to more than one such line are singular, and are special enough to yield only very few incidences.

In four dimensions the situation is, as expected, more involved. Similar to the three-dimensional case (as discussed in [7]), one can define a so-called *flecnode polynomial* $\mathsf{F}L_f$ in terms of f, of degree proportional to the degree of f, whose vanishing at a point $p \in Z(f)$ means that p is incident to a line ℓ that *osculates* up to order two to $Z(f)$, namely, ℓ approximates $Z(f)$ near p up to order two (with a third-order error term).

If $\mathsf{F}L_f$ does not vanish identically on $Z(f)$, we can use $\mathsf{F}L_f$ as a second partitioning polynomial, handle well the lines of L_0 that (are fully contained in $Z(f)$ but) are not fully contained in $Z(\mathsf{F}L_f)$, and are left with lines fully contained in $Z(f, \mathsf{F}L_f)$, which is a two-dimensional surface. We can then project $Z(f, \mathsf{F}L_f)$ onto a generic hyperplane, as we did for $Z(f, g)$, and finish up the analysis similarly to what was done above. Care is needed here, though, because when the degree of f is not a constant, the degrees of $Z(f, \mathsf{F}L_f)$ and of certain features thereof might become too large.

The interesting (and difficult) situation is when $\mathsf{F}L_f$ vanishes identically on $Z(f)$. In this case a result of Landsberg [8, 12] shows that $Z(f)$ is ruled. A simple analysis shows that the only nontrivial case is when each point of $Z(f)$ is incident to an infinite, 1-parameter family of ruling lines. Some old results of Severi [17], mentioned also in Rogora [15, Theorem 1], and in an unpublished thesis by Richelson, suggest that in this case $Z(f)$ must be a hyperplane, a quadric, or ruled by planes. If this is the case, we can get rid of the first two possibilities by requiring that no hyperplane or quadric contains more than q lines of L (a weaker and cleaner variant of assumption (i) of the theorem), and we hope that the remaining special structure should control the number of incidences.

While this extension and refinement of Theorem 1.2 is the main open problem that we propose (and plan to attack), there are many other interesting open problems. Clearly, extending the analysis to higher, arbitrary dimensions is a major challenge. We conjecture that the "leading term" in the bound in $\mathbb{R}^d$ should be $O(m^{2/(d+1)}n^{d/(d+1)})$, as it is for $d = 2, 3$, and 4. Also, it would be nice to find a simple additional assumption that implies that the number of points of P must be (close to) $O(n^{4/3})$, extending the bound for joints mentioned above.

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APPENDIX

A. THE ALGEBRAIC PROJECTION.

To define the desired projection, we make a short detour through the theory of resultants, recalling the relevant basic notions and results from classical algebraic geometry. A good reference are Sections 3.5 and 3.6 in Cox et al. [3].

Resultants: a quick review.

Let K be any field of characteristic 0; for our purpose, only $\mathbb{R}$ or $\mathbb{C}$ are relevant. For polynomials $f, g \in K[x]$ of positive degrees, the *resultant* of f and g with respect to x, denoted $Res(f, g, x)$, is the determinant of the Sylvester matrix of f and g (see [3, Definition 3.5.7]). A polynomial is called an *integer polynomial* provided that all of its coefficients are integers. Then we have (see [3, Proposition 3.5.8]):

THEOREM A.1. *Given $f, g \in K[x]$ of positive degrees, the resultant $Res(f, g, x) \in K$ is an integer polynomial in the coefficients of f and g. Furthermore, f and g have a common factor in $K[x]$ if and only if $Res(f, g, x) = 0$.*

If $K = \mathbb{C}$, these properties are equivalent to f and g having a common root if and only if $Res(f, g, x) = 0$. We extend the notion of a resultant for a pair f, g of 4-variate polynomials in $K[x, y, z, w]$ with respect to w, assuming that f and g have a positive degree in w. [2] We define the resultant of f and g with respect to w exactly as before, viewing f and g as polynomials in the variable w with coefficients in the field $K(x, y, z)$ of rational functions in x, y, z over K, denoted by $K(x, y, z)$ (for more details, see [3, Section 6]). For these extended resultants, we have (see [3, Proposition 3.6.1]):

THEOREM A.2. *Let $f, g \in K[x, y, z, w]$ have positive degrees in w. Then:*

(i) $Res(f, g, w)$ is in the ideal $\langle f, g\rangle \cap K[x, y, z]$.

(ii) $Res(f, g, w) = 0$ if and only if f and g have a common factor in $K[x, y, z]$ which has positive degree in w.

In particular, the resultant of f and g with respect to w is a polynomial in x, y, z.

Having reviewed these tools, we return to our setup, involving the 2-dimensional variety $V = Z(f, g)$ in $\mathbb{R}^4$. As above, $Z_{\mathbb{C}}(f, g)$ stands for the common zero set of f and g over $\mathbb{C}$.

THEOREM A.3. *Let $f, g \in \mathbb{C}[x, y, z, w]$ have positive degrees in w. Let $c = (c_1, c_2, c_3) \in \mathbb{C}^3$, such that there exists $w \in \mathbb{C}$, satisfying $f(c, w) = g(c, w) = 0$. Then the resultant of f and g with respect to w (which, as defined above, is a polynomial in x, y, z) vanishes at c, i.e., we have $Res(f, g, w)(c) = 0$.*

Proof. We pass to the field $\mathbb{C}$, and put $I = \langle f, g\rangle$, the ideal generated by f and g. Let $w_0 \in \mathbb{C}$ be such that $f(c, w_0) = g(c, w_0) = 0$. This implies that $(c, w_0) \in Z(I)$. By Theorem A.2, the resultant $r = Res(f, g, w)$ belongs to $I \cap \mathbb{C}[x, y, z]$ and thus also to I, implying that $r(c, w_0) = 0$. Since the variable w is absent in r, we deduce that $r(c) = 0$, as claimed. □

We can finally return to the task of projecting the real variety $V = Z(f, g)$ onto some generic (three-dimensional) hyperplane h. We may assume, after a suitable rotation of the coordinate frame, that h is the xyz-hyperplane. Since h is generic, we may also assume that both f and g have positive degrees in w, making the resultant $Res(f, g, w)$ well defined. Let $S = Z_{\mathbb{R}}(Res(f, g, w))$ denote the zero set of the resultant $Res(f, g, w)$ over the real numbers. S is called the "algebraic projection"[3] of V.

LEMMA A.4. *With the above notations, S is a 2-dimensional variety of degree $d \le D_1D_2$, containing the real projection of V onto the hyperplane h.*

Proof. It is clear that S is two-dimensional, being the zero set of a nontrivial polynomial in $\mathbb{R}^3$. It is also well known that the degree of $Res(f, g, w)$ is at most D_1D_2. To see that S contains the projection of V onto the hyperplane h, let (c_1, c_2, c_3) be any point of the projection of V onto h, which

[2]this notion extends to any number of variables.

[3]Note that in the case of a circle projected onto a line, the algebraic projection S is the entire line.

we have assumed to be the xyz-hyperplane. By Theorem A.3, the resultant of f and g vanishes on (c_1, c_2, c_3), implying that $(c_1, c_2, c_3) \in Z_{\mathbb{R}}(Res(f, g, w))$. $\square$