Similar Simplices in a d-Dimensional Point Set*

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Abstract

We consider the problem of bounding the maximum possible number $f_{k,d}(n)$ of k-simplices that are spanned by a set of n points in $\mathbb{R}^d$ and are similar to a given simplex. We first show that $f_{2,3}(n) = O(n^{13/6})$, and then tackle the general case, and show that $f_{d-2,d}(n) = O(n^{d-8/5})$ and[1] $f_{d-1,d}(n) = O^*(n^{d-72/55})$, for any d. Our technique extends to derive bounds for other values of k and d, and we illustrate this by showing that $f_{2,5}(n) = O(n^{8/3})$.

1 Introduction

Let P be a set of n points in $\mathbb{R}^d$, and let Δ be a prescribed k-dimensional simplex (k-simplex, for short), for some $2 \le k \le d-1$. Let $f(P,\Delta)$ denote the number of k-simplices spanned by P that are similar to Δ. Set

$$f_{k,d}(n) = \max f(P,\Delta),$$

where the maximum is taken over all sets P of n points in $\mathbb{R}^d$ and over all k-simplices Δ in $\mathbb{R}^d$. We wish to obtain sharp bounds on $f_{k,d}(n)$. It suffices to consider cases with $2 \le k \le d-1$, since, trivially, $f_{0,d}(n) = n$, $f_{1,d}(n) = \binom{n}{2}$, and $f_{d,d}(n) \le 2f_{d-1,d}(n)$.

The problem of obtaining sharp bounds on $f_{k,d}(n)$ is motivated by *exact pattern matching*: We are given a set P of n points in $\mathbb{R}^d$ and a "pattern set" Q of $m \le n$ points (in most applications m is much smaller than n; let us assume $m \ge d+1$), and we wish to determine whether P contains a similar copy of Q, under some allowed class of transformations, or, alternatively, to enumerate all such copies, See [12] for a comprehensive review of this and related problems. A commonly used approach to this problem is to take a d-simplex Δ

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[1]The notation $O^*(\cdot)$ hides polylogarithmic factors.

spanned by some points of Q, and find all congruent copies of Δ that are spanned by points of P. For each such copy Δ', take the similarity transformation(s) that map Δ to Δ', and check whether all the other points of Q map to points of P under that transformation. The efficiency of such an algorithm depends on the number of similar copies of Δ in P. Using this approach for congruences, de Rezende and Lee [19] developed an $O(mn^d)$-time algorithm to determine whether E contains a congruent copy of P. For $d = 3$, Brass [10] developed an $O(mn^{7/4}\beta(n)\log n + n^{11/7+\varepsilon})$-time algorithm, which improves an earlier result by Boxer [9]. See also [8, 11] for related work. To recap, for applications of this kind, the main quantity of interest is $f_{d-1,d}(n)$.

In the recent monograph [12, pp. 265–266], Brass et al. review the known bounds on $f_{k,d}(n)$ and state various conjectures and open problems. There are practically no known upper bounds, especially in $d \geq 3$ dimensions, with the sole exception of a bound of $O(n^{2.2})$ on $f_{2,3}(n)$ (and $f_{3,3}(n)$), given by Akutsu et al. [5].

The case of *congruent* simplices has also been studied; see Agarwal and Sharir [4] and references therein. Denote by $g_{k,d}(n)$ the maximum number of k-simplices that are spanned by a set of n points in $\mathbb{R}^d$ and are congruent to a given simplex Δ. Agarwal and Sharir have shown that $g_{2,3}(n) = O^*(n^{5/3})$ (and $\Omega(n^{4/3})$), $g_{2,4}(n) = O^*(n^2)$ (and $\Omega(n^2)$), $g_{2,5}(n) = \Theta(n^{7/3})$, and $g_{3,4}(n) = O^*(n^{9/4})$. A simple construction, attributed to Lenz [16], shows that $g_{k,d}(n) = \Omega(n^{d/2})$, for even values of d and for sufficiently large k. Erdős and Purdy [15] conjectured that this construction is asymptotically best possible, namely, that $g_{k,d}(n) = O(n^{d/2})$. Agarwal and Sharir also derive a recurrence for $g_{k,d}(n)$, for general values of k and d. The solution of this recurrence is $O(n^{\zeta(d,k)+\varepsilon})$, where $\zeta(d,k)$ is a rather complicated function of d and k. They show that $\zeta(d,k) \leq d/2$ for $d \leq 7$ and $k \leq d-2$, and conjecture that $\zeta(d,k) \leq d/2$ for all d and $k \leq d-2$, in accordance with the Erdős-Purdy conjecture just mentioned.

Returning to the case of similar simplices, we note that the only known lower bounds for $f_{k,d}(n)$ are the same bounds for $g_{d,d}(n)$, namely $\Omega(n^{d/2})$ for $d \geq 4$ even, and $\Omega(n^{d/2-1/6})$, for $d \geq 3$ odd; see also [1, 2, 4].

Our results. We first obtain the bound $O(n^{13/6})$ for $f_{2,3}(n)$, improving upon the bound of Akutsu et al. [5]. We note that Brass conjectures that $f_{3,3}(n) = o(n^2)$ [12, p. 265], but $f_{2,3}(n) = \Omega(n^2)$ (in fact, $f_{2,2}(n)$ is already $\Theta(n^2)$).

We then tackle the general case, obtaining the first nontrivial bounds for $f_{d-2,d}(n)$ and $f_{d-1,d}(n)$ (and thus also for $f_{d,d}(n)$); the trivial bounds are, respectively, $f_{d-2,d}(n) = O(n^{d-1})$ and $f_{d-1,d}(n) = O(n^d)$). Specifically, we show:

$$f_{d-2,d}(n) = O(n^{d-8/5}) \quad \text{and} \quad f_{d-1,d}(n) = O^*(n^{d-72/55}).$$

The above results imply that $f_{2,4}(n) = O(n^{12/5})$. Finally, we prove that $f_{2,5}(n) = O(n^{8/3})$. Note that this is the last interesting case for triangles because, by Lenz' construction, $f_{2,6}(n) = g_{2,6}(n) = \Omega(n^3)$.

Needless to say, none of these bounds is known (nor conjectured) to be tight. Our techniques are strongly based on bounds on the number of incidences between points and spheres (or circles). Considerable progress has been made on these problems in recent years; see [3, 6, 7, 17] and also [18] for a comprehensive survey of these and related results. We also use the recent bound of Elekes and Tóth [14] on the number of so-called rich non-degenerate hyperplanes. In this regard, our work can be regarded as an application of the recent developments in incidence problems, which raises several interesting basic open problems in this area, as discussed at the end of the paper.

2 Similar Triangles in $\mathbb{R}^3$

In this section we prove an improved bound on $f_{2,3}(n)$. Let Δ_0 be a fixed triangle, and let $\mathcal{S}(\Delta_0)$ denote the set of all triangles spanned by P and similar to Δ_0. As a warm-up exercise, we first derive a simple upper bound that is weaker, and then prove a tighter bound whose proof is considerably more involved.

2.1 A simpler and weaker bound

For each pair a, b of points of P, any triangle abc in $\mathcal{S}(\Delta_0)$, with $c \in P \setminus \{a, b\}$, has the property that c lies on a circle $\gamma_{a,b}$, which is orthogonal to ab and whose center lies at a fixed point on ab. Moreover, given a circle γ, there exist at most two (unordered) pairs a, b, such that $\gamma = \gamma_{a,b}$ (if there are two pairs, one is the reflection of the other through the center of γ). Hence, ignoring the multiplicity factor 2, $|\mathcal{S}(\Delta_0)|$ is at most the number of incidences between the points of P and the (at most) $\binom{n}{2}$ distinct circles $\gamma_{a,b}$.

As shown by Aronov et al. [6] (see also Agarwal et al. [3], and Marcus and Tardos [17]), the number of incidences between n points and c distinct circles in $\mathbb{R}^3$ (or, for that matter, in any dimension) is

$$O(n^{2/3}c^{2/3} + n^{6/11}c^{9/11}\log^{2/11}(n^3/c) + n + c). \tag{1}$$

Substituting $c = O(n^2)$, the second term dominates, and we obtain

$$f_{2,3}(n) = O(n^{24/11}\log^{2/11} n) = O^*(n^{24/11}) = O(n^{2.182}).$$

We remark that a similar approach was taken by Akutsu et al. [5], except that they used a weaker bound on point-circle incidences (albeit the best known at that time). Finally, taking Δ_0 to be an isosceles right triangle and P to be the set of vertices of a 2-dimensional grid, it is easy to verify that $f_{2,2}(n) = \Theta(n^2)$, which implies that $f_{2,d}(n) = \Omega(n^2)$ for any $2 \le d \le 5$ (as already noted, $f_{2,d}(n) = \Theta(n^3)$ for $d \ge 6$).

Remark: A consequence of (1) is the following useful variant, which will be used throughout the paper: The number of circles that contain at least k points of P is $O^*(n^3/k^{11/2}) + O(n^2/k^3 + n/k)$, and the number of incidences between the points of P and these circles is $O^*(n^3/k^{9/2}) + O(n^2/k^2 + n)$.

2.2 An improved bound

To simplify the presentation, let us assume[2] that Δ_0 is not isosceles, so its edges have distinct lengths. We denote by $\mathcal{S}(\Delta_0)$ the set of all triangles similar to Δ_0 and spanned by P. For each such triangle Δabc, we can order its vertices in a unique order, say a, b, c, so that a is incident to the two longest edges of the triangle, and b is the other endpoint of

[2]The proof is unaffected by this assumption, but the presentation is simplified.

the longest edge. We call a the *main vertex* of the triangle. Denote the sphere centered at a and containing b by $\sigma = \sigma_{a,b}$, and the sphere centered at a and containing c by $\tau = \tau_{a,c}$. Let $\gamma = \gamma_{a,b}$ be the circle that lies on τ and contains all the points c' such that $\Delta abc' \sim \Delta_0$. Clearly, $c \in \gamma$ (γ is the same circle constructed in the preceding proof).

Let Σ (resp., T) denote the set of all spheres $\sigma_{a,b}$ (resp., $\tau_{a,c}$) obtained from triangles $\Delta abc \in \mathcal{S}(\Delta_0)$. Define a relation Π between Σ and T, which contains all pairs $(\sigma_{a,b}, \tau_{a,c})$ of spheres for which $\Delta abc \in \mathcal{S}(\Delta_0)$. By construction, each sphere σ is associated with a unique sphere $\tau = \Pi(\sigma)$. We denote by Γ the set of all circles $\gamma_{a,b}$ (note that the pair (a, b) uniquely determines $\gamma_{a,b}$). In addition, we use the notation Σ_a, T_a and Π_a to denote, respectively, the subsets of Σ, T, and Π that are generated by triangles with a as the main vertex. For an integer $i \le n$, let $\Pi_{\le i} \subseteq \Pi$ denote the set of pairs of spheres (σ, τ) such that either σ or τ contains at most i points of P, and let $\Pi_{>i} = \Pi \setminus \Pi_{\le i}$ denote those pairs in which each of the spheres contains more than i points. Define

$$\Sigma_{\le i} = \{\sigma \in \Sigma \mid (\sigma,\ \Pi(\sigma)) \in \Pi_{\le i}\}$$

and $\Sigma_{>i} = \Sigma \setminus \Sigma_{\le i}$. Similarly define $T_{\le i}$ and $T_{>i}$.

The discussion above implies that each triangle $\Delta abc \in \mathcal{S}(\Delta_0)$ corresponds to an incidence between the point c and the circle $\gamma_{a,b}$. As already noted, the same circle $\gamma_{a,b}$ can arise for at most two pairs (a, b). Hence we have

$$\frac{1}{2}|\mathcal{S}(\Delta_0)| \le I(P, \Gamma) \le |\mathcal{S}(\Delta_0)|,$$

where $I(P, \Gamma)$ is the number of incidences between the points of P and the circles of Γ. It therefore suffices to bound $I(P, \Gamma)$.

Fix a threshold parameter k. We call a pair in $\Pi_{\le k}$ *light* and a pair in $\Pi_{>k}$ *heavy*. We claissfy the heavy pairs into non-degenerate and degenerate pairs, as follows: A heavy sphere σ (i.e., a sphere containing more than k points) is *degenerate* if there exists a circle $\gamma \subset \sigma$ (not necessarily from the family Γ) such that $|\gamma \cap P| \ge \beta|\sigma \cap P|$, for some sufficiently small constant $\beta > 0$; otherwise it is *non-degenerate*. A pair in $\Pi_{>k}$ is called *non-degenerate* if both the spheres in the pair are non-degenerate, and *degenerate* otherwise.

We bound separately the number of incidences between the points of P and the circles determined by each of these three types of pairs. Let I_L (resp., I_N, I_D) denote the number of incidences between P and the circles induced by light (resp., non-degenerate, degenerate) pairs. Then $I(P, \Gamma) = I_L + I_N + I_D$.

Handling light pairs. Let $a \in P$ and put

$$\Pi'_a = \Pi_a \cap \Pi_{\le k}, \qquad \Sigma'_a = \Sigma_a \cap \Sigma_{\le k}, \quad \text{and} \quad T'_a = T_a \cap T_{\le k}.$$

For a sphere pair $(\sigma, \tau) \in \Pi'_a$, put

$$P_\sigma = P \cap \sigma, \qquad P_\tau = P \cap \tau, \quad \text{and} \quad \Gamma_\tau = \{\gamma_{a,b} \mid b \in P_\sigma\}.$$

Recall that either $|\Gamma_\tau| = |P_\sigma| \le k$, or $|P_\tau| \le k$. All the circles of Γ_τ lie on τ and have the same radius. Hence, as follows, e.g., from [20], the number of similar triangles associated with the pair (σ, τ) is bounded by

$$I(P_\tau, \Gamma_\tau) \;=\; O\big((|P_\sigma||P_\tau|)^{2/3} + |P_\sigma| + |P_\tau|\big).$$

Summing over all $(\sigma, \tau) \in \Pi'_a$, we get

$$\begin{aligned} I'_a &:= \sum_{(\sigma,\tau)\in\Pi'_a} I(P_\tau, \Gamma_\tau) \\ &= O\left(\sum_{(\sigma,\tau)\in\Pi'_a} (|P_\sigma||P_\tau|)^{2/3}\right) \\ &+ O\left(\sum_{(\sigma,\tau)\in\Pi'_a} |P_\sigma|\right) \\ &+ O\left(\sum_{(\sigma,\tau)\in\Pi'_a} |P_\tau|\right). \end{aligned}$$

The last two sums are clearly $O(n)$. As for the first sum, in each term one of $|P_\sigma|$, $|P_\tau|$ is at most k. We may assume, without loss of generality, that $|P_\tau| \le k$, and then obtain

$$\begin{aligned} \sum_{(\sigma,\tau)\in\Pi'_a} (|P_\sigma||P_\tau|)^{2/3} &\le k^{1/3} \sum |P_\sigma|^{2/3}|P_\tau|^{1/3} \\ &\le k^{1/3} \left(\sum |P_\sigma|\right)^{2/3} \left(\sum |P_\tau|\right)^{1/3} \\ &= O(nk^{1/3}). \end{aligned}$$

Altogether, summing over all points $a \in P$, the number of point-circle incidences with circles generated by light pairs is

$$I_L = \sum_{a\in P} I'_a = O(n^2 k^{1/3}). \tag{2}$$

Handling non-degenerate heavy pairs. Let $\Pi_N \subseteq \Pi_{>k}$ denote the set of non-degenerate heavy pairs. We bound the number of incidences between the points of P and the circles generated by Π_N. Note that both spheres in each pair of Π_D contain more than k points of P. Set $\Sigma_N = \{\sigma \mid (\sigma, \Pi(\sigma)) \in \Pi_N\}$. Note that $|\Pi_N| = |\Sigma_N|$. We first obtain an upper bound on $|\Pi_N|$.

Lemma 2.1 $|\Pi_N| = O\left(\frac{n^4}{k^5} + \frac{n^3}{k^3}\right)$.

Proof: We apply to P and Σ_N the standard lifting transform to 4-space [13], so that each point $(x, y, z) \in P$ is mapped to $(x, y, z, x^2 + y^2 + z^2)$, and each sphere of Σ_N, centered at (x_0, y_0, z_0) and having radius r, is mapped to the hyperplane $w = 2x_0x + 2y_0y + 2z_0z + \left(r^2 - x_0^2 - y_0^2 - z_0^2\right)$. The lifting preserves the incidence relation, and has the additional property that cocircular points of P (on a circle that lies on one of the lifted spheres σ) are lifted to coplanar points in 4-space lying on a sub-2-plane of the hyperplane image of σ. Thus, a non-degenerate sphere is lifted to what Elekes and Tóth call a *β-degenerate* hyperplane [14] (with respect to the lifted image P^+ of the set P). We can therefore apply the result of [14], which asserts that the number of β-degenerate hyperplanes that contain at least k points of P^+, and hence, the number of spheres in Σ_N (and the number of pairs in Π_N), is bounded by $O(n^4/k^5 + n^3/k^3)$, provided that the constant β is chosen sufficiently small. This completes the proof of the lemma. $\Box$

We next partition Π_N into $O(\log n)$ classes, $\Pi_N^{(1)}, \Pi_N^{(2)}, \ldots$, where $\Pi_N^{(i)}$ consists of those pairs $(\sigma, \tau) \in \Pi_N$ such that

$$2^{i-1}k < \max\{|\sigma \cap P|, |\tau \cap P|\} \le 2^i k,$$

for $i = 1, \ldots, \log_2(n/k)$. We sum the incidence bounds within each class separately. Fix a class $\Pi_N^{(i)}$, and put $k_i = 2^i k$. Each pair $(\sigma, \tau) \in \Pi_N^{(i)}$, σ induces on τ a set of at most k_i congruent circles, and the number of points on τ is also at most k_i. Hence the number of incidences between these points and circles is $O(k_i^{4/3})$. By Lemma 2.1, the number of such pairs of spheres is $O(n^4/k_i^5 + n^3/k_i^3)$. Hence, summing over i, the overall number of incidences involving spheres in Σ_N is

$$I_N = O\left(\sum_i k_i^{4/3} \cdot \left(\frac{n^4}{k_i^5} + \frac{n^3}{k_i^3}\right)\right) = O\left(\frac{n^4}{k^{11/3}} + \frac{n^3}{k^{5/3}}\right). \tag{3}$$

Handling degenerate heavy pairs. Let $\Pi_D = \Pi_{>k} \setminus \Pi_N$ be the set of degenerate pairs. We apply the following pruning process on each pair $(\sigma, \tau) \in \Pi_D$. Suppose τ is a degenerate sphere containing more than k points of P (the case where σ is the heavy sphere is handled in an essentially symmetric manner). Then there is a circle $\gamma_1 \subset \tau$ containing at least $\beta|P_\tau|$ points of $P_\tau = P \cap \tau$. If we remove the points of $P \cap \gamma_1$, then one of the following may happen:

1. τ is incident to at most k of the remaining points of $P \setminus \gamma_1$.

2. τ becomes non-degenerate with respect to the remaining points.

3. τ is still degenerate and contains more than k points.

In the third case, we continue with the pruning process, until one of the first two events occurs, which will happen after at most $\log_{1/\beta} n$ iterations. At the end of the process, if τ contains k or fewer points, we include it, together with its remaining incidences, and with the sphere σ, as one of the pairs of $\Pi_{\le k}$, meaning that we only consider the remaining points when bounding the number of point-circle incidences on τ. Otherwise, τ is non-degenerate with respect to the set of at least k remaining points. In other words, when considering only the surviving points on τ (or the surviving circles, if σ was the heavy sphere), at the end of the pruning process, either (σ, τ) becomes a light pair, or (σ, τ) becomes a non-degenerate pair. Lemma 2.1 still holds for these latter pairs, because it relies on the bound of [14], and this bound still applies, as is easily verified.

We still have to count the incidences involving the removed points and/or circles on each of these spheres τ. Such a (yet uncounted) incidence $p \in \gamma$, on a degenerate sphere τ, is uncounted either because (a) γ was one of the circles whose points were removed from P_τ, or (b) γ was not such a circle, but p was removed because it lies on another circle $\gamma' \subset \tau$ that has been removed.

For an incidence of type (a), we have $\gamma \in \Gamma_\tau$. There can be at most two spheres τ for which this situation arises. Thus, the number of uncounted incidences of type (a) is at most twice the number of incidences between the points of P and the removed circles. Using the fact that each of these circles contains at least βk points, the results of [3, 7, 17] imply, as noted above, that the number of these incidences is $O^*(n^3/k^{9/2}) + O(n^2/k^2 + n)$.

To bound the number of incidences of type (b), we observe that, for each pair $(\sigma, \tau) \in \Pi_D$, the number of removed circles on τ is $O(\log n)$, and each $\gamma \in \Gamma_\tau$ can lose at most two incidences for each removed circle. Thus, the number of such incidences on τ is $O(|\Gamma_\tau| \log n)$. Summing over all degenerate spheres, the overall number of type (b) incidences is $O(n^2 \log n)$.

A similar analysis applies to the case where the degenerate sphere in the pair (σ, τ) is σ rather than τ. Here each pruning step removes circles from Γ_τ, whose centers lie on a common circle. It is easily checked that, for any point $p \in \tau$, at most two such circles can pass through p, so each pruning step loses at most $2|P_\tau|$ incidences, for a total of $O(|P_\tau| \log n)$ incidences. Summing over all spheres τ, we obtain the same bound $O(n^2 \log n)$ as above. Hence,

$$I_D = O^*\left(\frac{n^3}{k^{9/2}}\right) + O\left(\frac{n^2}{k^2} + n\right) + O(n^2 \log n) \quad (4)$$

$$= O^*\left(\frac{n^3}{k^{9/2}}\right) + O(n^2 \log n). \quad (5)$$

Adding (2)–(5), we get

$$I(P, \Gamma) \leq I_L + I_N + I_D = O\left(n^2 k^{1/3} + \frac{n^4}{k^{11/3}} + \frac{n^3}{k^{5/3}} + n^2 \log n\right),$$

for any k. By choosing $k = n^{1/2}$, we obtain the main result of this section:

Theorem 2.2 *Let P be a set of n points in $\mathbb{R}^3$, and let Δ_0 be some fixed triangle. Then the number of triangles similar to Δ_0 spanned by P is $O(n^{13/6}) = O(n^{2.167})$.*

3 Incidences Between Points and Spheres

Let P be a set of n distinct points in $\mathbb{R}^d$, and let S be a set of s distinct k-spheres in $\mathbb{R}^d$, for some fixed $1 \leq k \leq d-1$. Let $I(P, S)$ denote the number of incidences between the points of P and the spheres of S, and let $I_{k,d}(n, s)$ denote the maximum value of $I(P, S)$, for P, S as above. Clearly, for $k > 1$, we have $I_{k,d}(n, s) = sn$, since we can put all the points of P on some circle, and make all the spheres in S pass through that circle. Our goal is to derive a different bound, which, in the contexts that arise in this paper, leads to improved bounds on the number of similar simplices.

We first note that the case $k = 1$ (i.e., S is a set of circles) has been solved by Aronov et al. [6], who have shown that

$$I_{1,d}(n, s) = O^*(n^{2/3}s^{2/3} + n^{6/11}s^{9/11} + n + s),$$

for any $d \geq 2$. It therefore suffices to consider the case $k \geq 2$.

We fix some threshold parameter t, and classify each sphere in S as being either *t-light* (or simply *light*) if it contains at most t points of P, and *t-heavy* (or just *heavy*) otherwise. Clearly, the number of incidences with the light spheres is at most st.

Heavy spheres are further classified as follows. We say that a k-sphere σ is *j-degenerate*, for $1 \le j \le k$, if (i) there exists a j-sphere $\sigma' \subseteq \sigma$ which contains at least a β-fraction of the points of $P \cap \sigma$; i.e., $|P \cap \sigma'| \ge \beta |P \cap \sigma|$, where β is some sufficiently small constant; and (ii) j is the smallest integer with this property. If $j = k$, we also refer to σ as being *non-degenerate*.

By the result of Elekes and Tóth [14], the number of non-degenerate heavy spheres in S is

$$O\left(\frac{n^{k+2}}{t^{k+3}} + \frac{n^{k+1}}{t^{k+1}}\right),$$

and the number of incidences between these spheres and the points of P is

$$O\left(\frac{n^{k+2}}{t^{k+2}} + \frac{n^{k+1}}{t^{k}}\right).$$

A similar analysis holds for the j-degenerate spheres in S, for $j < k$. We replace each such sphere σ by a j-sphere $\sigma' \subset \sigma$ that contains a β-fraction of the points of $P \cap \sigma$, and reduce the problem to that of bounding the number of incidences between the points of P and the resulting j-spheres. Here however we face the problem that the j-spheres may appear with multiplicity.

4 Anchored Congruent Simplices

In this section we consider the following auxiliary problem, which arises in the analysis of similar simplices. Let P be a set of n points in $\mathbb{R}^d$, let Δ_1 be a fixed k-simplex, for $k \le d$, and let Δ_0 be a fixed $(k-1)$-subsimplex of Δ_1. Let $h(P, \Delta_1)$ denote the number of k-simplices Δ that satisfy the following properties: (i) Δ is spanned by P. (ii) Δ is congruent to Δ_0. (iii) $CH(\Delta \cup \{o\})$ is congruent to Δ_1, where o is the origin. We refer to such a simplex Δ as (a congruent copy of Δ_0) *anchored* at the origin. We denote by $h_{k,d}(n)$ the maximum value of $h(P, \Delta_1)$, over all sets P and simplices $\Delta_1 \supset \Delta_0$, as above.

For a fixed d, we derive upper bounds on the quantities $h_{k,d}(n)$, for $k = 1, 2, \ldots, d$, using induction on k. We start with the trivial bound $h_{1,d}(n) = O(n)$.

Let P and $\Delta_1 \supset \Delta_0$ be as above, with $\dim(\Delta_1) = k$. Let $a_0, a_1, \ldots, a_k$ denote the vertices of Δ_1, so that $a_1, \ldots, a_k$ are the vertices of Δ_0. Put $r_i = |a_0 a_i|$, for $i = 1, \ldots, k$.

Let $\Delta = p_1 p_2 \cdots p_k$ be an anchored congruent copy of Δ_0 spanned by P, so that $|op_i| = r_i$, for $i = 1, \ldots, k$. Then, for each i, the point p_i lies on a sphere σ_i of radius r_i centered at o. Fix one of these spheres, say σ_k, and fix the points $p_1, \ldots, p_{k-1} \in P$. Let $\gamma(p_1, \ldots, p_{k-1})$ denote the locus of all points $q \in \sigma_k$ which form with $p_1, \ldots, p_{k-1}$ an anchored congruent copy of Δ_0. We claim that $\gamma(p_1, \ldots, p_{k-1})$ is a $(d-k)$-dimensional sphere within σ_1. Indeed, it is the intersection of k $(d-1)$-spheres, one of which is σ_1, and the others are centered at the points p_j, $j < k$. Hence, $\gamma(p_1, \ldots, p_{k-1})$ is a sphere of dimension at least $d - k$, and it cannot have a larger dimension because the centers $o, p_1, \ldots, p_{k-1}$ are affinely independent.

It follows that the number of anchored congruent copies of Δ_0 is equal to the number of incidences between the points of P on σ_k and the $(d-k)$-spheres $\gamma(p_1, \ldots, p_{k-1})$, each counted with multiplicity. Let $\gamma = \gamma(p_1, \ldots, p_{k-1})$ be a fixed such sphere. Notice that the points $p_1, \ldots, p_{k-1}$ must all lie in the $(k-1)$-flat f that is orthogonal to γ and passes through its center o', and they all lie at fixed distances from o', and also at fixed distances from o (which also belongs to f). A somewhat naive bound on the multiplicity of γ is $h_{k-2,k-1}(n)$, but this bound is weak, because (i) the number of points of P in f is likely to be much smaller than n, and (ii) the fact that $p_1 p_2 \cdots p_{k-1}$ is "doubly anchored", at both o and o', puts additional constraints on the possible locations of the points $p_1, \ldots, p_{k-1}$.

Let us first explore the second property. Each point p_j, for $j = 1, \ldots, k-1$, lies on a $(k-3)$-sphere σ'_j, which is the intersection of the two respective $(k-2)$-spheres centered at o and o' and passing through p_j. Fix the first $k-3$ points $p_1, \ldots, p_{k-3}$, and consider the locus $\gamma' = \gamma'(p_1, \ldots, p_{k-3})$ of all points on σ'_{k-1} that lie at distance $|a_j a_{k-1}|$ from p_j, for $j = 1, \ldots, k-3$. Clearly, γ' is the intersection of $k-1$ $(k-2)$-spheres in $\mathbb{R}^{k-1}$, so it consists of $O(1)$ points. (MICHA SAYS: WHY??) Hence, we can bound the multiplicity of γ by the number of doubly-anchored congruent copies of a $(k-4)$-simplex spanned by $P \cap f$. ⟵

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triangle. For each pair of points $a, b \in P$, let $\sigma_{a,b}$ denote the 3-sphere orthogonal to ab and containing all the points c for which $abc \sim \Delta_0$. Let Σ be the set of resulting 3-spheres. As above, it is easily checked that no 3-sphere can arise in this way more than twice. Ignoring this constant multiplicity, we face the problem of bounding the number of incidences between a set Σ of $O(n^2)$ spheres and P.

As in earlier sections, we fix a parameter t and a sufficiently small constant $\beta > 0$. We define a 3-sphere to be light, heavy, non-degenerate, or degenerate as in the earlier sections. The number of incidences involving light 3-spheres is $O(n^2 t)$, so we concentrate on the heavy 3-spheres.

Consider first the *non-degenerate* heavy 3-spheres of Σ. In this case, by lifting the points and 3-spheres into $\mathbb{R}^6$, and then projecting them onto some generic 5-space, we can apply, as before, the Elekes-Tóth bound [14], to conclude that the number of such 3-spheres is $O(n^5/t^6 + n^4/t^4)$, and that the number of incidences between these 3-spheres and the points of P is $O(n^5/t^5 + n^4/t^3)$.

Consider next the 3-spheres in Σ that are degenerate, so each of them contains a 2-sphere that contains more than a β-fraction of the points on the 3-sphere. We replace the 3-spheres by the respective 2-spheres, bound the number of incidences with these 2-spheres, counted with the appropriate multiplicity, and lose only a constant factor using this bound. Consider first the case where the 2-spheres themselves are non-degenerate, in the sense that none of them contains a circle that contains more than a β-fraction of the points on the 2-sphere. Since these 2-spheres are (βt)-heavy, the number of distinct such 2-spheres is, as above, $O(n^4/t^5 + n^3/t^3)$, and the number of incidences with them is $O(n^4/t^4 + n^3/t^2)$.

Here however the 2-spheres may appear with multiplicity, but we claim that the maximum multiplicity of a 2-sphere is at most $O(n)$. Indeed, given a 2-sphere σ' with center o', if we fix one point a in the defining pair (a, b) of a 3-sphere $\sigma_{a,b}$ containing σ', the size of the triangle is determined, and the center o of $\sigma_{a,b}$ must then lie at a fixed distance from a, within the 2-plane π orthogonal to σ' and passing through its center o'. Also, o lies on another fixed circle in π centered at o'. These two circles intersect at most twice (assuming

$a \neq o'$, which can always be guaranteed), so at most two points b can form with a a pair (a, b) for which $\sigma_{a,b} \supset \sigma'$. Hence, the number of triangles similar to Δ_0 that fall into this subcase is $O(n^5/t^4 + n^4/t^2)$.

Finally, consider the subcase where the 2-spheres themselves are degenerate, so we replace each of them by a respective $(\beta^2 t)$-heavy circle, and bound the number of incidences between these circles and the points of P. The multiplicity of a circle is, trivially, at most $O(n^2)$. Hence, arguing as above, the number of triangles that arise in this subcase is $O^*(n^5/t^{9/2} + n^4/t^2)$. Hence, the overall number of triangles is $O(n^2 t + n^5/t^4 + n^4/t^2)$. Choosing $t = n^{2/3}$, we thus obtain

Theorem 4.1 *$f_{2,5}(n) = O(n^{8/3})$.*

As already noted, $d = 5$ is the last interesting case for triangles, since, already for the congruent case, $g_{2,6}(n) = \Theta(n^3)$ [4].

5 A General Bound for $f_{k,d}(n)$

We next consider the general case of bounding the maximum number $f_{k,d}(n)$ of mutually similar k-simplices in a set of n points in $\mathbb{R}^d$. For simplicity of presentation, we assume that d is even. The analysis proceeds by induction on k, where the base case is $k = d/2 - 1$, where we use the trivial bound $f_{d/2-1,d}(n) = O(n^{d/2})$. Since $\mathbb{R}^d$ contains $d/2$ mutually orthogonal 2-planes through the origin, it is easy to construct sets of n points (where we place $2n/d$ points in each of these 2-planes, on some circle centered at the origin), with $\Theta(n^{d/2})$ mutually similar $(d/2-1)$-simplices. To simplify the notation, put $M_k = f_{k,d}(n)$, for $k = d/2 - 1, \ldots, d-1$. So our induction base is $M_{d/2-1} = O(n^{d/2})$.

Suppose we already have an upper bound for M_{k-1}, and consider the step of bounding M_k. Let Δ_0 be a fixed k-simplex, let Δ_1 be some fixed ($(k-1)$-dimensional) facet of Δ_0, and let z be the vertex of Δ_0 not incident to Δ_1. Let $\tau = (p_1, \ldots, p_k)$ be a k-tuple of distinct points in P that span a simplex similar to Δ_1. The number of such tuples is at most M_{k-1}.

Any point $q \in P$ that forms with τ a k-simplex similar to Δ_0 must lie on a $(d-k)$-sphere $\sigma = \sigma_\tau$ orthogonal to the $(k-1)$-flat spanned by the points of τ. Moreover, let σ be a given $(d-k)$-sphere. Any k-tuple τ for which $\sigma = \sigma_\tau$ must be such that (i) the $(k-1)$-simplex s spanned by τ is determined up to *congruence*; (ii) s is contained in the $(k-1)$-flat f orthogonal to σ and passing through its center o; and (iii) o coincides with a fixed point rigidly attached to σ (the point corresponding to the foot of the perpendicular to Δ_1 from z in Δ_0).

In other words, given σ, its *multiplicity* (the number of times it arises as σ_τ for an appropriate tuple τ) can be estimated as follows. We have a set Q of $m \le n$ points in $\mathbb{R}^{k-1}$, and we want to bound the number of congruent full-dimensional simplices that are spanned by Q, where all these simplices are obtained from one another by rotation about (say) the origin o. We refer to such simplices as being *anchored* at the origin, and denote the maximum possible number of such mutually congruent anchored k-simplices by $h_{k-1,k-1}(m)$. We will shortly generalize this notation to cases where the dimension of the simplex is smaller than that of the ambient space.

To recap, we have a set S of at most M_{k-1} $(d-k)$-spheres, each appearing with mutiplicity at most $h_{k-1,k-1}(n)$ (but the bound M_{k-1} counts the spheres *with multiplicity*), and we need to bound the number of incidences between the points of P and the spheres in S, counted with the appropriate multiplicities.

To do so, we fix some threshold parameter t, and distinguish between *light* spheres, which contain at most t points of P, and *heavy* spheres, which contain more than t points of P. The number of incidences with light spheres is, obviously, at most $M_{k-1}t$.

Heavy spheres are further classified as follows. We say that a $(d-k)$-sphere σ is $(d-j)$-*degenerate*, for $k \le j \le d-1$, if (i) there exists a $(d-j)$-sphere $\sigma' \subseteq \sigma$ which contains at least a β-fraction of the points of $P \cap \sigma$; i.e., $|P \cap \sigma'| \ge \beta |P \cap \sigma|$, where β is some sufficiently small constant; and (ii) j is the largest integer with this property. If $j = k$, we also refer to σ as being *non-degenerate*.

By the result of Elekes and Tóth [14], the number of non-degenerate heavy spheres in S is

$$O\left(\frac{n^{d-k+2}}{t^{d-k+3}} + \frac{n^{d-k+1}}{t^{d-k+1}}\right),$$

and the number of incidences between these spheres and the points of P is

$$O\left(\frac{n^{d-k+2}}{t^{d-k+2}} + \frac{n^{d-k+1}}{t^{d-k}}\right). \tag{6}$$

Multiplying this by the mutiplicity bound

$$h_{k-1,k-1}(n) = O(n^{k-2}),$$

we get an overall contribution of

$$O\left(\frac{n^{d}}{t^{d-k+2}} + \frac{n^{d-1}}{t^{d-k}}\right). \tag{7}$$

A similar analysis holds for the $(d-j)$-degenerate spheres in S, for $j > k$. We replace each such sphere σ by a $(d-j)$-sphere $\sigma' \subset \sigma$ that contains a β-fraction of the points of $P \cap \sigma$, and reduce the problem to that of bounding the number of incidences between the points of P and the resulting $(d-j)$-spheres. Ignoring for the moment the issue of multiplicity, The Elekes-Tóth bound [14] implies, in complete analogy, that the number of these $(d-j)$-spheres is

$$O\left(\frac{n^{d-j+2}}{t^{d-j+3}} + \frac{n^{d-j+1}}{t^{d-j+1}}\right),$$

and the number of incidences between these spheres and the points of P is

$$O\left(\frac{n^{d-j+2}}{t^{d-j+2}} + \frac{n^{d-j+1}}{t^{d-j}}\right). \tag{8}$$

Here however we face the additional problem that the $(d-j)$-spheres may appear with multiplicity (in addition to the multiplicity of the original $(d-k)$-spheres in S). We tackle this issue as follows.

Let σ' be a $(d-j)$-sphere, and let f denote the $(j-1)$-flat orthogonal to σ' and passing through its center o'. Fix a point $p_1 \in P \cap f$. Let τ be a k-tuple $(p_1, p_2, \ldots, p_k)$ that contains

p_1 and generates σ'. The fixed distance from p_1 to any point on σ' determines the size of the simplex that is similar to Δ_0 and spanned by τ and by any point on σ'. Moreover, the center o of the $(d-k)$-sphere σ generated by τ satisfies the following properties: (i) The distances from o to $p_1, \ldots, p_k$ and to o' are all fixed. (ii) $\angle p_j o o' = \pi/2$, for each $j = 1, \ldots, k$.

Note that, if we regard o' as the origin (in f), and write $x = \vec{o'o}$, then the constraints can be expressed algebraically as

$$\|x - p_j\|^2 = r_j^2 \quad \text{and} \quad (x - p_j) \cdot x = 0,$$

for $j = 1, \ldots, k$, where r_j is the fixed distance from p_j to o. Hence

$$\|x\|^2 = \|p_j\|^2 - r_j^2,$$

for each j, and x is orthogonal to the affine hull of τ.

For our analysis, it suffices to consider the cases $j = k+1$, $j = k+2$. For larger values of j we use the naive bound $O(n^k)$ for the multiplicity, and get an overall bound of

$$\sum_{j=k+3}^{d-1} O\left(\frac{n^{d+k-j+2}}{t^{d-j+2}} + \frac{n^{d+k-j+1}}{t^{d-j}}\right) = \sum_{q=1}^{d-k-3} O\left(\frac{n^{d-q}}{t^{d-k-q}} + \frac{n^{d-q-1}}{t^{d-k-q-2}}\right).$$

Consider the case $j = k+1$ (so the ambient orthogonal space is $\mathbb{R}^k$). Fix the points $p_1, p_2, \ldots, p_{k-2}$, in $O(n^{k-2})$ ways. Each of the two last points p_{k-1}, p_k must lie on a respective circle γ_{k-1}, γ_k, which is the intersection of $k-2$ $(k-1)$-spheres, centered at the points $p_1, \ldots, p_{k-2}$ and having radii equal to the corresponding appropriately scaled edges of Δ_0, and an appropriate $(k-1)$-sphere centered at o'. Assuming that $o' \neq o$, these spheres do indeed intersect in a circle, because their centers are affinely independent. If $o' = o$ then there is a unique choice for σ, and the analysis proceeds as in the preceding case. Note that, once p_{k-1} has been determined, p_k must lie at an intersection of γ_k and another $(k-1)$-sphere centered at p_{k-1}, and again one can argue that there are at most two such intersection points, using the affine independence of $p_1, \ldots, p_{k-1}, o'$.

Hence, the multiplicity of σ' is at most the number of incidences between $O(n^{k-2})$ circles and n points in $\mathbb{R}^k$. Unfortunately, these circles may also come with multiplicity. For each such circle γ, the points $p_1, \ldots, p_{k-2}$ and o' must define a $(k-2)$-simplex which is congruent to a fixed simplex, lies in the $(k-2)$-flat orthogonal to γ and passing through its center o_γ, and is anchored at o_γ.

(MICHA SAYS: This is the doubly anchored situation.) ⟵

A naive bound on the number of such simplices is $O(n^{k-3})$, which we use as an upper bound on the multiplicity of γ.

Now fix a threshold parameter s, and distinguish between light circles, containing fewer than s points of P, and the other, heavy circles. The number of incidences with light circles is at most $O(n^{k-2}s)$. By the result of [6], the number of distinct heavy circles is at most $O^*(n^3/s^{11/2} + n^2/s^3 + n/s)$, and the number of incidences of these circles with the points of P is at most $O^*(n^3/s^{9/2} + n^2/s^2 + n)$. Multiplying by the multiplicity bound, the multiplicity of the $(d-j)$-sphere σ' is at most

$$O^*\left(n^{k-2}s + \frac{n^k}{s^{9/2}} + \frac{n^{k-1}}{s^2} + n^{k-2}\right).$$

Choosing $s = n^{4/11}$, we get a multiplicity bound of $O^*(n^{k-2+4/11})$. Combining this with (8), we get that the contribution of the case $j = k+1$ to the overall bound is

$$O^*\left(\frac{n^{d-7/11}}{t^{d-k+1}} + \frac{n^{d-18/11}}{t^{d-k-1}}\right). \tag{9}$$

The case $j = k+2$ is handled similarly (now the ambient orthogonal space is $\mathbb{R}^{k+1}$), except that we now fix the points $p_1, p_2, \ldots, p_{k-1}$, in $O(n^{k-1})$ ways, and this constrains the last point p_k to lie on a circle γ_k, which is the intersection of $k-1$ k-spheres, centered at the points $p_1, \ldots, p_{k-1}$ and having radii equal to the corresponding appropriately scaled edges of Δ_0, and an appropriate k-sphere centered at o'. Assuming that $o' \neq o$, these spheres do indeed intersect in a circle, because their centers are affinely independent. If $o' = o$ then there is a unique choice for σ, and the analysis proceeds as in the preceding case.

Hence, the multiplicity of σ' is at most the number of incidences between $O(n^{k-1})$ circles and n points in $\mathbb{R}^{k+1}$. Again, these circles may come with multiplicity. For each such circle γ, the points $p_1, \ldots, p_{k-1}$ and o' must define a $(k-1)$-simplex which is congruent to a fixed simplex, lies in the $(k-1)$-flat orthogonal to γ and passing through its center o_γ, and is anchored at o_γ.

(MICHA SAYS: Again, this is the doubly anchored situation.) ⟵

Here too we use the naive bound $O(n^{k-2})$ on the number of such simplices, as an upper bound on the multiplicity of γ.

As above, fixing a threshold parameter s, and distinguishing between light and heavy circles, we get the bound

containing fewer than s points of P, and the other, heavy circles. The number of incidences with light circles is at most $O(n^{k-2}s)$. By the result of [6], the number of distinct heavy circles is at most $O^*(n^3/s^{11/2} + n^2/s^3 + n/s)$, and the number of incidences of these circles with the points of P is at most $O^*(n^3/s^{9/2} + n^2/s^2 + n)$. Multiplying by the multiplicity bound, the multiplicity of the $(d-j)$-sphere σ' is at most

$$O^*\left(n^{k-1}s + \frac{n^{k+1}}{s^{9/2}} + \frac{n^k}{s^2} + n^{k-1}\right).$$

Choosing $s = n^{4/11}$, we get a multiplicity bound of $O^*(n^{k-1+4/11})$. Combining this with (8), we get that the contribution of the case $j = k+2$ to the overall bound is

$$O^*\left(\frac{n^{d-7/11}}{t^{d-k}} + \frac{n^{d-18/11}}{t^{d-k-2}}\right). \tag{10}$$

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An easy bound on this quantity is $O(n^{d-4})$. Indeed, if we fix $d-4$ of the points that span the simplex, then, together with o, they span a $(d-4)$-simplex s' (in $\mathbb{R}^{d-3}$), and each of the remaining two vertices can then be placed in at most two locations, depending on the orientation of the full simplex with respect to s', from which the bound follows.

To recap, we have a set Σ of $O(n^{d-2})$ 2-spheres, each occurring with multiplicity at most $O(n^{d-4})$, and we need to bound the number of incidences between Σ and P. We follow an approach similar to Section 2.2. We fix a parameter $t > 0$ and a sufficiently small constant $\beta > 0$ (that depends on d). As in Section 2.2, we call a 2-sphere *light* (resp., *heavy*) if it contains at most (resp., more than) t points of P. A heavy 2-sphere σ is *degenerate* if more than a β-fraction of the points of $P \cap \sigma$ are co-circular, and *non-degenerate* otherwise.

The number of incidences with the light 2-spheres is, trivially, at most $O(n^{d-2}t)$. We therefore focus on the heavy 2-spheres.

Incidences on non-degenerate heavy 2-spheres. Let M be the number of *distinct* non-degenerate 2-spheres in Σ. We can bound M by lifting the 2-spheres to $\mathbb{R}^{d+1}$, as in the proof of Theorem 2.2. We obtain a collection of M 3-flats, each containing at least t points of the lifted set P^+, and none of them contains a 2-plane with more than a β-fraction of its points (for the same reason as in the preceding proof). Project the lifted collection of points and 3-flats onto some generic 4-space E, and apply the Elekes-Tóth bound [14], to conclude that the number M of the projected 3-flats (hyperplanes) is $M = O(n^4/t^5 + n^3/t^3)$. Moreover, the number of incidences between the original 2-spheres and the points of P is at most $O(n^4/t^4 + n^3/t^2)$. Multiplying this bound by the maximum multiplicity $O(n^{d-4})$ of a 2-sphere, we obtain the bound $O(n^d/t^4 + n^{d-1}/t^2)$.

Incidences on degenerate heavy 2-spheres. We can replace each degenerate 2-sphere $\sigma \in \Sigma$ by a circle γ that contains more than a β-fraction of the points on σ, so it contains at least βt points. It then suffices to bound the number of incidences between these circles, *counted with multiplicity*, and the points of P, as $I(P, \{\gamma\}) \geq \beta I(P, \{\sigma\})$.

Let us first fix such a circle γ, and bound the multiplicity of γ, namely the number of $(d-2)$-tuples τ such that σ_τ contains γ. Any such tuple τ, together with the center o of γ, spans the $(d-2)$-dimensional flat h orthogonal to γ and passing through o. The number of such tuples is at most $O(n^{d-3})$. Indeed, if we fix a sub-tuple τ' of $d-3$ points of τ, then the size of the simplex is determined (because the distance from any of the fixed points to any point on γ is now fixed). Let o' denote the center of σ_τ. Then o' has to lie on a fixed $(d-3)$-sphere in h centered at o. Moreover, since o' is rigidly attached to the simplex, it follows that it must lie on a circle orthogonal to the $(d-4)$-flat spanned by τ', and this circle intersects the sphere in at most two points,[3] each of which corresponds to a unique τ.

Let C_t (resp., $C_{\geq t}$) denote the number of circles that are spanned by P and contain exactly (resp., at least) t points of P. As remarked above, it follows from the analysis of [6] that

$$C_{\geq \beta t} = O^*\left(\frac{n^3}{t^{11/2}} + \frac{n^2}{t^3} + \frac{n}{t}\right),$$

and that the number of incidences with these circles is $O^*(n^3/t^{9/2} + n^2/t^2 + n)$. The preceding argument implies that the number of $(d-2)$-simplices that correspond to heavy degenerate 2-spheres is at most

$$O(n^{d-3}) \cdot O^*\left(\frac{n^3}{t^{9/2}} + \frac{n^2}{t^2} + n\right) = O^*\left(\frac{n^d}{t^{9/2}} + \frac{n^{d-1}}{t^2} + n^{d-2}\right).$$

Hence the overall number of simplices is $O(n^{d-2}t + n^d/t^4 + n^{d-1}/t^2)$. By choosing $t = n^{2/5}$, we obtain $I(P, \Sigma) = O(n^{d-8/5})$, which also bounds the number of simplices similar to Δ_0 spanned by the points of P. Hence, we obtain the following.

[3]It is easily checked that one can always choose τ' so that the circle is not *contained* in the 2-sphere; if this were impossible, τ would not have spanned a $(d-3)$-space.

Theorem 5.1 $f_{d-2,d}(n) = O(n^{d-8/5})$.

A bound on $f_{d-1,d}(n)$**.** We can now obtain the main result of this section (and of the paper). Let Δ_0 be a fixed $(d-1)$-simplex, let Δ_1 be some fixed ($(d-2)$-dimensional) facet of Δ_0, and let z be the vertex of Δ_0 not incident to Δ_1.

Let $\tau = (p_1, \ldots, p_{d-1})$ be a $(d-1)$-tuple of distinct points in P that span a simplex similar to Δ_1. The number of such tuples is $M \le f_{d-2,d}(n) = O(n^{d-8/5})$.

Any point $q \in P$ that forms with τ a $(d-1)$-simplex similar to Δ_0 must lie of a circle $\gamma = \gamma_\tau$ orthogonal to the $(d-2)$-flat spanned by the points of τ. Moreover, let γ be a given circle. Any tuple τ for which $\gamma = \gamma_\tau$ must be such that (i) the $(d-2)$-simplex s spanned by τ is determined up to *congruence*; (ii) s is contained in the $(d-2)$-flat orthogonal to γ and passing through its center o; and (iii) o is rigidly attached to s (it is the point corresponding to the foot of the perpendicular to Δ_1 from z in Δ_0), and all possible simplices s are obtained from one another by rotation about o.

In other words, given γ, to estimate its *multiplicity*, we need to bound the number of congruent full-dimensional simplices that are spanned by a set Q of $m \le n$ points in $\mathbb{R}^{d-2}$, where the simplex is unique up to rotation about (say) the origin o. Arguing as above, this number is at most $O(m^{d-3})$, because any tuple of $d-3$ points of Q, together with o, determines a $(d-3)$-simplex (in $\mathbb{R}^{d-2}$), which can be completed into a full-dimensional one in only two ways.

We thus proceed as before, fixing a threshold parameter t, and distinguishing between *light* circles (those with at most t points), and the remaining *heavy* circles. The number of incidences with light circles is at most Mt, and the number of incidences with heavy circles is

$$O(n^{d-3}) \cdot O^*\left(\frac{n^3}{t^{9/2}} + \frac{n^2}{t^2} + n\right) = O^*\left(\frac{n^d}{t^{9/2}} + \frac{n^{d-1}}{t^2} + n^{d-2}\right).$$

By choosing $t = (n^d/M)^{2/11}$, we obtaining the main result:

Theorem 5.2 $f_{d-1,d}(n) = O^*(n^{d-72/55})$.

6 Similar Triangles in $\mathbb{R}^5$

Let P be a set of n points in $\mathbb{R}^5$, and let Δ_0 be a triangle. For each pair of points $a, b \in P$, let $\sigma_{a,b}$ denote the 3-sphere orthogonal to ab and containing all the points c for which $abc \sim \Delta_0$. Let Σ be the set of resulting 3-spheres. As above, it is easily checked that no 3-sphere can arise in this way more than twice. Ignoring this constant multiplicity, we face the problem of bounding the number of incidences between a set Σ of $O(n^2)$ spheres and P.

As in earlier sections, we fix a parameter t and a sufficiently small constant $\beta > 0$. We define a 3-sphere to be light, heavy, non-degenerate, or degenerate as in the earlier sections. The number of incidences involving light 3-spheres is $O(n^2 t)$, so we concentrate on the heavy 3-spheres.

Consider first the *non-degenerate* heavy 3-spheres of Σ. In this case, by lifting the points and 3-spheres into $\mathbb{R}^6$, and then projecting them onto some generic 5-space, we can apply, as before, the Elekes-Tóth bound [14], to conclude that the number of such 3-spheres is $O(n^5/t^6 + n^4/t^4)$, and that the number of incidences between these 3-spheres and the points of P is $O(n^5/t^5 + n^4/t^3)$.

Consider next the 3-spheres in Σ that are degenerate, so each of them contains a 2-sphere that contains more than a β-fraction of the points on the 3-sphere. We replace the 3-spheres by the respective 2-spheres, bound the number of incidences with these 2-spheres, counted with the appropriate multiplicity, and lose only a constant factor using this bound. Consider first the case where the 2-spheres themselves are non-degenerate, in the sense that none of them contains a circle that contains more than a β-fraction of the points on the 2-sphere. Since these 2-spheres are (βt)-heavy, the number of distinct such 2-spheres is, as above, $O(n^4/t^5 + n^3/t^3)$, and the number of incidences with them is $O(n^4/t^4 + n^3/t^2)$.

Here however the 2-spheres may appear with multiplicity, but we claim that the maximum multiplicity of a 2-sphere is at most $O(n)$. Indeed, given a 2-sphere σ' with center o', if we fix one point a in the defining pair (a, b) of a 3-sphere $\sigma_{a,b}$ containing σ', the size of the triangle is determined, and the center o of $\sigma_{a,b}$ must then lie at a fixed distance from a, within the 2-plane π orthogonal to σ' and passing through its center o'. Also, o lies on another fixed circle in π centered at o'. These two circles intersect at most twice (assuming $a \neq o'$, which can always be guaranteed), so at most two points b can form with a a pair (a, b) for which $\sigma_{a,b} \supset \sigma'$. Hence, the number of triangles similar to Δ_0 that fall into this subcase is $O(n^5/t^4 + n^4/t^2)$.

Finally, consider the subcase where the 2-spheres themselves are degenerate, so we replace each of them by a respective $(\beta^2 t)$-heavy circle, and bound the number of incidences between these circles and the points of P. The multiplicity of a circle is, trivially, at most $O(n^2)$. Hence, arguing as above, the number of triangles that arise in this subcase is $O^*(n^5/t^{9/2} + n^4/t^2)$. Hence, the overall number of triangles is $O(n^2 t + n^5/t^4 + n^4/t^2)$. Choosing $t = n^{2/3}$, we thus obtain

Theorem 6.1 *$f_{2,5}(n) = O(n^{8/3})$.*

As already noted, $d = 5$ is the last interesting case for triangles, since, already for the congruent case, $g_{2,6}(n) = \Theta(n^3)$ [4].

7 Discussion

Examining the proof of the general bound in Section 2, we note that there are three sources for potential improvements. First, the proof starts with the naive estimate $f_{d-3,d}(n) = O(n^{d-2})$; one should be able to get a better, nontrivial bound. Indeed, the previous section shows that this is the case of $d = 5$. Two other possibilities for improvements are in the estimation of the multiplicities of the circles and 2-spheres (and 3-spheres for $f_{2,5}(n)$) that arise in the analysis. We look at the flat h orthogonal to the circle, 2-sphere, or 3-sphere, and make the worst case assumption that $|h \cap P| = n$. With a more careful analysis (e.g., using the Elekes-Tóth bound), we expect to be able to improve this considerably. Also, when counting multiplicities, we are probably not exploiting all the restriction on the possible positions of the congruent sub-simplex. We are currently exploring these possible improvements, and expect to include them in the full version.

Another open problem is to improve the bound $f_{2,3}(n) = O(n^{13/6})$. A potential source for such an improvement is the fact that, when we lift P into $\mathbb{R}^4$, the resulting set P^+ lies on the convex 3-dimensional paraboloid, and the hope is that the Elekes-Tóth bound could be improved for such point sets.

Another observation is that we can relate $f_{k,d}(n)$ to $g_{k-2,d}(n)$ (the maximum number of $(k-2)$-simplices *congruent* to a given simplex), as follows. Let Δ_0 be a given k-simplex. Fix a pair a, b of points in P. If we use a, b as two (fixed) vertices of a k-simplex similar to Δ_0, then the size of that simplex is fixed, so the number of such simplices is at most $g_{k-2,d}(n)$, implying that

$$f_{k,d}(n) = O(n^2 g_{k-2,d}(n)).$$

(In fact, the bound is probably smaller, because all the possible $(k-2)$-simplices that go with a fixed edge ab are "anchored" about ab, so their number should be smaller.) Recall the Erdős-Purdy conjecture that $g_{k-2,d}(n) = O(n^{d/2})$ (for even d). If the conjecture were true then we would have $f_{k,d}(n) = O(n^{2+d/2})$, which, for large values of d, is significantly smaller than the general bounds derived in this paper.

Another comment to observe is that the proof technique is essentially a careful analysis of incidences between points and spheres of various dimensions. While the case of circles has already been studied fairly intensively, the case of higher dimensional spheres has not received much attention. The bounds that we obtain via the Elekes-Tóth bound seem to be weak. For example, using this technique for estimating the number of k-rich circles would yield a bound of $O(n^3/k^4 + n^2/k^2)$, whereas the bound using (1) is $O(n^3/k^{11/2} + n^2/k^3)$. One would hope that similar improvements could be obtained for incidences with higher-dimensional spheres too. We propose to study this problem in more general settings, and regard the present paper as an initial step in this direction.

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