An Improved Bound for k-Sets in Four Dimensions*

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Abstract

We show that the number of halving sets of a set of n points in $\mathbb{R}^4$ is $O\left(n^{4-1/18}\right)$, improving the previous bound of [9] with a simpler (albeit similar) proof.

1 Introduction

Let S be a finite set of $n \geq d+1$ points in $\mathbb{R}^d$ and let $1 \leq k \leq n-1$ be an integer parameter. A *k-set* of S is a k-element subset of S that can be strictly separated from its complement by a hyperplane. The *k-set problem* asks for sharp bounds on the maximum number $F_k^{(d)}(n)$ of k-sets of any set of n points in $\mathbb{R}^d$. The dimension d is usually considered to be a constant, while k and n are arbitrarily large. It is not hard to see that the number of k-sets is maximized for point sets *in general position*, i.e., such that no $d+1$ points lie in a common hyperplane. In this setting, the following variant of the problem turns out to be essentially equivalent and technically more convenient to study: An oriented $(d-1)$-dimensional simplex σ spanned by d points of S is called a *j-facet* of S, for $0 \leq j \leq n-d$, if there are exactly j points of S in the positive open halfspace determined by σ. We denote the number of j-facets of S by $G_j(S)$ and seek sharp bounds on the maximum $G_j^{(d)}(n)$ of the numbers $G_j(S)$ over all sets S of n points in general position in $\mathbb{R}^d$. In dimension 2, the number of k-sets of S is equal to $G_{k-1}(S)$, and in dimension 3, it is equal to $\frac{1}{2}(G_{k-2}(S)+G_{k-1}(S))+2$; see [3]. In higher dimensions, there are no longer any exact linear relations between these numbers, but for any fixed dimension d, the numbers $F_k^{(d)}(n)$ and $G_k^{(d)}(n)$ lie within constant multiplicative factors of one another (see, e.g., [8]).

A special case arises when $n-d$ is even and $j=(n-d)/2$. Then $G_{(n-d)/2}^{(d)}(n)$ counts the maximum possible number of so-called *halving facets* of S. If we reverse the orientation of a halving facet, we obtain again a halving facet. Thus, we can forget about the orientation and just speak of the underlying unoriented simplices, which are called *halving simplices*. Bounds on the number of halving simplices can be translated to bounds on the number of j-facets for any j [2], so it is sufficient to study the former quantity.

The study of k-sets and j-facets began almost 40 years ago [5, 7], and tight bounds on the above quantities are still elusive, even in the plane, where the maximum number of halving edges is known to be at most

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$O(n^{4/3})$ [4], and at least $\Omega(n \cdot 2^{c\sqrt{\log n}})$ for some constant c [10, 12]. In three dimensions the upper bound is $O(n^{5/2})$ [11], and the lower bound is $\Omega(n^2 \cdot 2^{c\sqrt{\log n}})$. In fact, in any dimension d, the known lower bound is $\Omega(n^{d-1} \cdot 2^{c\sqrt{\log n}})$, which is obtained by "lifting" the 2-dimensional constructions of [10, 12]. In $d \geq 5$ dimensions, the known upper bounds become considerably weaker, and are of the form $O(n^{d-\delta_d})$, where $\delta_d = 1/(4d-3)^d$, leaving a fairly big gap between the upper and lower bounds. Moreover, the proof of these bounds uses the so-called colored Tverberg theorem, for which there is no known elementary proof; the only known proof, given in [13], uses methods from algebraic topology. See [8] for a review of this approach.

This also used to be the case for $d = 4$, until the recent work of Matoušek et al. [9], who obtained, with an elementary proof, the improved bound $O(n^{4-2/45})$. In this note we follow their footsteps, and present, with a somewhat simpler proof, a further improved bound of $O(n^{4-1/18})$ on the number of halving simplices in a set of n points in $\mathbb{R}^4$.

As in essentially all known proofs of upper bounds for the number of halving simplices in any dimension, the analysis in this paper, as the preceding one in [9], only uses a simple local property of halving simplices, the so-called *antipodality property*, first observed by Lovász [7], which we define in more detail next.

2 Preliminaries

We begin by reviewing the basic ingredients of the proof, many of which are borrowed from the preceding study [9].

Lemma 2.1. *Let S be a set of n points in $\mathbb{R}^d$, and let T be the collection of all halving simplices of S. Then T is* antipodal*, in the sense that the following holds for any $d-1$ points $p_1, \ldots, p_{d-1} \in S$: Whenever $a, b \in S$ are two distinct points such that both $ap_1 \cdots p_{d-1}, bp_1 \cdots p_{d-1} \in T$, then there is a third point $c \in S$ such that $cp_1 \cdots p_{d-1} \in T$ and such that the triangle abc intersects the affine hull of $p_1 \cdots p_{d-1}$ (see Figure 1).*

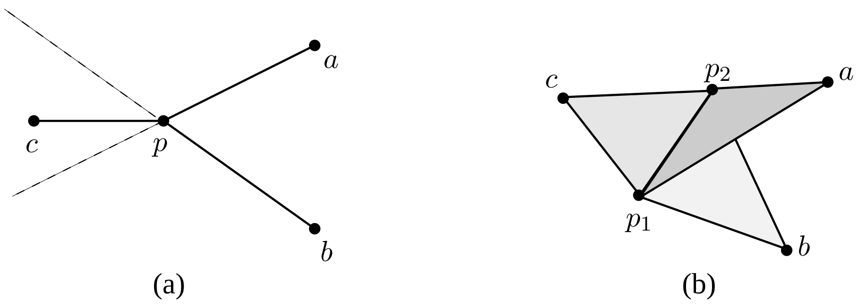

Figure 1: The antipodality property (a) in two dimensions and (b) in three dimensions. (The view of both configurations coincides if we look at (b) in the direction p_1p_2.

In the statements of the following two lemmas, a *generic* 2-plane is a two-dimensional plane π in $\mathbb{R}^4$ that lies in general position with respect to S. In particular, no point of S lies on π, no edge connecting two points of S meets π, and a triangle Δ spanned by S can meet π only at a single point which lies in the relative interior of Δ.

Lemma 2.2 (Matoušek et al. [9])**.** *Let S be a set of n points in general position in $\mathbb{R}^4$, and let T be the collection of the halving simplices of S. If $t = |T| > Cn^{11/3}$, for some absolute constant $C > 0$, then there is a generic 2-plane π that intersects $\Omega(t^3/n^8)$ simplices of T.*

Lemma 2.3. *Let S and T be as above. Then no generic 2-plane intersects more than $O(n^{4-1/6})$ simplices of T.*

Lemma 2.3 constitutes the improvement in the present paper; it replaces a weaker statement in [9], where the upper bound is only $O(n^{4-2/15})$, with what we believe to be a simpler and shorter proof.

We emphasize that the proof only uses antipodality of the set of halving simplices, so, as in the preceding work, the result also holds for arbitrary so-called 4-*uniform antipodal hypergraphs* in $\mathbb{R}^4$ (in the terminology of [9]). There are two main challenges raised by the present and the preceding studies: (a) Improve further the upper bound for 4-uniform antipodal hypergraphs in $\mathbb{R}^4$. (b) Find additional properties of halving simplices which can help in tightening the upper bound.

3 Bounding the number of simplices stabbed by a 2-plane

Let S and T be as above, and let π be a generic 2-plane, which crosses some simplices of T. Note that if a 3-simplex $\tau \in T$ and π intersect (necessarily generically), then the intersection is a line-segment of positive length whose endpoints lie in the relative interior of two triangles bounding τ.

We repeat some of the terminology of [9]. Let E be the set of line segments $\{\tau \cap \pi \mid \tau \in T\}$, and let V be the set of endpoints of these edges. Then $G = (V, E)$ is a (straight-edge) geometric graph in the plane π, but with a particular kind of vertex and edge labelling: Each point $q \in V$ is the intersection of π with some triangle spanned by a triple of distinct points $a, b, c \in S$, and we label q by the (unordered) triple abc. Similarly, each edge $e \in E$ is the intersection of π with some simplex spanned by four points $a, b, c, d \in S$, and we label e by the pair of labels of its endpoints, which are two distinct sub-triples of $abcd$. No two objects receive the same label. In particular, $m := |V| \le \binom{n}{3}$. Moreover, rephrasing what has just been noted, if two points abc and xyz of V are connected by an edge, then the triples abc and xyz must share a common pair of indices, say, $a = x$ and $b = y$, and the edge is labeled by the pair (abc, abz).

As shown in [9], the geometric graph $G = (V, E)$ is also *antipodal*. That is, the following holds for each vertex u of V: For any pair of edges (u, v), (u, w) of E incident to u, there is a third edge (u, z) such that u lies in the triangle vwz. An equivalent formulation of this property is that, if (u, v) and (u, w) are two angularly consecutive edges of E emanating from u, then there is another edge (u, z) in E lying in the wedge opposite to the one enclosed between (u, v) and (u, w). See Figure 1(a).

The crossings between edges of G will be of central importance in our analysis. We recall the following fundamental result, first proved by Ajtai et al. [1] and independently by Leighton [6] (see also [8]):

Theorem 3.1 (Crossing Lemma)**.** *If $G = (V, E)$ is a simple graph, then in any drawing of G in the plane, there are at least $\Omega(|E|^3/|V|^2)$ crossings between the (not necessarily straight) arcs representing the edges of G, provided that $|E| \ge 4|V|$. Consequently, we always have $|E| = O(|V| + |V|^{2/3}X^{1/3})$, where X is the minimum number of crossings in any drawing of G.*

For antipodal geometric graphs we also have the following result by Dey [4].

Lemma 3.2 (Dey)**.** *The number of crossings between the edges of an antipodal geometric graph $G = (V, E)$ is at most $|V|^2$.*

Proof. We present the proof of the lemma (borrowed from [9]), because we will later need the notions of *convex* and *concave chains* that the proof exploits. We remark that the proof presented here is simpler than the original proof of Dey; see also [8, p. 288].

By choosing an appropriate coordinate system for the plane π, we may assume that no edge in E is vertical. Let us consider an edge $(u, v) \in E$ with left endpoint u and right endpoint v. If there exists an edge with left endpoint v and with slope larger than the slope of (u, v), then let (v, w) be the edge that has the smallest slope among all such edges and we call (v, w) the *convex successor* of (u, v); otherwise, the convex successor is not defined. The antipodality property guarantees that no two edges can have the same convex successor. Thus, if we define a convex chain as a maximal sequence $e_1, \ldots, e_k \in E$ such that each e_{i+1} is the convex successor of e_i, then these chains form a partition of the edge set E, and clearly, each chain is an x-monotone convex polygonal curve. Note that if (u, v) is the rightmost edge of a convex chain, (u, v) must have the largest slope among all the edges with right endpoint v, for otherwise, the antipodality property would imply that (u, v) has a convex successor. Thus, every vertex v is the right endpoint of at most one convex chain, so there are at most $|V|$ such chains. Similarly, there are at most $|V|$ *concave chains*, which are defined analogously (by reversing the direction of the y-axis). If two edges in E cross, then we can extend one of them to a convex chain and the other one to a concave chain, and charge the crossing to the pair of chains. Since a convex curve and a concave curve can cross at most twice, and since a crossing in G can be charged to two different pairs of chains, the total number of crossings is at most $|V|^2$. □

As observed in [9], by applying Lemma 3.2 to the graph of halving edges of n points in the plane, we see that there are only $O(n^2)$ crossings between the halving edges. Together with the Crossing Lemma, this yields a simplified proof of Dey's bound of $O(n^{4/3})$ for the number of halving edges.

In our setting, however, direct application of Lemma 3.2 does not yield a sharp bound: The number of vertices is $|V| = O(n^3)$, so Lemma 3.2 only implies that the number of crossings in G is $O(n^6)$. Combining this bound with the bound of the Crossing Lemma, we only obtain the trivial bound $|E| = O(n^4)$. We circumvent this difficulty as follows.

3.1 Bounding the number of edges of G

For each pair a, b of points of S, let G_{ab} denote the graph (V_{ab}, E_{ab}), where V_{ab} is the set of all the vertices labeled by a and b (so they are of the form abc, for $c \in S \setminus \{a, b\}$), and E_{ab} is the set of all edges labeled by a and b (so they are of the form (abc, abd), for $c, d \in S \setminus \{a, b\}$; such an edge represents (i.e., is the intersection of π with) the simplex $abcd$). Clearly, $n_{ab} = |V_{ab}| \leq n - 2 < n$.

The graph G is the union of all the graphs G_{ab}, for $a, b \in S$. As argued above (in the proof of Lemma 3.2), the edges of G can be decomposed into $O(n^3)$ convex pairwise edge-disjoint chains, and, independently, into $O(n^3)$ concave pairwise edge-disjoint chains. As noted above, this implies that the number X of crossings between the edges of G is $O(n^6)$.

We next partition each (convex or concave) chain γ of G into a sequence of maximal contiguous subchains, each contained in some subgraph G_{ab} (in the worst case, a subchain might consist of a single edge). Consider the passage from one subchain to a different one. For an appropriate choice of symbols, this takes place at a node abc of γ, so that γ enters abc by an edge (abx, abc) and leaves it by an edge (abc, acy); thus abc is a common delimiter of a subchain of G_{ab} and of a subchain of G_{ac}.

For each node abc, denote by Q_{abc} the number of chains passing through abc for which the incoming edge and the outgoing edge belong to different subchains, as above. (Given only abc, there are $O(1)$ possibilities for the pair of subgraphs that are involved in such a transition; however, given abc and γ, the pair of subgraphs is unique.) Clearly, $Q_{abc} = O(n)$.

We decompose the set of vertices $V = \bigcup_{a,b} V_{ab}$ into a logarithmic number of subsets, placing in the k-th subset $V^{(k)}$, for $k = 1, 2, \ldots$, those vertices abc for which $2^{k-1} \le Q_{abc} < 2^k$ (vertices with $Q_{abc} = 0$ are ignored). We fix k, and apply the following construction only to the vertices of $V^{(k)}$. We will then repeat the analysis for each k separately.

So let k be fixed, and let abc be a vertex of $V^{(k)}$. We take each of the $\Theta(Q_{abc}^2) = \Theta(4^k)$ pairs of an incoming edge and an outgoing edge of two such transitory chains, so that the pair of selected edges is itself involved in such a transition, and "shortcut" each pair to obtain $\Theta(4^k)$ new straight edges. Specifically, we take an incoming edge (abx, abc) of one transitory chain and an outgoing edge (abc, acy) of another (or the same) transitory chain, and connect the two other endpoints of these edges by a straight segment, to obtain the edge (abx, acy); see Figure 2. For technical reasons, we retain either only the shortcuts where the new edge lies below the middle vertex abc, or only the shortcuts where the new edge lies above the middle vertex, choosing the larger of the two subcollections. We lose at most a factor of 2 by imposing this restriction. Without loss of generality, we consider only shortcuts where the new edge lies below the middle vertex. We note that (i) the new edges are not necessarily edges of G, (ii) some of the shortcuts may arise from pairs of convex chains, some from pairs of concave chains, and some from mixed pairs of a convex and a concave chain, (iii) the endpoints abx, acy of a shortcut edge are not necessarily in $V^{(k)}$, only the middle vertex abc is, and (iv) there are only $O(1)$ ways in which a new edge can be formed: the middle vertex must be labeled by a, by one of b, x, and by one of c, y.

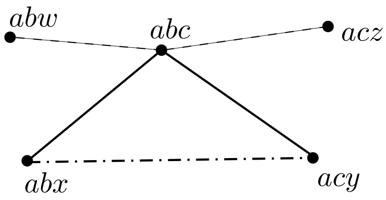

Figure 2: Constructing a shortcut edge of G_a^*.

Altogether we get $P = \Theta\left(\sum_{a,b,c} Q_{abc}^2\right) = \Theta\left(4^k |V^{(k)}|\right)$ new edges. We form n new graphs G_a^*, one for each $a \in S$, and place each new edge, of the form (abx, acy), in the respective graph G_a^*, labeled with the (unique) symbol common to the two endpoints. Let P_a denote the number of edges of G_a^*, so $\sum_a P_a = P$.

Next, we apply the Crossing Lemma (Theorem 3.1) to each of the graphs G_a^*. Let Z_a denote the number of crossings in G_a^*. Observing that the number of vertices of G_a^* is $O(n^2)$, we get

$$P_a = O\left(n^{4/3} Z_a^{1/3} + n^2\right).$$

Counting crossings in G_a^*. To estimate Z_a, consider a crossing between two edges of G_a^*, say, (abx, acy) and (adu, aev), with respective middle vertices abc and ade. Since (abx, acy) and (adu, aev) cross each other, one of the following situations must arise:

(a) There exists a crossing between an edge among $\{(abx, abc), (abc, acy)\}$ and $\{(adu, ade), (ade, aev)\}$, say, it is a crossing between (abx, abc) and (ade, aev); see Figure 3(a). We charge the crossing in G_a^* to the latter crossing, and note that such a crossing is charged at most 4^k times. Indeed, given the crossing edges (abx, abc) and (ade, aev), there are $O(2^k)$ choices for each of y and v. The overall number of crossings in this case, over all $a \in S$, is therefore $O(4^k n^6)$, namely $O(4^k)$ times the number of crossings in G.

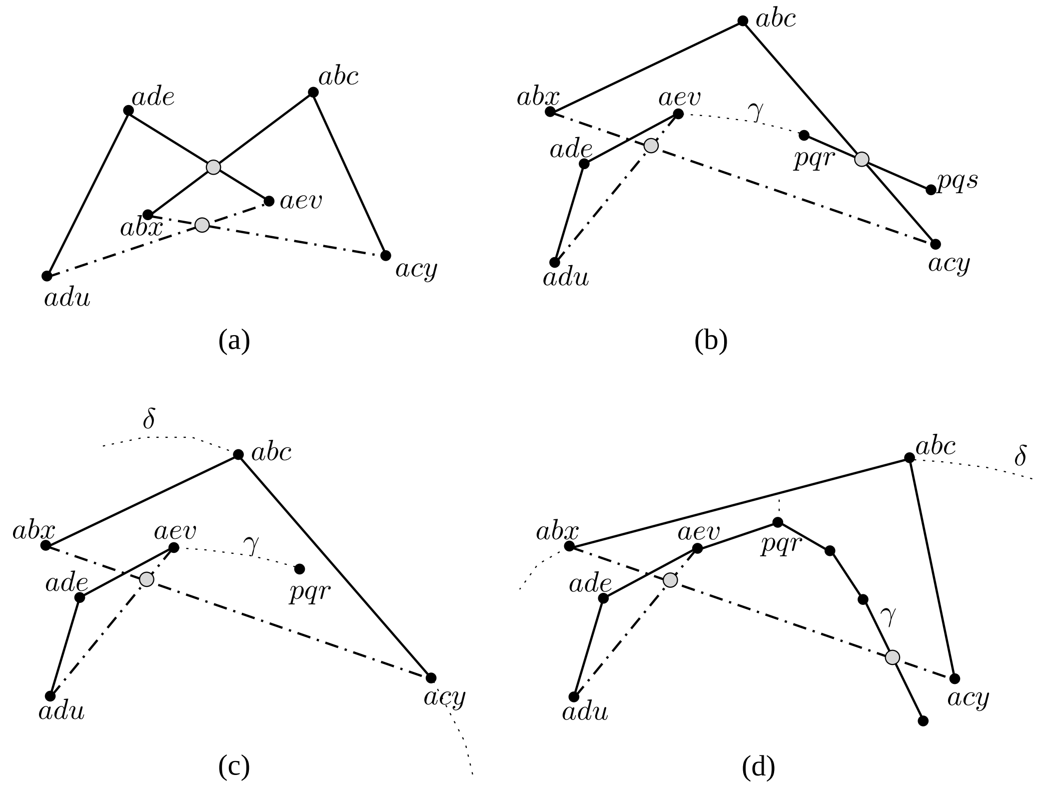

Figure 3: Charging a crossing in G_a^* (a) to a crossing in G; (b) to a further crossing in G involving the chain γ; (c) to an endpoint of γ; or (d) to a vertex of γ closest to the top of the triangle.

(b) There does not exist a crossing as in (a). Since the boundaries of the triangles (abx, abc, acy) and (adu, ade, aev) must intersect at least twice, there must exist another intersection between one of the "bases" (abx, acy), (adu, aev) and a "short" edge of the other triangle. Without loss of generality, assume that the short edge (ade, aev) crosses the base (abx, acy). In this case one of the endpoint of the short edge must lie inside the triangle (abx, abc, acy); without loss of generality, assume that aev lies inside the triangle. See Figure 3(b,c,d).

The edge (ade, aev) belongs to some (unique) global concave chain γ. We follow γ from aev onwards (away from ade) until one of the following events occurs:

(b.i) We reach an edge (pqr, pqs) of γ which crosses one of the edges (abx, abc), (abc, acy), say, (abc, acy); see Figure 3(b). In this case we charge our original crossing to the new crossing, and claim, as in (a), that the new crossing is charged at most $O(4^k)$ times. Indeed, given the new crossing, we first "guess" which of the two endpoints, abc, acy, of the crossed edge is the middle vertex abc (allowing only vertices in $V^{(k)}$ to be guessed), and then guess x, in $O(2^k)$ ways, and thereby obtain the first shortcut edge (abx, acy). We then trace the unique concave chain γ containing (pqr, pqs) backwards until it crosses (abx, acy). Since a concave chain can cross a straight segment at most twice, there are only two possible crossings of the above kind (and a more careful examination of the configuration shows that there is only one crossing). We thus retrieve the edge (ade, aev). We then guess which of its two endpoints, the one that lies inside the triangle (abx, abc, acy) or the one that lies outside, is the middle vertex ade (allowing, as above, only vertices in $V^{(k)}$ to be guessed), and then guess u, in $O(2^k)$ ways, and retrieve the original charging crossing. Hence, here too, the overall number of crossings in this case is $O(4^k n^6)$.

(b.ii) The chain γ terminates at some vertex pqr, still inside the triangle (abx, abc, acy). Then pqr lies below one of the edges (abx, abc), (abc, acy), say, (abc, acy); see Figure 3(c). In this case we charge our original crossing to the pair (pqr, δ), where δ is the unique concave chain containing (abc, acy), and claim, as above, that such a pair is charged at most $O(4^k)$ times. Indeed, given the pair (pqr, δ), we first retrieve the edge (abc, acy), which is the unique edge of δ lying above pqr. Then we "guess" which of its two endpoints is the middle vertex abc, and then guess x, in $O(2^k)$ ways, and thereby obtain the first shortcut edge (abx, acy). We next retrieve the unique concave chain γ that terminates at pqr (the uniqueness of γ follows from the antipodality of G, as argued in the proof of Lemma 3.2), and then, as in (b.i), trace γ backwards until it crosses (abx, acy). As above, this produces the edge (ade, aev), and we complete the picture by guessing ade and u. Again, the overall number of crossings in this case is $O(4^k n^6)$; here $O(n^6)$ is the product of the number of vertices and the number of global chains, both being $O(n^3)$.

(b.iii) Finally, we face the case where the chain γ exits the triangle (abx, abc, acy) by crossing its "base" (abx, acy) again; see Figure 3(d). In this case γ must contain a vertex pqr that lies inside the triangle (abx, abc, acy), below one of its short edges, say (abx, abc), so that, in a small neighborhood of pqr within γ, pqr is vertically nearest to the edge (abx, abc). In this case we charge our original crossing to the pair (pqr, δ), where δ is the unique concave chain containing (abx, abc), and claim, as above, that such a pair is charged at most $O(4^k)$ times. Indeed, given the pair (pqr, δ), we first retrieve the edge (abx, abc), which is the unique edge of δ lying above pqr. Then we "guess" abc and y, in $O(2^k)$ ways, and thereby obtain the first shortcut edge (abx, acy). We next retrieve the unique concave chain γ that contains pqr and satisfies the local minimal vertical distance condition at pqr. Finding γ can be done by drawing through pqr a line parallel to the edge (abx, abc); then γ is the unique concave chain through pqr tangent to that line; see [11] for a similar argument. Then, similar to the preceding cases, we trace γ forward or backwards (here the two directions are meaningful) until it crosses (abx, acy). As above, this produces the edge (ade, aev), and we complete the picture by guessing ade and u. Again, the overall number of crossings in this case is $O(4^k n^6)$.

The final stretch. The preceding analysis yields $\sum_a Z_a = O(4^k n^6)$. Applying Hölder's inequality to the previous bound, we get

$$P = \sum_a P_a = O\left(n^{4/3}\left(\sum_a Z_a^{1/3}\right) + n^3\right) = O\left(n^{4/3}\left(\sum_a Z_a\right)^{1/3} \cdot n^{2/3} + n^3\right) = O(4^{k/3} n^4).$$

On the other hand,

$$P = \sum_a P_a = \Theta\left(\sum_{a,b,c} Q_{abc}^2\right) = \Omega\left(2^k \sum_{a,b,c} Q_{abc}\right),$$

implying that

$$\sum_{a,b,c} Q_{abc} = O(n^4/2^{k/3}).$$

When k is small, we use instead the trivial upper bound

$$\sum_{a,b,c} Q_{abc} = O(2^k n^3).$$

That is,

$$\sum_{a,b,c} Q_{abc} = \min\left\{O(n^4/2^{k/3}),\, O(2^k n^3)\right\}. \tag{1}$$

We now sum this bound over all k, and notice that the two terms are equal when $2^k = n^{3/4}$. This is easily seen to imply that the overall bound is $O(n^{15/4})$.

We now fix a graph G_{ab}, and bound the number of concave and convex chains into which G_{ab} is decomposed. Put $Q_{ab} = \sum_c Q_{abc}$. Sweep a vertical line ℓ from left to right through π. At $x = -\infty$, ℓ meets no chains of G_{ab} (it meets no edges of the graph). As ℓ sweeps past a vertex abc, at most $1 + Q_{abc}$ chains can terminate there, and at most $1 + Q_{abc}$ chains can start there. (We add 1 to include also the global chains that start and end at abc and belong to G_{ab}, if any.) Hence, the total number of subchains of G_{ab} is at most $n + Q_{ab}$.

Regarding each crossing between a pair of edges of G_{ab} as a crossing between the concave subchain containing one of the edges and the convex subchain containing the other edge, the number of crossings X_{ab} between the edges of G_{ab} is thus $O((n + Q_{ab})^2) = O(n^2 + Q_{ab}^2)$. The Crossing Lemma therefore implies that the number E_{ab} of edges of G_{ab} satisfies

$$E_{ab} = O\left(n^{2/3}X_{ab}^{1/3} + n\right) = O\left(n^{4/3} + n^{2/3}Q_{ab}^{2/3}\right).$$

Summing over all graphs G_{ab}, the number E of edges of G satisfies

$$E = \sum_{a,b} E_{ab} = O\left(n^{10/3} + n^{2/3}\left(\sum_{a,b} Q_{ab}^{2/3}\right)\right) = O\left(n^{10/3} + n^{4/3}\left(\sum_{a,b} Q_{ab}\right)^{2/3}\right) =$$

$$O\left(n^{10/3} + n^{4/3} \cdot n^{5/2}\right) = O(n^{23/6}).$$

This establishes Lemma 2.3. Combining this with the lower bound $E = \Omega(t^3/n^8)$, provided by Lemma 2.2, we get

$$t = O\left(n^{4-1/18}\right).$$

This is an improvement over the previous bound $O(n^{4-2/45})$ of [9], with a simpler proof.

In summary, we have:

Theorem 3.3. *The number of halving simplices in a set of n points in $\mathbb{R}^4$ is $O\left(n^{4-1/18}\right)$.*

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