Balanced Lines, Halving Triangles, and the Generalized Lower Bound Theorem

Micha Sharir* Emo Welzl†

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1 Introduction

The following three facts are related to each other.

Fact A Let R and B be two disjoint finite planar sets, so that $|R \cup B|$ is even and $R \cup B$ is in general position (i.e., no three points are collinear). Points in R and B are referred to as 'red' and 'blue,' respectively. A line ℓ is *balanced* (w.r.t. (R, B)) if ℓ passes through a red point and a blue point, and on both sides of ℓ, the number of red points minus the number of blue points is the same.

The number of balanced lines is at least $\min\{|R|, |B|\}$.

This number is attained, if R and B can be separated by a line.

Fact B $n \in \mathbf{N}$. Let Q be a set of $2n + 1$ points in 3-space in general position (i.e., no four points are coplanar). A *halving triangle of* Q is a triangle spanned by three points in Q such that the plane containing the three points equipartitions the remaining points of Q.

The number of halving triangles is at least n^2.

This number is attained, if Q is in convex position.

Fact C $d \in \mathbf{N}$. Let $\mathcal{P}$ be a convex polytope[1] which is the intersection of $d+4$ halfspaces in general position in d-space[2] (i.e., no $d + 1$ bounding hyperplanes meet in a common point). Let its edges be oriented according to a generic linear function (edges are directed from smaller to larger value; 'generic' means that the function evaluates to distinct values at the vertices of $\mathcal{P}$).

The number of vertices with $\lceil \frac{d}{2} \rceil - 1$ *outgoing edges is at most the number of vertices with* $\lceil \frac{d}{2} \rceil$ *outgoing edges.*[3]

*School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel and Courant Institute of Mathematical Sciences, New York University, New York, NY 10012, USA; sharir@math.tau.ac.il.

†Theoretical Computer Science, ETH Zürich, CH-8092 Zürich, Switzerland; emo@inf.ethz.ch.

[1]By 'polytope' we imply that it is bounded!

[2]Therefore, either $\mathcal{P}$ is empty, or it is a simple convex d-polytope with at most $d + 4$ facets. All vertices are incident to d edges. Our set-up is chosen in this way, in order to have a clean relation to the other statements.

[3]In fact, for all $1 \leq j \leq \lceil d/2 \rceil$, the number of vertices with $j - 1$ outgoing edges is at most the number of

This is tight if $\mathcal{P}$ is empty.

(A) has been recently proved[4] by J. Pach and R. Pinchasi [7], answering a question of G. Baloglou's. (C) is a very special case of the Generalized Lower Bound Theorem (GLBT) for simple polytopes, which—in turn—is part of the g-Theorem proved by R. P. Stanley [8] (thereby answering a conjecture by P. McMullen, who later provided also an alternative proof [6]); cf. also [10]. It was recently shown that (B) and (C) can be derived from each other [9]. In Section 2 we present a simple proof of the equivalence (A⇔B). That is, (A)–(C) are equivalent to each other.[5] In Section 3, we give an alternative proof of the equivalence (A⇔C). Clearly, that is already implied by (A⇔B⇔C), but we include here an argument for this specific setting for the sake of completeness.

On one hand, this means that the result of [7] admits a proof that is considerably simpler than their original proof, via the GLBT. On the other hand, Pach and Pinchasi's proof has merits of its own, because (i) no purely combinatorial proof of the GLBT (such as that in [7]) has been previously known (not even for the special case (C) equivalent to the balanced line problem), and (ii) that proof is based on allowable sequences in the dual, and thus (A) applies also for oriented matroids.

2 Balanced Lines and Halving Triangles

We first transform the balanced lines problem (A) to yet another problem (D) involving halving triangles in three dimensions, which appears to be new.

Assume that the points of $R \cup B$ (as in (A)) lie in the plane $z = 1$. Project these points onto the unit sphere centered at the origin O by mapping each point $r \in R$ to $r^* = r/\|r\|$, and each point $b \in B$ to $b^* = -b/\|b\|$. Let S_0 denote the resulting set of projected points, and put $S = S_0 \cup \{O\}$. By a small perturbation of $R \cup B$ that does not change the combinatorial type of this set, we may assume that S is in general position.

Observe the following properties, whose proofs are straightforward:

(i) The xy-plane π_0 separates S_0 into sets of cardinalities $|R|$ and $|B|$.

(ii) For $r \in R$ and $b \in B$, the line passing through r and b is a balanced line iff the triangle Or^*b^* is a halving triangle of S. In particular, this establishes a correspondence between the balanced lines in $R \cup B$ and those halving triangles of S that are incident to O and are crossed by π_0 (i.e., π_0 intersects their relative interior).

(iii) The point O is an extreme point of S if and only if R and B are separated by a line.

Moreover, we can apply a reverse transformation as follows. Let Q be any set of $2n + 1$ points in 3-space in general position. Let $q_0 \in Q$ be a fixed point, and let π_0 be a plane of Q that passes through q_0 and through no other point of Q. Let π be a plane parallel to

vertices with j outgoing edges. And for d odd, and $j = \lceil d/2 \rceil$, these numbers are even equal. But that will not be relevant in our context.

[4] The statement in [7] is restricted to the case $|R| = |B| = n$. Then a balanced line must have the same number of red and blue points on each side, and there are at least n such balanced lines.

[5] Of course, true statements are always equivalent; we mean that these facts can be derived from each other in a fashion that is significantly simpler compared to the proofs of the individual statements.

π_0. Map each point $q \in Q \setminus \{q_0\}$ to the point of intersection of π with the line that passes through q and q_0. Denote by R (resp. B) the subset of points on π that are images of points of Q that lie in the side of π_0 that contains (resp. does not contain) π.

(iv) A triangle $q_0q_1q_2$, for $q_1, q_2 \in Q$, is a halving triangle crossed by π_0 if and only if the line that passes through the images of q_1 and q_2 is a balanced line for (R, B).

These properties imply the equivalence (A⇔D) of the result of Pach and Pinchasi and the following assertion (D).

Fact D $n \in \mathbf{N}$. Let Q be a set of $2n+1$ points in 3-space in general position. Let $q_0 \in Q$ be a fixed point, and let π_0 be a plane of Q that passes through q_0 and through no other point of Q, and separates $Q \setminus \{q_0\}$ into two sets of cardinalities k and $2n-k$.

There are at least $\min\{k, 2n-k\}$ *halving triangles of* Q *that are incident to* q_0 *and are crossed by* π_0.

This number is attained, if q_0 is an extreme point of Q.

Let us first show that, indeed, *for* q_0 *extreme, the number of halving triangles of* Q *that are incident to* q_0 *and are crossed by* π_0 *equals* $\min\{k, 2n-k\}$. Project $Q_0 = Q \setminus \{q_0\}$ centrally from q_0 onto a plane parallel to a supporting plane of Q at q_0; denote the projected set by Q_0^*. The plane π_0 projects to a line λ that separates Q_0^* into sets of cardinalities k and $2n-k$. It is then easy to check that, for points $q_1, q_2 \in Q_0$, the triangle $q_0q_1q_2$ is a halving triangle of Q crossed by π_0 if and only if the segment $q_1^*q_2^*$, connecting the images q_1^*, q_2^* of q_1, q_2, is a halving edge[6] of Q_0^* that is crossed by the line λ. By Lovász' lemma [3, 5], the number of such edges is exactly $\min\{k, 2n-k\}$.

We proceed to a *proof of implication (D ⇒ B).* Suppose (D) holds. Consider a set Q of $2n+1$ points. Let π_q, for $q \in Q$, be pairwise parallel planes such that $\pi_q \cap Q = \{q\}$ for each $q \in Q$. Every halving triangle Δ of Q is crossed by exactly one of these planes which is also incident to a vertex of Δ (a plane crosses a triangle if it contains one of the three vertices, and separates the other two). Hence, there are at least

$$\sum_{i=1}^{2n+1} \min\{i-1, 2n+1-i\} = n^2$$

halving triangles, which implies (B). (By the preceding argument, equality is attained when Q is in convex position.)

Finally, let us provide the *proof of implication (B ⇒ D).* Suppose that assertion (D) is false. Thus there exist a set Q of $2n+1$ points, a parameter $0 \le k \le 2n$, a point $q_0 \in Q$ and a plane π_0 passing through q_0 and partitioning $Q \setminus \{q_0\}$ into two sets of cardinalities k and $2n-k$, such that the number c of halving triangles of Q incident to q_0 and crossed by π_0 is strictly smaller than $\min\{k, 2n-k\}$. First, we project $Q_0 = Q \setminus \{q_0\}$ from q_0 onto a sphere centered at q_0; let Q_0' denote the resulting set of projected points, and $Q' = Q_0' \cup \{q_0\}$. In this way, the collection of halving triangles incident to q_0 did not change, nor did the number of points on either side of π_0. Therefore Q', q_0 and π_0 still provide a configuration contradicting (D). Now let π_q, for $q \in Q_0'$, be planes parallel to π_0 with $\pi_q \ni q$ for each q.

[6] An edge whose containing line equipartitions $Q_0^* \setminus \{q_1^*, q_2^*\}$.

If necessary, rotate π_0 slightly about q_0 so that $\pi_q \cap Q' = \{q\}$ for each $q \in Q_0'$. As in the previous argument, every halving triangle of Q' is crossed by exactly one of the planes in $\{\pi_0\} \cup \{\pi_q \mid q \in Q_0'\}$ (which is also incident to a vertex of the triangle). Since all points apart from q_0 are extreme in Q', the number of halving triangles of Q' is exactly

$$\underbrace{c - \min\{k, 2n-k\}}_{<0} + \underbrace{\sum_{i=1}^{2n+1} \min\{i-1, 2n+1-i\}}_{=n^2} < n^2 \ . \tag{1}$$

The equivalence (B⇔D), and thus (B⇔A) is established.

Remark 1 Consider $Q_0' \cup \{q_0\}$ as in the argument just given. Let π' be another plane through q_0 that partitions Q_0' into sets of cardinalities k' and $2n-k'$, and let c' be the number of halving triangles incident to q_0 and crossed by π' (this is also the number of such halving triangles in the original Q). Since the left-hand side of (1) is equal to the number of halving triangles of $Q_0' \cup \{q_0\}$, it follows that

$$c - \min\{k, 2n-k\} = c' - \min\{k', 2n-k'\} \ .$$

Hence, if there were a configuration contradicting (D), then there would also be one with a plane π_0 that equipartitions $Q \setminus \{q_0\}$, and, thus, if there were a configuration contradicting (A), then there would also be one with $|R| = |B|$. That is, the 'special case' of (A) treated in [7] (see footnote 4) immediately entails the more general formulation in (A).

3 Balanced Lines and the GLBT

We want to exhibit a more direct relation between (A) and (C). We will not do so with (C) itself, though, but replace it by the following assertion (E), which is known to be equivalent to (C) by the Gale transform [9].

Fact E $m \in \mathbf{N}$. Let S be a set of m points in 3-space, and let ρ be a directed ray pointing at its apex x, such that $S \cup \{x\}$ is in general position, and ρ is disjoint from S and from all segments connecting points in S. An oriented triangle spanned by three points in S is called a *j-triangle of S*, if there are exactly j points of S on its positive side.[7] We say that *ρ enters a j-triangle Δ of S*, if it intersects Δ from the positive side to the negative side of it (i.e., x is on the negative side of Δ). If ρ crosses Δ from the negative to the positive side, then we say that *ρ leaves* Δ. Let $g_j(x,S)$ be the number of j-triangles entered by ρ minus the number of j-triangles left by ρ.

$g_{\lceil (m-4)/2 \rceil}(x,S) \geq 0.$

Equality holds if x is extreme in $S \cup \{x\}$.

First note that if m is odd, then $\lceil (m-4)/2 \rceil = m - 3 - \lceil (m-4)/2 \rceil$. Hence, for every $\lceil (m-4)/2 \rceil$-triangle entered there is one that is left (the same triangle with opposite orientation). Therefore, $g_{\lceil (m-4)/2 \rceil}(x,S) = 0$ for all x. So the statement is interesting for even m only.

[7]The orientation of the triangle declares one side of the plane it spans as the positive side. Obviously, the opposite orientation of a j-triangle is an $(m-3-j)$-triangle.

In fact, as already suggested by the above notation, it can be shown that $g_j(x,S)$ is a function of x that is independent of the choice of the ray ρ pointing at it. From this it immediately follows that $g_j(x,S)=0$ for all j, if x is extreme in $S\cup\{x\}$. Using the Gale transform (see [9]), one can show that if x is not extreme in $S\cup\{x\}$, then there is a simple polytope $\mathcal{P}$ in $\mathbf{R}^{m-4}$ with at most m facets, such that the so-called g-vector of $\mathcal{P}$ is exactly the vector $(g_j(x,S))_{j=0}^{\lfloor (m-4)/2\rfloor}$; see [9] for details (if x is extreme, $\mathcal{P}$ is the empty polytope). The nonnegativity of this vector is the GLBT.[8] We refer to [9] for the equivalence[9] (E⇔C); see also [4].

We have prepared the ground for a *proof of equivalence (D⇔E)*. Assume the set-up of statement (D); recall that Q has $2n+1$ points. Put $Q_0 = Q\setminus\{q_0\}$, and let $H(q)$, for any $q\in\pi_0$, denote the number of halving triangles of $Q_0\cup\{q\}$ that are incident to q and are crossed by π_0. We draw a line ℓ in π_0 passing through q_0, move a point q along ℓ from infinity to q_0, and keep track of the changes in $H(q)$ during this motion (see [2] for related results obtained via the this continuous motion paradigm, and [1, Chapter 3.6-3.8] for a thorough treatment of the combinatorial changes occurring in such a motion).

Initially, q is an extreme point of $Q_0\cup\{q\}$ and so $H(q)=\min\{k,2n-k\}$.

As q moves along ℓ, $H(q)$ changes only when q becomes coplanar with three points $a,b,c\in Q_0$, so that the plane passing through these four points bounds two open halfspaces, one of which contains $n-1$ points of Q_0 and the other $n-2$. Three cases may arise, as illustrated in Figure 1.

(a) The four points a,b,c,q are in convex position, say in this counterclockwise order (Figure 1(a)).

(b) The four points are not in convex position but q is an extreme point of the quadruple, and, say, c lies in the interior of qab (Figure 1(b)).

(c) The four points are not in convex position and q is the middle point (Figure 1(c)).

Each case is further divided into two subcases, depending on whether q reaches the plane of abc from the side containing $n-2$ points of Q_0 (subcase (i)), or from the side containing $n-1$ points of Q_0 (subcase (ii)). Let δ denote the line of intersection of π_0 and the plane of abc (drawn as a dashed line in Figure 1).

In case (a.i), the triangles abc, qac were halving triangles of $Q_0\cup\{q\}$ before q reached the plane of abc, and the triangles qab, qbc are halving triangles after q leaves that plane. If δ does not cross the quadrangle $abcq$ (at the time of coplanarity) then π_0 does not cross the triangle qac before q reaches the plane of abc, and does not cross qab, qac after q leaves that plane. Hence $H(q)$ does not change in this case. On the other hand, if δ crosses $abcq$ then π_0 crosses the triangle qac before q reaches the plane of abc, and crosses exactly one of the triangles qab, qac afterwards. Hence $H(q)$ does not change in this case either. Case (a.ii) is treated in a fully symmetric manner, and $H(q)$ does not change in this subcase as well. In cases (b.i) and (b.ii) the local behavior at q is the same as in the corresponding subcases (a.i) and (a.ii), so $H(q)$ does not change in these cases either.

[8] And the characterization of all possible g-vectors is the g-Theorem.

[9] In fact, in the set-up of (C), the number of vertices with $j-1$ outgoing edges is at most the number of vertices with j outgoing edges, for all $j\le\frac{d}{2}$, and this is true in even and odd dimension.

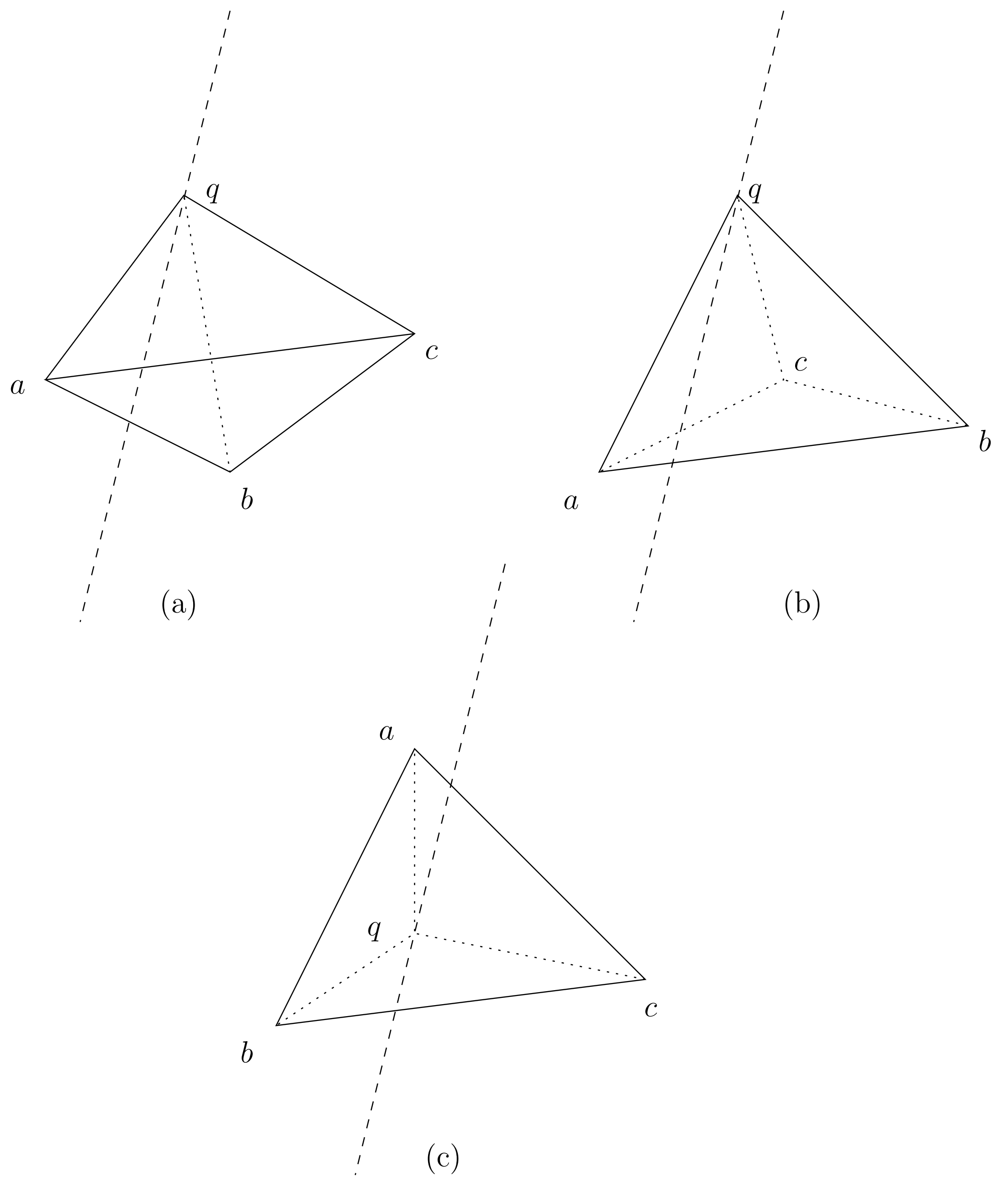

Figure 1: Three cases of coplanarity: (a) The four points are in convex position. (b) The four points are not in convex position but q is an extreme point of the quadruple. (c) The four points are not in convex position and q is the middle point.

In case (c.i), the triangle abc was a halving triangle of $Q \cup \{q\}$ before q reached the plane of abc, and the triangles qab, qbc and qac are halving triangles after q leaves that plane. The line δ always crosses exactly two of these three triangles, which means that $H(q)$ *increases* by 2 in this subcase. By a symmetric reasoning, $H(q)$ *decreases* by 2 in subcase (c.ii). In each of these subcases, abc spans, depending on its orientation, an $(n-2)$-triangle and an $(n-1)$-triangle of Q_0. In case (c.i), q *enters* the $(n-2)$-triangle[10] spanned by abc, or more precisely, the ray on which q moves to q_0 enters this $(n-2)$-triangle. In case (c.ii), q *leaves* the $(n-2)$-triangle spanned by abc.

We have shown

$$H(q_0) = \min\{k, 2n-k\} + 2g_{n-2}(q_0, Q_0) \ .$$

So $H(q_0) \geq \min\{k, 2n-k\}$ iff $g_{n-2}(q_0, Q_0) \geq 0$. The latter is the assertion of (E) (note that $n-2 = \lceil (2n-4)/2 \rceil$). This completes the proof.

Remark 2 This implication does not hold if we consider the number of j-triangles of Q, for $j \leq n-2$, that are incident to q_0 and are crossed by π_0. In this case, $H(q)$ changes by $+2$ when q enters a $(j-1)$-triangle of Q_0 or when q leaves a j-triangle of Q_0, and $H(q)$ changes by -2 when q leaves a $(j-1)$-triangle of Q_0 or when q enters a j-triangle of Q_0. In this case we have

$$H(q_0) = 2\min\{j+1, n-2-j, k, 2n-k\} + 2\Big(g_j(q_0, Q_0) - g_{j-1}(q_0, Q_0)\Big),$$

which does not lead to the same implication as in the preceding proof.

4 Discussion

The purpose of this paper is to show the relation between the balanced lines problem (A) of [7], some problems involving halving triangles in 3-space, and the Generalized Lower Bound Theorem. This sheds some extra light on the result in [7]. It explains the difficulty in obtaining a purely combinatorial proof of (A), as experienced in [7]. It highlights the additional merit of the proof of [7], in providing, implicitly, the first purely combinatorial proof of the special case of the Generalized Lower Bound Theorem described in (C).

In doing so, we also obtained the property (D), which seems to be new, and can be regarded as another application of the machinery developed in [9].

Several interesting challenges remain.

- Can one obtain a direct and simpler proof of the balanced line result (A)?
- Can one obtain a purely combinatorial proof of the Generalized Lower Bound Theorem, beyond the special case established (indirectly) here?

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[10] That is, q approaches the plane of abc from the side that contains $n-2$ points of Q_0.

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