From Joints to Distinct Distances and Beyond: The Dawn of an Algebraic Era in Combinatorial Geometry

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A brief history of the universe

- György Elekes passed away in Septemer 2008

A brief history of the universe

- More that 10 years ago, he was thinking of
Erdős's distinct distances problem:

Estimate the smallest possible number of distinct distances determined by any set of s points in the plane

Show that this is always at least $\Omega(s/\sqrt{\log s})$
(Cannot be improved: The integer grid yields only these many distances)

A 1946 classic
A hard nut; Slow steady progress

A brief history of the universe, Cont'd

- Elekes found a nice transformation of the distinct distances problem to an incidence problem between points and curves (actually, lines) in 3D

- Formulated a couple of deep conjectures on the new setup (If proven, they yield the almost tight lower bound $\Omega(s/\log s)$)

- Sent the stuff to me

A brief history of the universe, Cont'd

- And we all went to sleep (Elekes in more ways than one)

- I woke up by an earthquake (the first one, 3 months after Elekes's death):

arXiv:0812.1043 (Dec 2008)
Title: Algebraic Methods in Discrete Analogs of the Kakeya Problem
Authors: Larry Guth, Nets Hawk Katz

History of the universe unfolds

- New algebraic machinery applied to an incidence problem in 3D (the joints problem, posed by me and others back in 1992)

- And solving it completely, after two decades of frustration

- We (H. Kaplan, E. Shustin, me) started to work on it, simplifying, extending, generalizing

- And then I tried to apply it to Elekes's transformed problem

History of the universe continues to unfold

- Discovered more geometric properties of the transformation

- Applied "successfully" the new algebraic machinery

- Bounds that are even better than those in the joints problem

- But not strong enough to have a real impact on distinct distances

- Presented the paper at SOCG (2010)

The end is near

- And then the earth shook again:

  arXiv:1011.4105 (Nov 2010)
  Title: On the Erdős distinct distances problem in the plane
  Authors: Larry Guth, Nets Hawk Katz

- They picked up the lead of the "Elekes–Sharir program" (That is, Elekes's transformation and conjectures) and managed to solve them completely

A new era is dawning

- This yielded the nearly tight bound $\Omega(s/\log s)$ for the distinct distances problem, after 65 years of frustration

- A new methodology that many (Kaplan, Matoušek, me, Solymosi, Tao, Zahl, . . ., ???) are trying to apply to other problems

- New proofs of old results (simpler, different)

And new results:

- Unit distances in three dimensions
  [Zahl], [Kaplan-Matoušek-Sharir]

- Point-circle incidences in three dimensions
  [in progress]

- Complex Szemerédi-Trotter incidence bound and related bounds
  [Solymosi-Tao]

- Range searching with semi-algebraic ranges
  [An algorithmic application; in progress]

Old-new Machinery from Algebraic Geometry

- Low-degree polynomial vanishing on a given set of points

- Polynomial ham sandwich cuts

- Polynomial partitioning

- Miscellany (Thom-Milnor, Bézout, Harnack, Warren, and co.)

- And just plain good old stuff from the time when men were men, women were women, and algebraic geometry was algebraic geometry (circa end of 19th century)

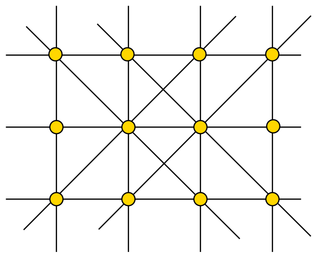
It is all about incidences between points and lines (or curves, or surfaces) in three or higher dimensions

Beat the Szemerédi-Trotter incidence bound in the plane

Szemerédi–Trotter:
Incidences between points and lines in the plane

P: Set of m distinct points in the plane
L: Set of n distinct lines

$I(P, L) =$ Number of incidences between P and L
$\quad = |\{(p, \ell) \in P \times L \mid p \in \ell\}|$



$I(m, n) = \max\{I(P, L) \mid |P| = m,\ |L| = n\}$

[Szemerédi–Trotter, 1983]: $I(m, n) = \Theta(m^{2/3}n^{2/3} + m + n)$

Do better in “truly 3-dimensional” scenarios

Miraculously, and without realizing it, this is what Elekes managed to do

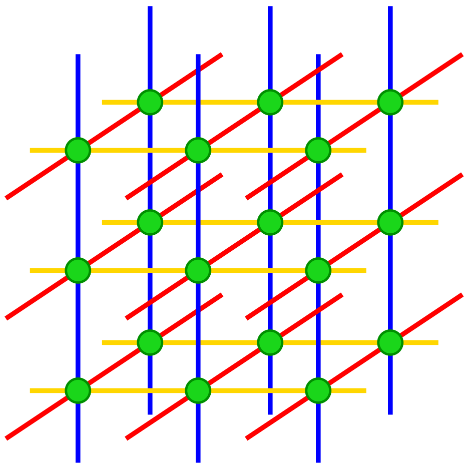
It started with Joints (in 3-space)

L – Set of n lines in $\mathbb{R}^3$

Joint: Point incident to (at least) three non-coplanar lines of L

The Joints Problem [Chazelle et al., 92]. Show:
The number of joints in L is $O(n^{3/2})$

Worst-case tight:



Joints

The maximum number of joints in a set of n lines in 3D is $\Theta(n^{3/2})$
[Guth, Katz 08]

And in d dimensions
(Joint = point incident to at least d lines, not all on a hyperplane)
$\Theta(n^{d/(d-1)})$
[Kaplan, Sharir, Shustin 10],
[Quilodrán 10]
(Similar, and very simple proofs)

And now to something completely different: Distinct Distances

An [Erdős 46] classic

How many distinct distances are always determined by any set of s points in the plane?

$d(S) =$ Number of distinct distances determined by a set S

$$g(s) = \min\left\{ d(S) \mid |S| = s \right\}$$

Distinct Distances

The $\sqrt{s} \times \sqrt{s}$ grid gives $g(s) = O(s/\sqrt{\log s})$ [Erdős 46]

Erdős conjectured this to be tight

Inching upwards over the ages, best lower bound was
$g(s) = \Omega(s^{0.8641}) \approx \Omega\left(s^{\frac{48-14e}{55-16e}}\right)$ [Katz, Tardos 04]

From Distinct Distances to Incidences in 3D
Elekes's Transformation

S: Ground set of s points in the plane

$x = d(S)$: Number of distinct distances in S

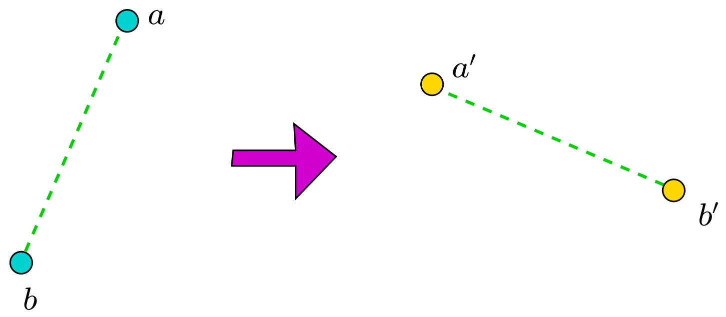
$\delta_1, \ldots, \delta_x$: The distinct distances

$E_i = \{(a, b) \mid dist(a, b) = \delta_i\}$

From Distinct Distances to Incidences in 3D

There is a rotation (rigid motion) mapping a to a' and b to b'

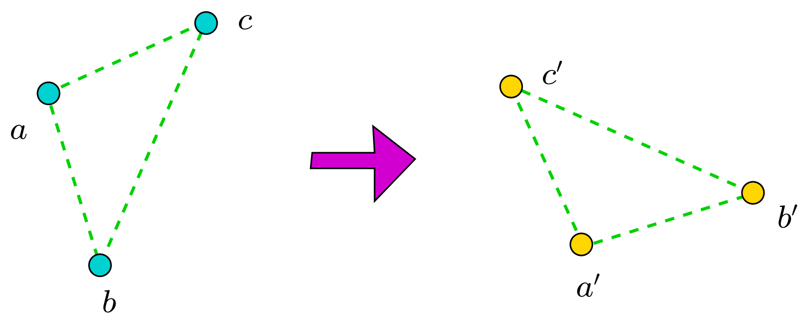
$\Leftrightarrow dist(a,b) = dist(a',b')$



Every quadruple (a,b,a',b') in $E_i \times E_i$ generates such a rotation

From Distinct Distances to Incidences in 3D

Multiplicity of a rotation τ:
$|\tau(S) \cap S|$ = Number of points of S mapped by τ to other points of S



From Distinct Distances to Incidences in 3D

N_k: Number of rotations of multiplicity k

$N_{\geq k}$: Number of rotations of multiplicity at least k

$$\sum_{k\geq 2} \binom{k}{2} N_k = \sum_{i=1}^{x} |E_i|(|E_i| - 1)$$

Both sides count the 5-tuples (τ, a, b, c, d), with $\tau(a) = c$ and $\tau(b) = d$

From Distinct Distances to Incidences in 3D

$$\sum_{k\geq 2}\binom{k}{2}N_k = \sum_{i=1}^{x}|E_i|(|E_i|-1)$$

Cauchy-Schwarz for RHS:

$$\sum_{i=1}^{x}|E_i|(|E_i|-1) = \Omega\left(\frac{1}{x}\left(\sum_i |E_i|\right)^2\right) = \Omega\left(\frac{s^4}{x}\right)$$

And rearranging LHS:

$$\sum_{k\geq 2}\binom{k}{2}N_k = N_{\geq 2} + \sum_{k\geq 3}(k-1)N_{\geq k}$$

so $\quad N_{\geq 2} + \sum_{k\geq 3}(k-1)N_{\geq k} = \Omega\left(\frac{s^4}{x}\right)$

From Distinct Distances to Incidences in 3D

$$N_{\geq 2} + \sum_{k \geq 3} (k-1) N_{\geq k} = \Omega\left(\frac{s^4}{x}\right)$$

Challenge: Upper bound LHS by $\approx s^3$

Main Conjecture (Elekes): $N_{\geq k} = O(s^3/k^2)$

If true $\Rightarrow \frac{s^4}{x} = O\left(\sum_k \frac{s^3}{k}\right) = O\left(s^3 \log s\right)$

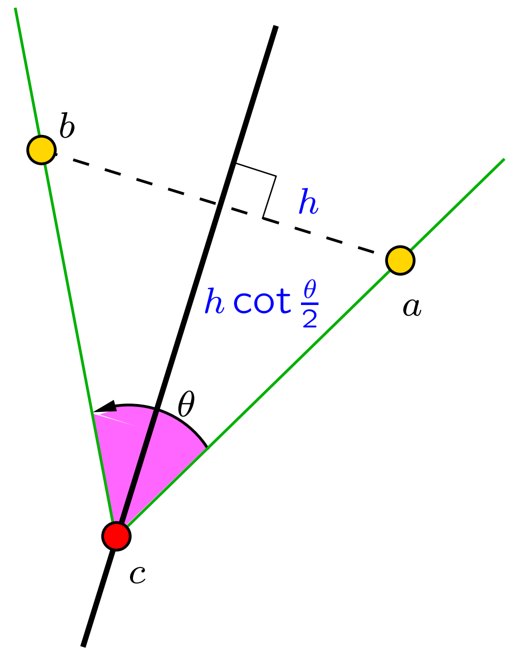
Or $x = \Omega(s/\log s)$

From Distinct Distances to Incidences in 3D

A rotation (rigid motion) has three degrees of freedom

If not pure translation:
Can always be represented as pure rotation (around some center)

Represent τ by $\left(c, \cot\frac{\theta}{2}\right)$, where
c = center of rotation
θ = angle of rotation



From Distinct Distances to Incidences in 3D

Rotation $\mapsto$ Point in 3D

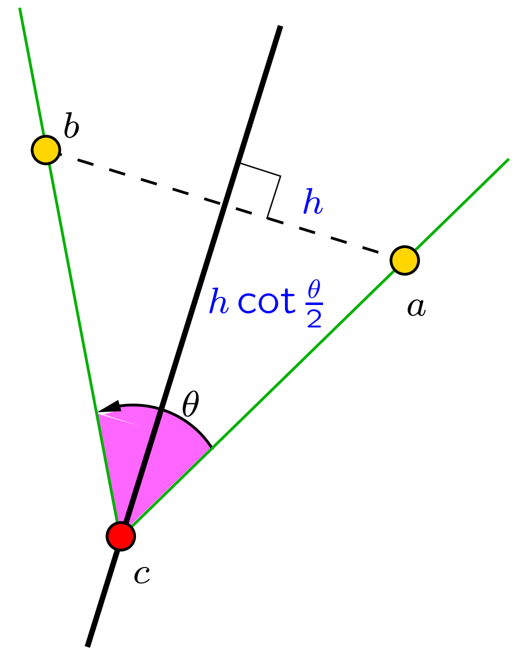
$\ell_{a,b}$: Locus of rotations that map a to b

$\ell_{a,b} \mapsto$ Line in 3D

(An observation of Guth and Katz that we missed:

Elekes thought they were helices

I thought they were parabolas)



The New Setup

$n = s^2$ lines in 3-space

m rotations (points) in 3-space

A rotation with multiplicity k is incident to k lines

Reduced Problem: Show that the number of points incident to $\geq k$ lines is

$$O\left(\frac{s^3}{k^2}\right) = O\left(\frac{n^{3/2}}{k^2}\right)$$

The New Setup

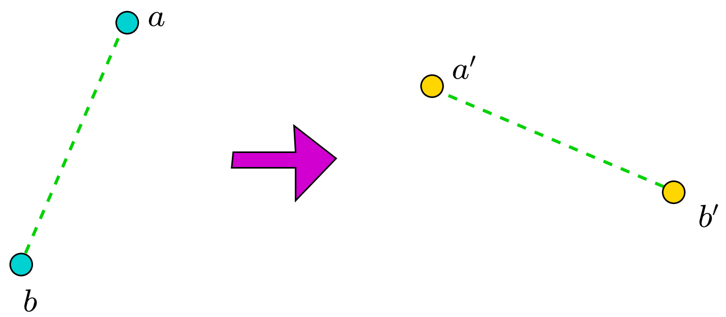
Main Conjecture [Elekes-Sharir, 2010]:
Now **Main Theorem** [Guth-Katz, 2010]:

$$N_{\geq k} = O\left(s^3/k^2\right) = O\left(n^{3/2}/k^2\right)$$

In more generality than the Elekes setup:
Arbitrary points and lines (with some restrictions)

(We [Elekes-Sharir] only managed to prove $N_{\geq k} \approx O\left(\frac{s^3}{k^{12/7}}\right)$,
and only for $k \geq 3$)

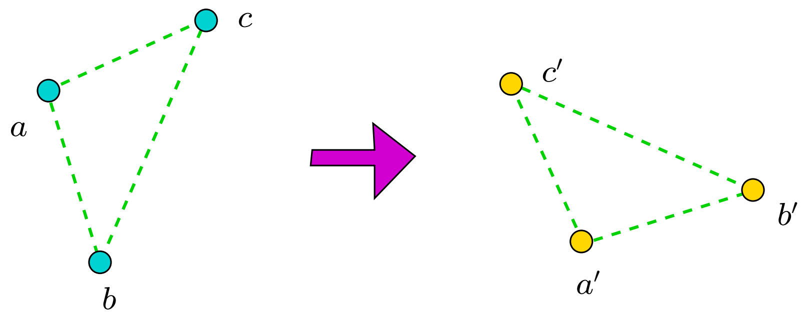
The case $k = 2$



•• Number of rotations (rigid motions) which map (at least) two points of S to two other points of S is $O(s^3)$

• Hard special case; requires separate treatment
(Left open in [Elekes-Sharir 2010])
Challenge: Find direct proof?

The case $k = 3$



●● Number of rotations (rigid motions) which map (at least) three points of S to three other points of S is $O(s^3)$

(Already shown in [Elekes-Sharir 2010])

Line-Point Incidences in $\mathbb{R}^3$

Theorem: (implied by [Guth-Katz 10])
For a set P of m points
And a set L of n lines in $\mathbb{R}^3$, such that
(i) Each point of P is incident to at least three lines
(ii) No plane contains more than $O(n^{1/2})$ lines
(Holds in the Elekes setup)

$$\max I(P, L) = \Theta(m^{1/2}n^{3/4} + n)$$

[Elekes, Kaplan, Sharir 09]
Same setup, but **(ii)** No plane contains more than $O(n)$ points

$$\max I(P, L) = \Theta(m^{1/3}n) \quad \text{for } m \geq n$$

New ingredients of the proof

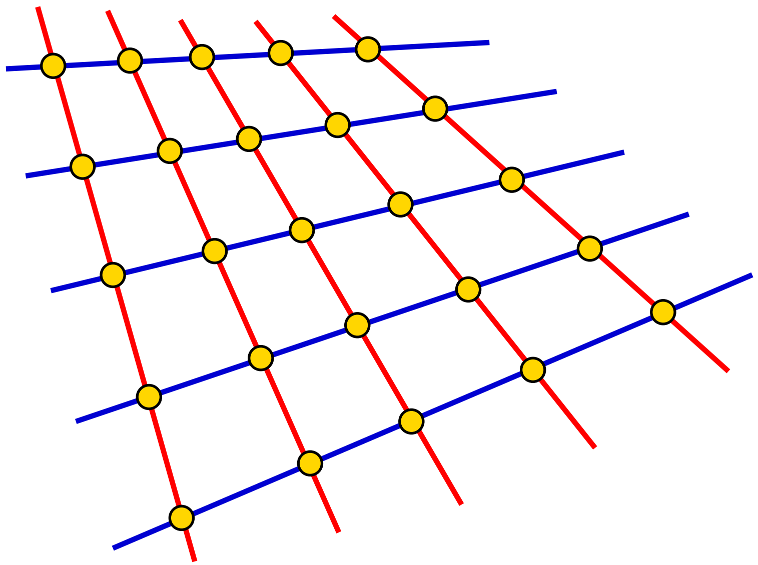
- Separate treatment of $k = 2$:

A set L of n lines in 3D, with at most $O(n^{1/2})$ lines in a plane or on a regulus, has at most $O(n^{3/2})$ intersection points

- Uses 19th century algebraic geometry [Salmon 1882 / Cayley]

Related to ruled surfaces

Avoid "complete bipartite" scenarios



Solution: Don't put all your eggs in the same basket
(Only $\sqrt{n}$ of them)

New ingredients of the proof: The case $k \geq 3$

Theorem: For a set L of n lines in 3D, so that no plane contains more than $O(\sqrt{n})$ lines, the number of points incident to at least k lines of L is at most $O(n^{3/2}/k^2)$

Follows from the incidence bound

$$mk = O(m^{1/2}n^{3/4}) \quad \Rightarrow \quad m = O(n^{3/2}/k^2)$$

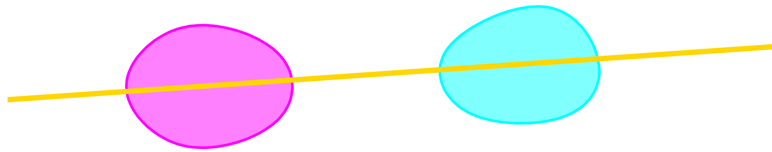
Prove the incidence bound using polynomial partitions

Via the polynomial ham sandwich theorem

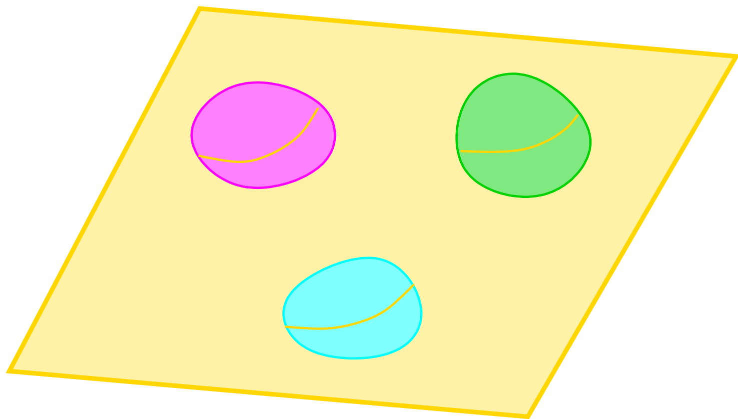
Standard Ham Sandwich Cut

Any d sets in $\mathbb{R}^d$ can be simultaneously bisected by a common hyperplane

2D:

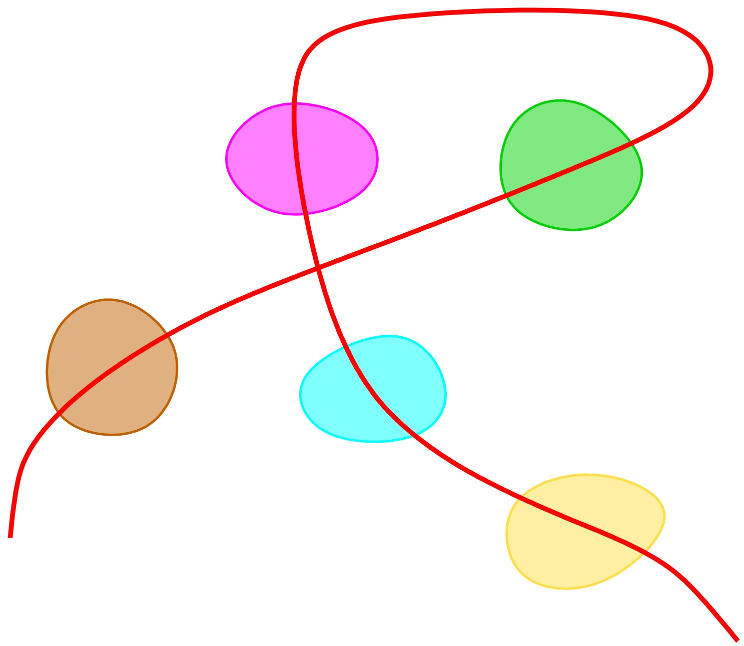


3D:



Polynomial Ham Sandwich Cut

Bisect more sets by the zero set of a higher-degree polynomial



Polynomial Ham Sandwich Cut

[Stone, Tukey, 1942]

d = dimension
D = degree
Put $M = \binom{d+D}{d} - 1$
Let $S_1, \ldots, S_M$ be any M sets in $\mathbb{R}^d$
Then there exists a polynomial p of degree at most D which bisects each of $S_1, \ldots, S_M$

$$\text{Bisects}: \quad |S_i \cap \{p < 0\}|, \ |S_i \cap \{p > 0\}| \leq |S_i|/2$$

Moral: If you want to bisect M sets in d dimensions,
Use a polynomial of degree $D \approx M^{1/d}$

Polynomial Ham Sandwich Cut

Frightening as it sounds, it is trivial nonetheless:

$M = \binom{d+D}{d} - 1$ = Num of nonconstant monomials of a d-variate polynomial of degree D

Apply the Veronese map $V : \mathbb{R}^d \mapsto \mathbb{R}^M$

$V(x) =$ tuple of the value at x of all M monomials of degree $\leq D$

Then bisect each of $V(S_1), \ldots, V(S_M)$ by a hyperplane h in $\mathbb{R}^M$ (Standard Ham Sandwich Cut)

Polynomial Ham Sandwich Cut (PHSC)

$p = h \circ V$ is the desired bisecting polynomial:

Linear combination of monomials

Polynomial partitioning of a point set via PHSC

A set S of n points in $\mathbb{R}^d$ can be partitioned into t subsets, each consisting of at most n/t points, by a polynomial p of degree $D = O\left(t^{1/d}\right)$, so that the subsets lie in distinct connected components of $\mathbb{R}^d \setminus Z$

Sketch: Apply PHSC to S, then to the two halves, then to the four quarters, etc., until obtaining the desired number of subsets

Sequence of bisecting polynomials $p_1, p_2, \ldots$
Of degrees $\approx 1^{1/d}, 2^{1/d}, 4^{1/d}, \ldots, t^{1/d}$

Multiply them to get the desired $p = p_1 p_2 \cdots$
Of degree $\approx 1^{1/d} + 2^{1/d} + 4^{1/d} + \cdots + t^{1/d} = O\left(t^{1/d}\right)$

Polynomial partitioning

- A new kind of space decomposition
- Competes (very favorably) with cuttings, simplicial partitioning
- Many advantages (and some (temporary?) disadvantages)
- Main new tool to take home

Szemerédi–Trotter planar incidence bound

Proof Via polynomial partitioning

Recall:

P: Set of m distinct points in the plane
L: Set of n distinct lines

$I(P, L) = O(m^{2/3}n^{2/3} + m + n)$

For simplicity, assume $m = n$

Partition P into $t = \Theta(n^{2/3})$ subsets, each consisting of at most $n/t = O(n^{1/3})$ points

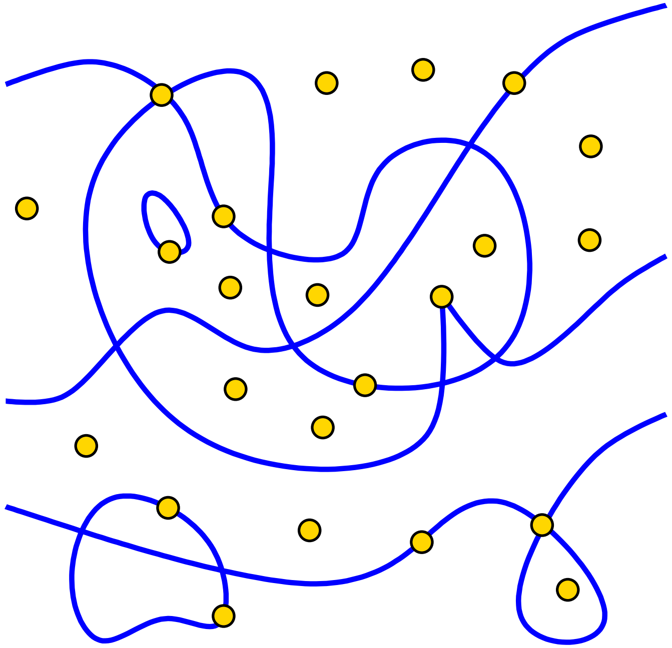
Using a polynomial q of degree $D = O(t^{1/2}) = O(n^{1/3})$

$Z = Z(q)$: Zero set of q

Proof, cont'd

Partitions P into subsets $P_0, P_1, \ldots, P_t$:

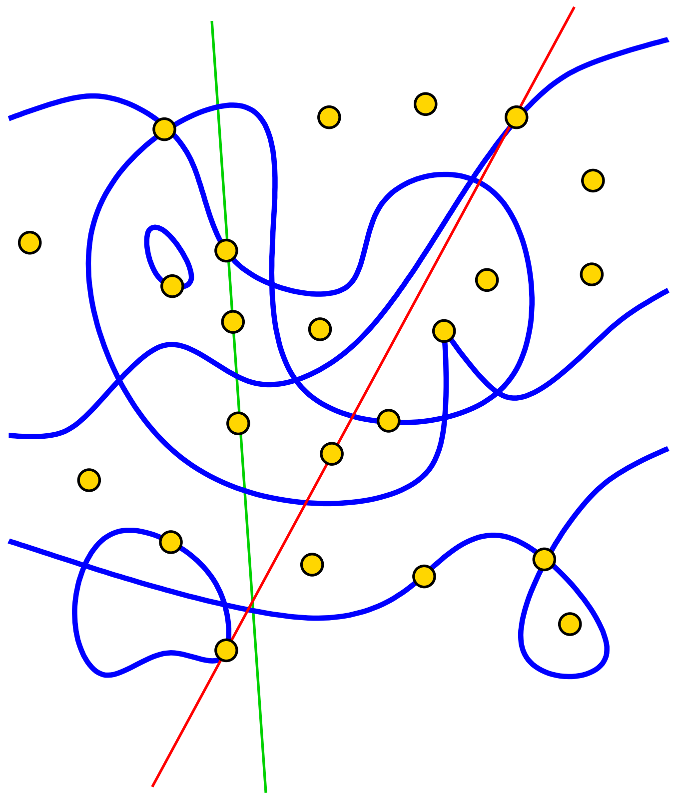
- $P_0 \subset Z$
- $|P_i| = O(n/t) = O(n^{1/3})$ for $i \geq 1$
- Each component of $\mathbb{R}^2 \backslash Z$ contains at most one P_i



Proof, cont'd

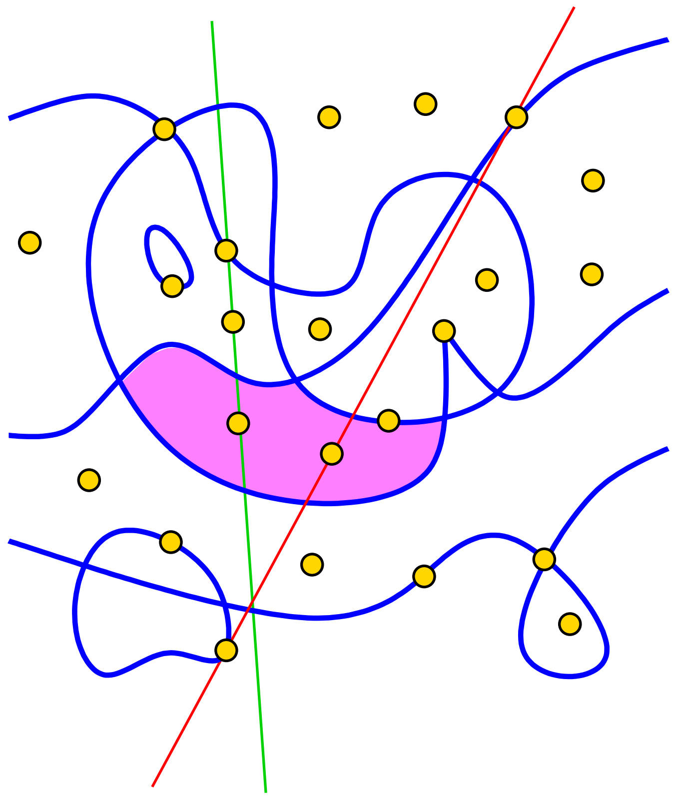
A line $\ell \in L$ interacts with at most $D + 1 = O(n^{1/3})$ subsets P_i

(In between, has to cross Z, in at most D points)



Proof, cont'd

- ℓ has just one incidence with P_i: Only $O(nD) = O(n^{4/3})$ such incidences

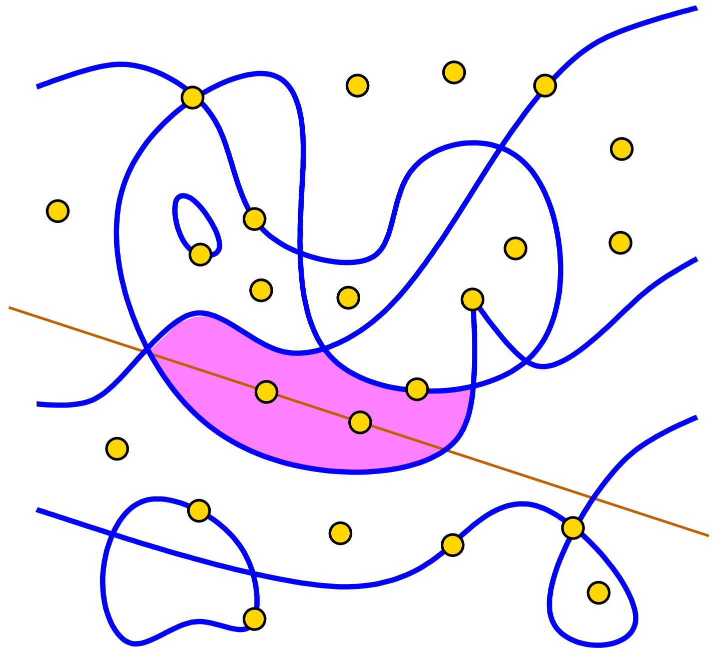


Proof, cont'd

- ℓ has at least two incidences with P_i:

Each $p \in P_i$ is incident to at most $|P_i| - 1$ such lines

For a total of $\leq |P_i|(|P_i| - 1)$ incidences



Num of incidences:

$O((n^{1/3})^2)$ within one P_i

Times $O(n^{2/3})$ sets P_i

$= O(n^{4/3})$

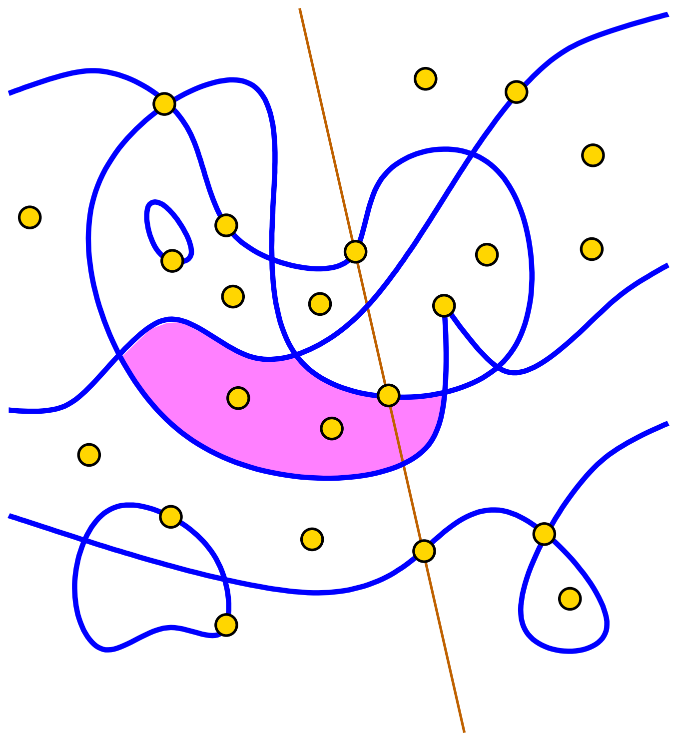
Proof, cont'd

- Incidences with P_0 (points on Z):

ℓ intersects Z in at most D points
(unless $\ell \subset Z$)

$O(nD) = O(n^{4/3})$ incidences

(The case $\ell \subset Z$ is also easy:
Only D such lines;
$O(nD) = O(n^{4/3})$ incidences)



Incidences in three dimensions

- Same approach

- Partitioning by a polynomial of degree D:
  $\approx D^2$ subsets in the plane
  $\approx D^3$ subsets in 3-space
  Finer partition $\Rightarrow$ Better bound

- But harder to handle points and lines on the zero set Z

- Still, much simpler variant of the technique in
[Elekes, Kaplan, Sharir 10]

The Dawn of a New Era

- New, unexpected, powerful machinery

- Still early in the game to assess full impact;
Hope is unbounded

- Better bridges between the two communities
(Algebraic / combinatorial geometry)

Thank You