Assignment 4  
Advanced Topics in Computational Geometry  

Due: May 29, 2012

Problem 1

Show that the complexity of the union of $m$ convex polytopes in three dimensions, that have a total of $n$ facets, such that they all contain the origin in their interior, is $O(n^2\alpha(n))$. Give a construction that shows that this bound is tight in the worst case.

Problem 2

(a) Let $L$ be a set of $n$ lines in the plane (in general position). Show that $\sum_c |c|^2 = \Theta(n^2)$, where the sum is over all cells $c$ of $A(L)$, and where $|c|$ is the complexity of $c$ (e.g., number of vertices). (Hint: Use the zone theorem.)

(b) Let $H$ be a set of $n$ planes in three dimensions. Show that $\sum_c |c|^2 = \Theta(n^3)$, where the sum is over all cells $c$ of $A(H)$, and where $|c|$ is the total complexity of $c$ (i.e., number of vertices, edges, and faces of $c$). (Hint: Use the zone theorem.)

(c) The analysis does not work in four or higher dimensions. Explain why, and show that for any dimension $d$ we have instead $\sum_c |c| \cdot |c|_{d-1} = \Theta(n^d)$, where $|c|_{d-1}$ is the number of $(d-1)$-dimensional faces (facets) bounding $c$.

(d) Using (a), show that the total complexity of $m$ arbitrary faces of $A(L)$ is $O(m^{1/2}n)$. (Hint: Cauchy-Schwarz!)

Problem 3

Let $L$ be a set of $n$ lines in the plane (in general position). Call a face $f$ of $A(L)$ balanced if the highest vertex of $f$ (in the $y$-direction) is not the leftmost or rightmost vertex of $f$; otherwise $f$ is tilted.

Using the Clarkson-Shor technique, show that the number of balanced faces at level at most $k$ is $O(k^2)$, and that the bound is tight in the worst case.
Problem 4

Let $\mathcal{F}$ be a collection of $n$ disks in three dimensions, none of which is vertical. Regard these disks as the graphs of $n$ partially defined bivariate functions, and analyze the complexity of the lower envelope of these functions, applying a (simpler) variant of the technique shown in class (or any technique of your choice). Give a construction showing that the complexity of the envelope can be $\Omega(n^2)$, even when the disks are disjoint.