Problem 1

Let $L$ be a set of $n$ lines in the plane. For each triangle $\Delta$, let $L_\Delta$ denote the set of lines of $L$ that cross $\Delta$.

(a) Show that the number of different sets $L_\Delta$ is $O(n^6)$.

(b) Show that the number of triples $u, v, w$ of vertices of $A(L)$ such that no line crosses the triangle $uvw$ (ignoring lines that touch it at $u$, $v$, or $w$) is $O(n^3)$, and that this bound is tight in the worst case.

(c) Using (b), show that the number of triples $u, v, w$ of vertices of $A(L)$ such that the triangle $uvw$ is crossed by at most $k$ lines of $L$, is $O(n^3k^3)$.

Problem 2

Let $P$ be a set of $n$ points on the line, and let $R$ be a random sample of points of $P$, such that each point is chosen independently with probability $r/n$ (so the expected size of $R$ is $r$). Let $\varphi(R)$ denote the largest cardinality of $I \cap P$, over all intervals $I$ that do not contain any point of $R$. Show that, with high probability, $\varphi(R) \leq \frac{cn}{r} \log r$, for some sufficiently large constant $c$. (Show that the probability of the complementary event is at most $1/rk$, where $k$ depends on the constant $c$.)

(Hint: Fix an interval $I$ for which $|I \cap P| \geq \frac{cn}{r} \log r$, and estimate the probability that it does not contain any point of $R$. Define the collection of such intervals carefully and use the probability union bound.)

Problem 3

(a) Let $S$ be a set of $n$ non-vertical line segments in the plane (in general position). We insert the elements of $S$ one by one in a random order, and maintain the lower envelope of $S$ as we go, so that after each insertion we update the envelope, to reflect the presence of the new segment.

Show that the expected number of vertices that the algorithm generates is $O(n\alpha(n))$. 

(b) Design an efficient algorithm that computes the lower envelope, using the above randomized incremental insertion, so that its expected running time is $O(n\alpha(n) \log n)$.  
(Hint: Maintain the vertical decomposition of the portion of the plane below the envelope into a collection of vertical semi-unbounded trapezoids.)

**Note:** Do not cite what we did in class. This is a special simple variant of the general case, and the exercise is to solve it “from scratch” in a (somewhat) simpler manner.

**Problem 4**

Consider the incremental algorithm for constructing a single cell in an arrangement of $n$ arcs, as presented in class. Use the DAG that the algorithm maintains as a data structure for answering the following type of queries: Given a query point $q$, does it belong to the cell?

(a) Describe how the query is answered.

(b) Show that, for any fixed $q$, the expected cost of the query is $O(\log n)$.

(Hint: Estimate the probability that the trapezoid containing $q$ after $i - 1$ insertions is different from the trapezoid after the insertion of the $i$-th arc, using a simple variant of the analysis given in class.)

(c) What is the expected size of the structure and the expected preprocessing cost?

**Problem 5**

Let $P$ be a simple polygon with $n$ edges, and let $K$ be the Minkowski sum of $P$ with a unit disk.

(a) Describe the shape and structure of the boundary of $K$, and show that its complexity (number of its vertices, segments, and arcs) is $O(n)$.

(Hint: Use (possibly with modifications) Problem 4 of the previous assignment.)

(b) Solve Problem 4(d) of the previous assignment.