Problem 1

Davenport-Schinzel sequences of order 2 and triangulations: Let $P$ be any convex polygon with $n$ vertices. A triangulation of $P$ is a collection of $n - 3$ non-intersecting chords connecting pairs of vertices of $P$ and partitioning $P$ into $n - 2$ triangles. Set up a correspondence between such triangulations and $DS(n - 1, 2)$ sequences, as follows. Number the vertices $1, 2, \ldots, n$ in their order along $\partial P$. Let $T$ be a given triangulation. Include in $T$ the edges of $P$ too. For each vertex $i$, let $T(i)$ be the sequence of vertices $j < i$ connected to $i$ in $T$ and arranged in decreasing order, and let $U_T$ be the concatenation of $T(2), T(3), \ldots, T(n)$.

(a) Show that $U_T$ is a $DS(n - 1, 2)$ sequence of maximum length.

(b) Show that any $DS(n - 1, 2)$-sequence of maximum length can be realized in this manner, perhaps with an appropriate renumbering of its symbols.

(c) Use (a) and (b) to show that the number of different $DS(n, 2)$ sequences of maximum length is $\frac{1}{n-1} \binom{2n-4}{n-2}$ (where two sequences are different if one cannot obtain one sequence from the other by renumbering its symbols).

(Note: Obviously, you have to show that this is the number $F(n)$ of triangulations of $P$. Show it, e.g., by deriving a recurrence relation on $F(n)$ and solving it inductively.)

Problem 2

(a) Show that $\lambda_3(n) \geq 5n - 8$.

(b) Show that the lower bound in (a) can be realized as the lower envelope sequence of $n$ segments.
Problem 3

Let $F$ be a collection of $n$ partially-defined and continuous functions over the reals. Suppose that $F$ is the disjoint union of $c$ subcollections $F_1, \ldots, F_c$, such that (i) Any pair of functions in $F_i$ intersect in at most $s_i$ points, for $i = 1, \ldots, c$. (ii) Any pair of functions of $F$ intersect in at most $s$ points. Here $c$, $s$, and the $s_i$'s are all constants.

(a) Show that the complexity of the lower envelope of $F$ is $O(\lambda_q(n))$, where $q = \max_{i=1}^c s_i$.

(b) Show that the envelope can be computed in time $O(\lambda_{q+1}(n) \log n)$.

(c) As a corollary, what is the complexity of the lower envelope of a collection of $n$ line segments and unit-radius circles? (For the purpose of the lower envelope, each circle can be replaced by its lower semi-circle.)

Problem 4

Let $P = \{p_1(t), \ldots, p_n(t)\}$ be a set of $n$ moving points in the plane, so that each point $p_i$ moves at some constant speed $v_i$ along some line $\ell_i$. At time $t$, let $q = q(t)$ be the point of $P$ that satisfies: (i) $q$ lies in the right halfplane $x \geq 0$, and (ii) the segment $\overrightarrow{oq}$ ($o$ is the origin) has the largest slope (among all such segments for points in the right halfplane). If no point of $P$ lies in $x \geq 0$ at time $t$ then $q$ is undefined. There are discrete times when more than one point satisfies (i) and (ii).

(a) Obtain an upper bound on the number of discrete changes in $q(t)$ (including transitions from being defined to being undefined and vice versa).

(b) Using (a), and its symmetric version for points in the left halfplane $x \leq 0$, to obtain an upper bound on the number of discrete changes in the left or right neighbors of $o$ on the upper boundary of the convex hull of $\{o\} \cup P$ (including transitions where $q$ starts or stops being a vertex of the upper hull).

Problem 5

Given $n$ ellipses in the plane, all containing the origin. What is the combinatorial complexity of their union? What is the complexity of their intersection?