Counting Plane Graphs: Cross-Graph Charging Schemes*

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Abstract. We study cross-graph charging schemes for graphs drawn in the plane. These are charging schemes where charge is moved across vertices of different graphs. Such methods have been recently applied to obtain various properties of triangulations that are embedded over a fixed set of points in the plane. We show how this method can be generalized to obtain results for various other types of graphs that are embedded in the plane. Specifically, we obtain a new bound of[1] $O^*\left(187.53^N\right)$ for the maximum number of crossing-free straight-edge graphs that can be embedded over any specific set of N points in the plane (improving upon the previous best upper bound 207.85^N in Hoffmann et al. [14]). We also derive upper bounds for numbers of several other types of plane graphs (such as connected and bi-connected plane graphs), and obtain various bounds on expected vertex-degrees in graphs that are uniformly chosen from the set of all crossing-free straight-edge graphs that can be embedded over a specific point set.
We then show how to apply the cross-graph charging-scheme method for graphs that allow certain types of crossings. Specifically, we consider graphs with no set of k pairwise-crossing edges (more commonly known as k-quasi-planar graphs). For $k = 3$ and $k = 4$, we prove that, for any set S of N points in the plane, the number of graphs that have a straight-edge k-quasi-planar embedding over S is only exponential in N.

1 Introduction

Background. Consider the following problem — given a set S of labeled points in the plane, no three collinear, what is the number of graphs that have a straight-edge crossing-free embedding over S? That is, we consider graphs whose vertex set is (or is mapped to) S and whose edges are drawn as straight segments connecting the corresponding pairs of points, so that these segments do not cross

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[1] In the notations $O^*()$, $\Theta^*()$, and $\Omega^*()$, we neglect polynomial factors.

each other (at a point in their relative interiors). For example, if S is a set of N points in convex position, the answer is known to be $\Theta\left((6+4\sqrt{2})^N\right) \approx \Theta\left(11.66^N\right)$ [11]. The more general problem asks for the maximum number of crossing-free straight-edge graphs that can be embedded over any specific set of N points in the plane. The first exponential bound, 10^{13N}, on the number of such graphs was proved by Ajtai et al. [4] back in 1982. Since then, progressively (and significantly) smaller upper bounds have been derived (for example, see [14, 18, 23]). Upper bounds on numbers of more specific types of crossing-free straight-edge graphs, such as Hamiltonian cycles, spanning trees, perfect matchings, and triangulations, were also studied (e.g., see [6, 7, 20, 21, 25]). Worst-case lower bounds for these numbers have also been addressed (e.g., see [3, 9, 12]).[2]

Research on the above problems has led to the development of several useful combinatorial techniques, many of which are interesting in their own right. One such distant achievement was the introduction of the *Catalan numbers* by Euler and Lamé [10, 15]. A more recent development was the first proof for the *crossing lemma*, presented by Ajtai et al. [4]. In this paper we discuss another novel combinatorial technique that has recently emerged from research on the above counting problems. Namely, this is the concept of *cross-graph charging schemes*.

The idea of applying charging schemes to obtain graph properties probably originated from the attempts of Heesch to prove the *four colors theorem* [13]. Later, his ideas were used in Appel and Haken's famous proof of the theorem [5], and their extensions have become a common technique in graph theory (e.g., see [2, 17]). This technique involves giving charges to vertices (or edges, or faces, for graphs drawn in the plane) of a graph G, and then moving these charges between various vertices (or edges, or faces) of G. The novel approach of moving such charges between vertices and edges *of different graphs over the same point set* originated by Sharir and Welzl in 2006 [25], in studying the maximum number of triangulations that can be embedded over a specific set of N points in the plane. Since then, this technique has been extended in [18, 19, 23, 24] to study various combinatorial and algorithmic properties of triangulations.

In this paper, we extend the idea of cross-graph charging schemes beyond the realm of triangulations. We first show how to apply this technique to bound the maximum number of crossing-free straight-edge graphs that can be embedded over a specific set of points in the plane. Then we show how to extend this idea to several other types of graphs, including families of non-planar graphs (this seems to be the first derivation of reasonable bounds for such graph types). It seems likely that these techniques can be further extended to other types of problems (that is, problems not involving bounding or counting the number of embedded graphs), and we hope that the present study will motivate such applications. Before discussing our results any further, we require some formal definitions of the concepts related to these problems.

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[2] We try to keep a comprehensive list of the various up-to-date bounds in a dedicated webpage `http://www.cs.tau.ac.il/~sheffera/counting/PlaneGraphs.html` (version of May 2012).

Notations and results. A *planar graph* is a graph that can be embedded in the plane in such a way that its vertices are embedded as points and its edges are embedded as Jordan arcs that connect the respective pairs of points and can meet only at a common endpoint. A *crossing-free straight-edge graph* is a plane embedding of a planar graph such that its edges are embedded as non-crossing straight line segments; we sometimes refer to such graphs simply as *plane graphs.* In Sect. 2 we only consider plane graphs. In Sect. 3 we allow certain types of crossings by considering *quasi-planar graphs*; here too we assume that the edges are embedded as (possibly crossing) straight line segments. In both sections we only consider embeddings where the points are in general position, that is, where no three points are collinear. For upper bounds on the number of graphs, this involves no loss of generality, because the number of graphs can only grow when a degenerate point set is slightly perturbed into general position.

A *triangulation* of a finite point set S in the plane is a maximal plane graph on S (that is, no additional straight edges can be inserted without crossing any of the existing edges). For a set S of points in the plane, we denote by $\mathcal{T}(S)$ the set of all triangulations of S, and put $\mathsf{tr}(S) := |\mathcal{T}(S)|$. Similarly, we denote by $\mathcal{P}(S)$ the set of all plane graphs of S, and put $\mathsf{pg}(S) := |\mathcal{P}(S)|$. Finally, let $\mathsf{tr}(N) = \max_{|S|=N} \mathsf{tr}(S)$ and $\mathsf{pg}(N) = \max_{|S|=N} \mathsf{pg}(S)$. So another way of formulating our problem is — find a small constant b (ideally, find the smallest) such that $\mathsf{pg}(N) = O^*(b^N)$. (By the results mentioned above, we know that such a b exists.)

Notice that every plane graph is contained in at least one triangulation. Also, by Euler's formula, every triangulation has fewer than $3|S|$ edges, and thus, every triangulation contains fewer than $2^{3|S|} = 8^{|S|}$ plane graphs. From the above we have the inequality $\mathsf{pg}(S) < 8^{|S|} \cdot \mathsf{tr}(S)$, which implies $\mathsf{pg}(N) < 8^N \cdot \mathsf{tr}(N)$. Every several years an improved upper bound for $\mathsf{tr}(N)$ is discovered (e.g., see [8, 22, 25]), and currently, the best known bound is $\mathsf{tr}(N) < 30^N$ [23]. Combining this bound with the above inequality implies $\mathsf{pg}(N) < 240^N$. Currently, the best known lower bound is $\mathsf{pg}(N) = \Omega(41.18^N)$ [3].

The inequality $\mathsf{pg}(N) < 8^N \cdot \mathsf{tr}(N)$ seems rather weak, since it potentially counts some plane graphs many times. Razen, Snoeyink, and Welzl [18] were the first to address this inefficiency, deriving the slightly improved inequality $\mathsf{pg}(N) = O\left(7.9792^N\right) \cdot \mathsf{tr}(N)$. A more significant improvement of $\mathsf{pg}(N) < 6.9283^N \cdot \mathsf{tr}(N)$ was recently obtained by Hoffmann et al. [14]. This implies the bound $\mathsf{pg}(N) < 207.85^N$.

As far as we know, our cross-graph charging-scheme method is currently the only method that does not rely on the ratio between $\mathsf{pg}(N)$ and $\mathsf{tr}(N)$ and yields a non-astronomical bound. An initial, more direct application of this method implies only a bound of 3207.42^N. On the other hand, by combining this method with the current bound on the number of triangulations (indirectly, by using an upper bound on the maximum number of plane graphs with at least cN edges, which is derived in [14] and relies on $\mathsf{tr}(N)$) we obtain $\mathsf{pg}(N) = O^*\left(187.53^N\right)$.

Our method relies on charging schemes between objects from different plane graphs over the same point set (hence the name *cross-graph charging schemes*).

Given a set S of N points in the plane, we consider the set $S \times \mathcal{P}(S)$ and call each of its elements a *ving* (*v*ertex *in* *g*raph, similar to the definition of a *vint* (*v*ertex *in* *t*riangulation) from [23–25]). Intuitively, a ving is an instance of a vertex in a specific plane graph. Our charging schemes are between vings from different graphs (sharing a common vertex).

A *k-quasi-plane* graph is a straight-edge graph over a set of points in the plane that may contain crossings, but does not contain any set of k pairwise crossing edges (some other works, such as [2], refer to such graphs as *k-quasi-planar geometric graphs*). Notice that a 2-quasi-plane graph is simply a plane graph.

For a set S of points in the plane, we denote by $\mathcal{Q}_k(S)$ the set of all k-quasi-plane graphs on S, and put $\mathsf{qp}_k(S) := |\mathcal{Q}_k(S)|$. Moreover, we let $\mathsf{qp}_k(N) = \max_{|S|=N} \mathsf{qp}_k(S)$. As far as we know, there are no known singly exponential upper bounds on $\mathsf{qp}_k(N)$, for any $k \geq 3$. We show that an appropriate extension of our technique easily implies the bounds $\mathsf{qp}_3(N) \leq 2^{26N}$ and $\mathsf{qp}_4(N) \leq 2^{145N}$. These bounds are probably very far from being tight, but our purpose here is to show that the number of 3-quasi-plane graphs, say, that can be embedded over a specific point set is only exponential in the number of points (note that, the first bound, on $\mathsf{qp}_3(N)$, is significantly smaller than the first exponential bound — 10^{13N} — that was obtained for the number of plane graphs [4]). We also show that the main conjecture about quasi-planar graphs (namely, that the number of their edges is linear for any fixed k, e.g., see [1, 2, 16]) would imply, if true, that $\mathsf{qp}_k(N)$ is (only) exponential in N for any fixed k.

2 An Upper Bound on the Number of Plane Graphs

In this section we derive upper bounds on the number of plane graphs. In Subsect. 2.1, we derive the initial bound $\mathsf{pg}(N) \leq 4096^N$. In Subsect. 2.2, we exploit some geometric aspects of the problem, to improve the bound to $\mathsf{pg}(N) \leq 3207.42^N$. Even though this is far worse than the recent bound $\mathsf{pg}(N) < 207.85^N$ [14], it constitutes a significant progress in deriving bounds that do not depend on $\mathsf{tr}(N)$. In Subsect. 2.3, we extend our technique to obtain the bound $\mathsf{pg}(N) = O^*\left(187.53^N\right)$, which is currently the best known upper bound for this quantity. This extension is a combination of our technique with some recently obtained bounds on the number of certain types of plane graphs (from [14]). These latter bounds do depend on the number of triangulations, but the way we exploit these bounds makes our new bound (that is, $O^*\left(187.53^N\right)$) depend *non-linearly* on $\mathsf{tr}(N)$; see below for details. In Subsect. 2.4, we present a variety of additional results that can be obtained by simple extensions of our technique.

2.1 The Infrastructure and an Initial Bound

Given two vertices p, q of a plane graph G, we say that p *sees* q in G if the (straight) edge pq does not cross any edge of G. The *degree* of a ving (p, G) is the degree (number of neighbors) of p in G; a ving of degree i is called an *i-ving*.

We say that a ving $v = (p, G)$ is an *x-ving* if we cannot increase the degree of p by inserting additional (straight) edges to G (that is, every vertex that is not connected to p in G cannot see p in G). We say that a ving $u = (p, G')$ *corresponds* to the x-ving $v = (p, G)$ if G is obtained by inserting into G' all the edges that connect p to the points that it sees in G' and is not connected to them. Notice that every ving corresponds to a unique x-ving. Given a plane graph $G \in \mathcal{P}(S)$, we denote by $v_i(G)$ the number of i-vings in G, for $i \geq 0$, and by $v_x(G)$ the number of x-vings in G. Finally, the expected value of $v_x(G)$, for a graph chosen uniformly at random from $\mathcal{P}(S)$, is denoted as $\hat{v}_x(S)$. More formally, $\hat{v}_x = \hat{v}_x(S) := \mathbb{E}\{v_x(G)\} = \dfrac{\sum_{G \in \mathcal{P}(S)} v_x(G)}{\mathsf{pg}(S)}$.

The following lemma, inspired by similar lemmas in [23–25], presents a connection between $\hat{v}_x$ and upper bounds for $\mathsf{pg}(N)$.

Lemma 1. *For $N \geq 2$, let $\delta_N > 0$ be a real number, such that $\hat{v}_x(S) \geq \delta_N N$ holds for every set S of N points in the plane. Then $\mathsf{pg}(N) \leq \dfrac{1}{\delta_N} \mathsf{pg}(N-1)$.*

Proof. Let S be a set that maximizes $\mathsf{pg}(S)$ among all sets of N points in the plane. Note that we can get some plane graphs of S by choosing a point $q \in S$ and a plane graph G of $S \setminus \{q\}$, inserting q into G, and then connecting q to all of the vertices that it can see in G. In fact, a plane graph G of S can be obtained in exactly $v_x(G)$ ways in this manner (in particular, if $v_x(G) = 0$, G cannot be obtained at all in this fashion). This is easily seen to imply that

$$\hat{v}_x \cdot \mathsf{pg}(S) = \textstyle\sum_{G \in \mathcal{P}(S)} v_x(G) = \sum_{q \in S} \mathsf{pg}(S \setminus \{q\}) \ .$$

The leftmost expression equals $\hat{v}_x \cdot \mathsf{pg}(N)$, and the rightmost expression is at most $N \cdot \mathsf{pg}(N-1)$. Hence, with $\hat{v}_x \geq \delta_N N$, we have $\mathsf{pg}(N) = \mathsf{pg}(S) \leq \dfrac{N}{\hat{v}_x} \cdot \mathsf{pg}(N-1) \leq \dfrac{1}{\delta_N} \cdot \mathsf{pg}(N-1)$. ◻

We thus seek a lower bound for $\hat{v}_x$, of the kind assumed in Lemma 1. For this purpose, we use a charging scheme similar in spirit to the one presented in [23–25]. The following lemma establishes such a bound, which is rather weak. Nevertheless, it has the advantage of being a "stand-alone" bound, independent of the bound on the number of triangulations on S. In the following subsections, we will derive a considerably improved bound, which does depend on the known bounds on the number of triangulations of S (albeit in a nonlinear manner).

Lemma 2. *For every point set S of N points in the plane, $\hat{v}_x(S) \geq \dfrac{N}{4096}$.*

Proof. We use a charging scheme where every i-ving $v = (p, G)$ is given $7 - i$ units of charge. The sum of the charges of the vings in any fixed plane graph $G \in \mathcal{P}(S)$ is $\sum_i (7-i) v_i(G) = 7 \sum_i v_i(G) - \sum_i i v_i(G) = 7N - \sum_i i v_i(G)$. Since G can have at most $3N - 6$ edges, we have $\sum_i i v_i(G) \leq 6N - 12$. This implies that the total charge in any fixed graph is at least $7N - \sum_i i v_i(G) \geq N + 12$. Therefore, on average, every ving has a charge larger than 1.

Next, every i-ving moves its entire charge to its corresponding x-ving. (In general, the x-ving lies in a graph different than the one containing the i-ving.) This results with all of the charge being placed only on x-vings. If we can show that every x-ving gets charged at most t units in this manner, we will get the lower bound $\hat{v}_x \geq N/t$, as is easily verified.

To upper bound the charge that an x-ving $v = (p, G)$ can get, we need to consider the degree d of v. Notice that the number of i-vings that charge v is exactly $\binom{d}{i}$ (that is, the number of ways to remove $d - i$ edges that are adjacent to p in G). Therefore, the total charge to v is

$$\sum_{0 \leq i \leq d} \binom{d}{i}(7 - i) = 7 \sum_{0 \leq i \leq d} \binom{d}{i} - \sum_{0 \leq i \leq d} \binom{d}{i} i = 7 \cdot 2^d - d \cdot 2^{d-1} = 2^{d-1}(14 - d).$$

This expression maximizes when d is either 12 or 13, and is then 4096. Thus, on average, a plane graph of S has more than $\frac{N}{4096}$ x-vings. $\square$

Finally, by combining Lemmas 1 and 2 and using an obvious induction on N, we obtain

Theorem 3. $\mathsf{pg}(N) \leq 4096^N$.

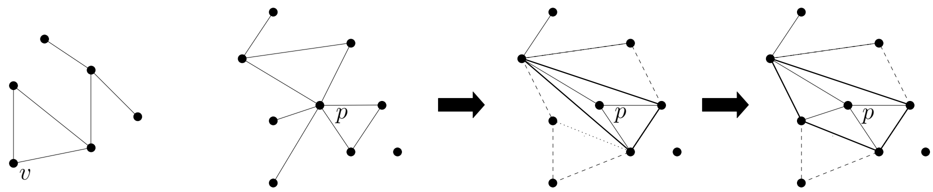
2.2 First Improvement

Interestingly, the bound in Subsect. 2.1 hardly relies on the geometric properties of the problem. Specifically, it only uses Euler's formula for plane graphs[3], and the trivial property, already noted, that in a plane graph, connecting a ving to any subset of the vertices that it sees results in a (larger) plane graph. In this subsection we obtain an improved bound by observing and exploiting some additional geometric properties of x-vings.

Lemma 4. *For every set S of N points in the plane,* $\hat{v}_x(S) \geq \dfrac{N}{3207.42}$.

Proof. We start by applying the same charging scheme as in the proof of Lemma 2, but then perform another step of moving charges across x-vings, as follows. We say that an x-ving $v = (p, G)$ is an x_i-ving if v is also an i-ving. According to the analysis in the proof of Lemma 2, only x_{12}-vings and x_{13}-vings are charged 4096; the next highest charge is 3072. At the other end, an x_3-ving is charged only 44, and an x_2-ving is charged only 24. Note that an x-ving (p, G) can be an x_2-ving only if p is part of the boundary of the convex hull of S and the two neighbors of p along this boundary are connected in G (e.g., see Fig. 1(a)). Note also that x_1-vings (and x_0-vings) do not exist. Consider an x_i-ving $v = (p, G)$, where $i > 3$, and let S_v be the set of i vertices that are connected to p in G. Let P_v be the star-shaped polygon (with respect to p) that is obtained by removing from G all the edges that are incident to p, ordering the vertices of S_v in their

[3] In fact, it only uses the fact that the number of edges in a plane graph is at most three times the number of vertices.



(a) (b)

Fig. 1. (a) The ving involving v is an x_2-ving. (b) An x_6-ving $v = (p, G)$ and an x_3-ving it reduces to: The enclosing polygon P_v (whose new edges are drawn dashed), and a triangulation of P_v, with the triangle containing p highlighted. A corresponding x_4-ving is also depicted.

angular (cyclic) order around p, and connecting every pair of consecutive vertices by an edge (some of these connecting edges may already exist in G, and adding the others cannot create a crossing). Triangulate P_v arbitrarily and let Δ denote the triangle that contains p. We remove from G all the edges incident to v, add the edges of Δ, and connect p to the three vertices of Δ to obtain a new graph G', and notice that $v' = (p, G')$ is an x_3-ving (once again, some of the edges of Δ, but not all of them, may already exist in G). Notice that we did not add the missing edges of P_v to G'. We say that v is *reduced* to v'. An example for such a reduction is depicted in Fig. 1(b).

Given a specific x_3-ving $v = (p, G)$, we now consider how many x_{12}-vings and x_{13}-vings can be reduced to it. Let Δ denote the triangle spanned by the three vertices u, v, w that p is connected to. (By construction, only x_3-vings where all the edges of Δ belong to G should be considered.) Denote by a, b, c the number of additional vertices that p would be able to see after the removal of each of the three respective edges of Δ in G. For example, if we remove the first edge of Δ then p would see $3 + a$ vertices (including the three vertices of Δ that are connected to p), if we remove all three edges of Δ then p would be able to see $3 + a + b + c$ vertices, etc. After such an edge removal, we can connect p to all of the new vertices that it sees, and obtain an x-ving that reduces to v. For every set of values of a, b, c, out of the seven possible edge removal combinations, at most four could yield an x_{12}-ving or an x_{13}-ving (for example, four combinations are obtained when $a = 9$, $b = 1$, and $c = 0$). This can be verified with a simple case analysis, depending on how many of a, b, c are equal to 9 or 10 (so that the corresponding quantity $3 + a$, $3 + b$, or $3 + c$ is 12 or 13). Thus, at most four x_{12}-vings and x_{13}-vings can be reduced to any specific x_3-ving. From every x_{12}-ving and every x_{13}-ving, we move a charge of 810.4 to some x_3-ving that it is reduced to. Now, every x-ving is charged at most $3285.6 = 44 + 4 \cdot 810.4 = 4096 - 810.4$ (x_i-vings are charged at most 0 when $i > 13$, at most 3072 when $3 < i < 12$, and 24 when $i = 2$). This already gives us the bound $\hat{v}_x(S) \geq N/3285.6$.

To further improve this bound to the one asserted in the lemma, we have to send some of the charge of x_{12}- and x_{13}-vings also to x_4-vings, as hinted in Fig. 1(b). For the full details, see the full version of this paper. ⊔⊓

By combining Lemmas 1 and 4, we obtain our second upper bound:

Theorem 5. $\mathsf{pg}(N) \leq 3207.42^N$.

2.3 Second Improvement

Given a set S of N points in the plane, we let $\mathsf{pg}_c^>(S)$ (resp., $\mathsf{pg}_c^\leq(S)$) denote the number of plane graphs with more than cN edges (resp., at most cN edges) that can be embedded over S, for some parameter $0 < c < 3$. Additionally, let $\hat{v}_{x,m}(S)$ denote the expected (i.e., average) value of $v_x(G)$ over all plane graphs $G \in \mathcal{P}(S)$ with at most m edges.

In [14], Hoffmann et al. establish the following theorem:

Theorem 6. *For any set S of N points in the plane and $19/12 \leq c \leq 3$,*

$$\mathsf{pg}_c^>(S) = O^*\left(\left(\frac{5^{5/2}}{8\left(c+t-\frac{1}{2}\right)^{c+t-\frac{1}{2}}(3-c-t)^{3-c-t}(2t)^t\left(\frac{1}{2}-t\right)^{\frac{1}{2}-t}}\right)^N \mathsf{tr}(S)\right),$$

where $t = \frac{1}{2}\left(\sqrt{(7/2)^2+3c+c^2} - 5/2 - c\right)$.

We begin by stating the following variants of Lemmas 1 and 2.

Lemma 7. *Let S be a set of N points in the plane and let $0 < c < 3$ be a parameter, such that $\hat{v}_{x,cN}(S) \geq \delta N$ for some constant $\delta > 0$. Then $\mathsf{pg}_c^\leq(S) \leq \frac{1}{\delta} \cdot \mathsf{pg}(N-1)$.*

Proof. By applying the same proof as in Lemma 1, we obtain the relation $\mathsf{pg}_c^\leq(S) \leq (1/\delta) \cdot \mathsf{pg}_{cN/(N-1)}^\leq(N-1)$. The lemma follows by noting that $\mathsf{pg}_{cN/(N-1)}^\leq(N-1) \leq \mathsf{pg}(N-1)$. The reason for replacing c by $cN/(N-1)$ is that the graphs obtained by removing x-vings have only $N-1$ vertices (and fewer than cN edges). ◻

We let $c = 1.968549$; see the full version of this paper for an explanation why we use this value of c. Substituting this value of c into Theorem 6, and using $\mathsf{tr}(N) < 30^N$ from [23], we get $\mathsf{pg}_c^>(N) = O^*\left(187.53^N\right)$. For a proof of the following lemma, see the full version of this paper.

Lemma 8. *For every point set S of N points in the plane, either $\mathsf{pg}(S) = O^*(187.53^N)$ or $\hat{v}_{x,cN}(S) > N/187.53$ (or both).*

By combining Lemmas 7 and 8, we get the following improved bound.

Theorem 9. $\mathsf{pg}(N) = O^*\left(187.53^N\right)$.

Proof. Let S be a set of N points in the plane that maximizes $\mathsf{pg}(S)$ (that is, $\mathsf{pg}(S) = \mathsf{pg}(N)$). As mentioned above, $\mathsf{pg}_c^>(S) \leq \mathsf{pg}_c^>(N) = O^*\left(187.53^N\right)$. Hence,

$$\mathsf{pg}(N) = \mathsf{pg}(S) = \mathsf{pg}_c^\leq(S) + \mathsf{pg}_c^>(S) \leq \mathsf{pg}_c^\leq(S) + O^*\left(187.53^N\right) \quad . \qquad (1)$$

By Lemma 8 we have either $\mathsf{pg}_c^{\leq}(S) = O^*\left(187.53^N\right)$ or $\hat{v}_{x,cN}(S) > N/187.53$. The former case immediately implies the asserted bound, and in the latter case we have, by Lemma 7,

$$\mathsf{pg}(N) \leq 187.53 \cdot \mathsf{pg}(N-1) + O^*\left(187.53^N\right) \ ,$$

and the asserted bound follows by induction on N. ◻

Remarks: (1) As opposed to previous bounds for $\mathsf{pg}(N)$, the dependence of the bound on $\mathsf{tr}(N)$ is non-linear, as can be seen in the analysis in the full version of this paper.
(2) The bound in Theorem 9 can be slightly improved by passing some of the charge to x_5-vings and x_6-vings.

2.4 Additional types of plane graphs and degree-related results

In this subsection we present various additional bounds that can be obtained by using the above technique. Specifically, we extend the technique to some other types of plane graphs, and show how to derive degree-related properties of random plane graphs (embedded over a fixed set S). Proofs for the lemmas and theorems of this subsection can be found in the full version of this paper.

Let $\mathsf{pg}(N,i)$ be the maximum number of plane graphs with no vertex of degree smaller than i, that can be embedded over a specific N-point set in the plane.

Theorem 10. $\mathsf{pg}(N,1) = O^*\left(186.46^N\right)$ *and* $\mathsf{pg}(N,2) = O^*\left(180.20^N\right)$.

Since every connected graph has no isolated vertices, we obtain the following.

Corollary 11. *For every point set S of N points in the plane, the number of connected plane graphs that can be embedded over S is $O^*\left(186.46^N\right)$.*

Although this is only a slight improvement over our bound of $O^*(187.53^N)$ on the total number of plane graphs, this is nevertheless, as far as we know, the first time that a bound on the number of connected plane graphs is asymptotically smaller than the bound on the total number of plane graphs. In a similar manner, since every bi-connected graph has no vertices of degree 0 or 1, we obtain

Corollary 12. *For every point set S of N points in the plane, the number of bi-connected plane graphs that can be embedded over S is $O^*\left(180.20^N\right)$.*

Cross-graph charging can also be used to obtain other properties of random plane graphs (embedded over a fixed point set). We present the following lemmas as examples of such properties. The first lemma, which is a variant of Lemma 2, lower bounds the expected number of isolated vertices, leaves (i.e, 1-vings), and 2-vings in a graph uniformly chosen from $\mathcal{P}(S)$. Let the expected value of $v_i(G)$, for a graph chosen uniformly at random from $\mathcal{P}(S)$, be denoted as $\hat{v}_i(S)$.

Lemma 13. *For every set S of N points in the plane,*

$$\hat{v}_0(S) \geq \frac{N}{4096}, \quad \hat{v}_1(S) \geq \frac{3N}{1024}, \quad \text{and} \quad \hat{v}_2(S) \geq \frac{33N}{2048}.$$

Lemma 14. *For every set S of N points in the plane, $\hat{v}_2(S) + \hat{v}_3(S) \geq N/24$.*

3 Quasi-Plane Graphs

In this section we show that our techniques can easily be extended to obtain bounds for k-quasi-plane graphs. We derive the bounds $\mathsf{qp}_3(N) \le 2^{26N}$ and $\mathsf{qp}_4(N) \le 2^{145N}$.

The number of 3-quasi-plane graphs. We use the notation given in the introduction. A 3-quasi-plane graph does not contain three pairwise crossing edges. Ackerman and Tardos [2] proved that such graphs have at most $6.5N - 20$ edges, and that this is tight up to some additive constant. Using this result, we can apply our method in a straightforward manner. As before, we denote by $\mathcal{Q}_k(S)$ the set of all k-quasi-plane graphs embedded on a fixed labeled set S of N points in the plane, and put $\mathsf{qp}_k(S) := |\mathcal{Q}_k(S)|$. Moreover, we let $\mathsf{qp}_k(N) = \max_{|S|=N} \mathsf{qp}_k(S)$.

Given a k-quasi-plane graph $G \in \mathcal{Q}_k(S)$, we say that a ving $v = (p, G)$ is an *x-ving* if we cannot add to G any additional edges that are adjacent to p without violating the k-quasi-planarity property of G. We denote by $v_i(G)$ the number of i-vings in G, and by $v_x(G)$ the number of x-vings in G (as in the previous cases). The expected value of $v_x(G)$, for a graph chosen uniformly from $\mathcal{Q}_k(S)$, is denoted as $\hat{v}_x^k(S)$. More formally,

$$\hat{v}_x^k = \hat{v}_x^k(S) := \mathbb{E}\{v_x^k(G)\} = \frac{\sum_{G \in \mathcal{Q}_k(S)} v_x(G)}{\mathsf{qp}_k(S)} \quad .$$

Lemma 15. *For $N \ge 2$ and $k \ge 2$, let $\delta_N^k > 0$ be a real number, such that $\hat{v}_x^k(S) \ge \delta_N^k N$ holds for every set S of N points in the plane. Then $\mathsf{qp}_k(N) \le \frac{1}{\delta_N^k} \mathsf{qp}_k(N-1)$.*

Proof. Identical to the proof of Lemma 1. ◻

Lemma 16. *For every set S of N points in the plane, $\hat{v}_x^3(S) \ge N/2^{26}$.*

Proof. We use a charging scheme where every i-ving $v = (p, G)$ will be charged $14 - i$ units. The sum of the charges of the vings in any fixed 3-quasi-plane graph $G \in \mathcal{P}(S)$ is $\sum_i (14 - i) v_i = 14 \sum_i v_i - \sum_i i v_i = 14N - \sum_i i v_i$. Since G can have at most $6.5N - 20$ edges, we have $\sum_i i v_i \le 13N - 40$. This implies that the total charge in any fixed graph is at least $14N - \sum_i i v_i \ge N + 40$. Therefore, on average, every ving has a charge larger than 1.

Next, we move all of the charge to x-vings in the same manner as in Lemma 2. Connecting a new edge to p (while not violating the 3-quasi-plane property) does not affect the set of additional edges that can be connected to p, since a pair of edges connected to p cannot participate in a triplet of pairwise crossing edges. Consider the charge that an x_d-ving (a notation analogous to that used for plane graphs) $v = (p, G)$ can have. By the observation just made, the number of i-vings that charge v is exactly $\binom{d}{i}$, as before. Therefore, v is charged exactly

$$\sum_{i=0}^{d} \binom{d}{i} (14 - i) = 14 \sum_{i=0}^{d} \binom{d}{i} - \sum_{i=0}^{d} \binom{d}{i} i = 14 \cdot 2^d - d \cdot 2^{d-1} = 2^{d-1}(28 - d).$$

This expression maximizes when d is either 26 or 27, and is then 2^{26}. Therefore, on average, a 3-quasi-plane graph on S has more than $\frac{N}{2^{26}}$ x-vings. □

By combining Lemmas 15 and 16, we obtain an upper bound on the number of 3-quasi-plane graphs. As far as we know, this is the first exponential upper bound for $\mathsf{qp}_3(N)$.

Theorem 17. $\mathsf{qp}_3(N) \leq 2^{26N}$.

Quasi-plane graphs with $k \geq 4$. Ackerman [1] proved that every 4-quasi-plane graph that is embedded over a set of N points in the plane has at most $36N - 72$ edges, even when the edges are not necessarily straight. This implies that $\mathsf{qp}_4(S)$ is also exponential in N. Specifically:

Theorem 18. $\mathsf{qp}_4(N) \leq 2^{145N}$.

Proof. Since Lemma 15 applies for every k, we only need to replace Lemma 16. We can derive the bound $\hat{v}_d^3(S) \geq \frac{N}{2^{145}}$ by using the same analysis as in the proof of Lemma 16, except that an i-ving will be charged $73 - i$ units. In this case, the analysis implies that an x_d-ving is charged $2^{d-1}(146 - d)$. This expression is maximized for $d = 144$ and $d = 145$, and is then 2^{145}. □

A common conjecture (e.g., see $[1, 2, 16]$) is that every k-quasi-plane graph with N vertices has at most $c_k N$ edges, where c_k is a constant depending on k (in fact, the conjecture is also made for the more general case where the edges are not necessarily straight). Proving the conjecture will immediately imply that $\mathsf{qp}_k(N)$ is exponential in N for every fixed k. This consequence is easily obtained by adapting the proof of Theorem 18, and giving each i-ving a charge of $2c_k+1-i$. Valtr [26] proved that any k-quasi-plane graph with N vertices has $O(N \log N)$ edges. Combining this bound with the cross-graph charging technique only yields the superexponential bound $\mathsf{qp}_k(N) = (N/\log N)^{O(N)}$.

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