Compilation

Lecture 8

Optimizations

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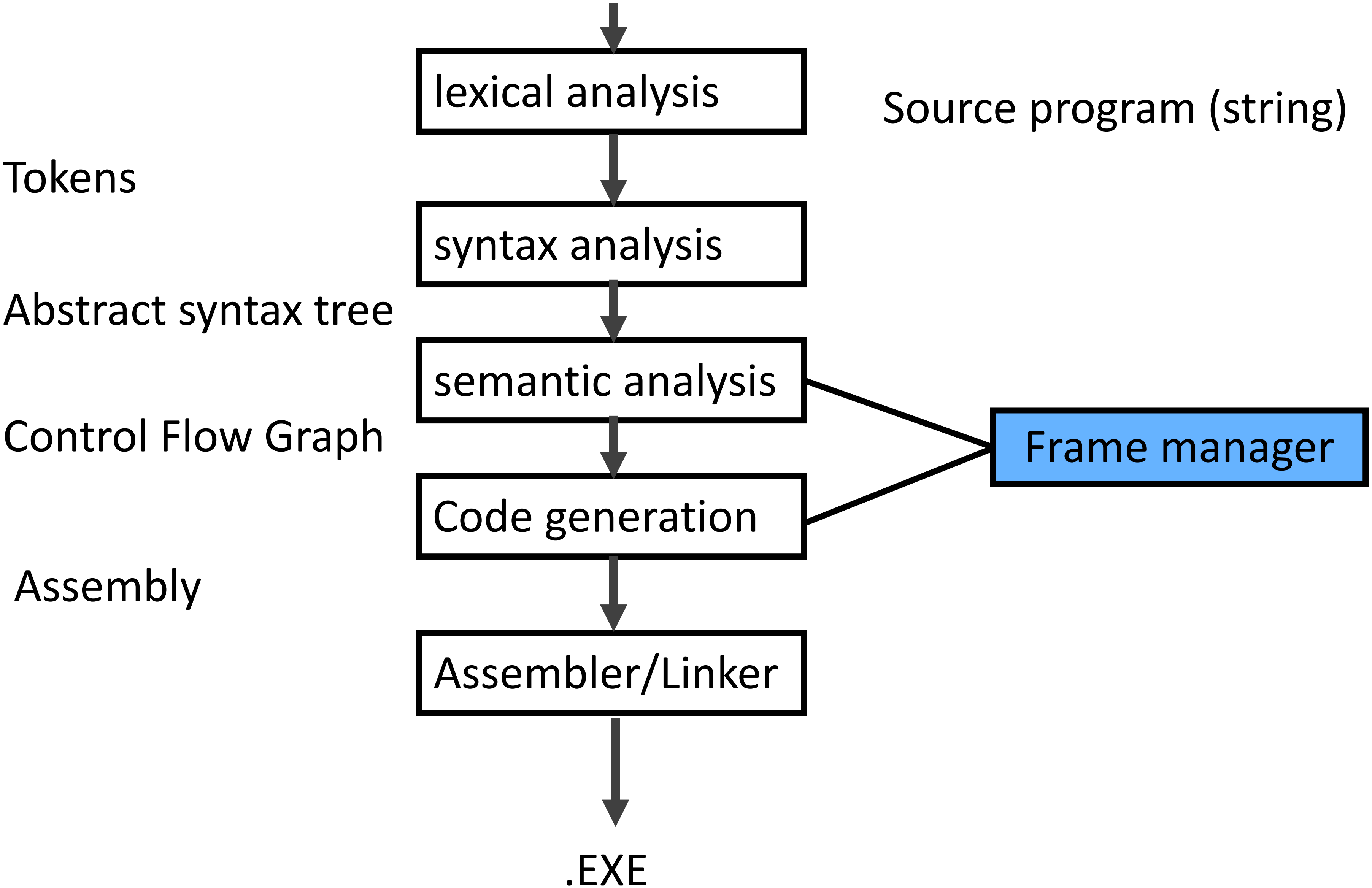
Basic Compiler Phases



IR Optimization

Optimization points

source code → Front end → IR → Code generator → target code

User
profile program
change algorithm

Compiler
intraprocedural IR
Interprocedural IR
IR optimizations

Compiler
register allocation
instruction selection
peephole transformations

now

IR Optimization

- Making code better

IR Optimization

- Making code “better”

Overview of IR optimization

- Formalisms and Terminology
  - Control-flow graphs
  - Basic blocks
- Local optimizations
  - Speeding up small pieces of a procedure
- Global optimizations
  - Speeding up procedure as a whole
- The dataflow framework
  - Defining and implementing a wide class of optimizations

Program Analysis

- In order to optimize a program, the compiler has to be able to reason about the properties of that program
- An analysis is called **sound** if it never asserts an incorrect fact about a program
- All the analyses we will discuss in this class are sound
  - *(Why?)*

Soundness & Precision

```
int x;
int y;

if (y < 5)
    x = 137;
else
    x = 42;

Print(x);
```

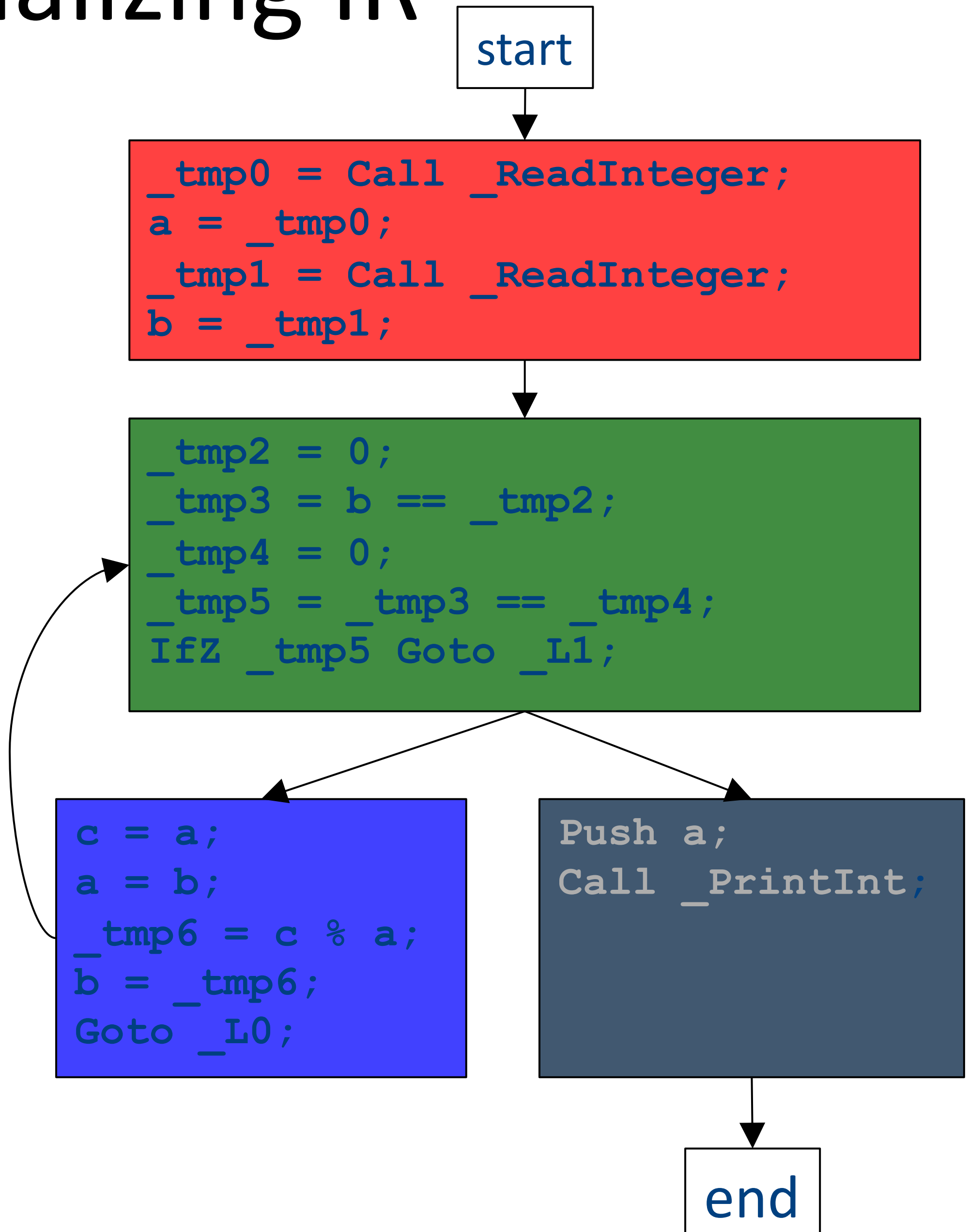
"At this point in the program, **x** is either 137, 42, or 271"

A formalism for IR optimization

- Every phase of the compiler uses some new abstraction:
  - Scanning uses regular expressions
  - Parsing uses CFGs
  - Semantic analysis uses proof systems and symbol tables
  - IR generation uses ASTs
- In optimization, we need a formalism that captures the structure of a program in a way amenable to optimization

Visualizing IR

```
main:
    _tmp0 = Call _ReadInteger;
    a = _tmp0;
    _tmp1 = Call _ReadInteger;
    b = _tmp1;
_L0:
    _tmp2 = 0;
    _tmp3 = b == _tmp2;
    _tmp4 = 0;
    _tmp5 = _tmp3 == _tmp4;
    IfZ _tmp5 Goto _L1;
    c = a;
    a = b;
    _tmp6 = c % a;
    b = _tmp6;
    Goto _L0;
_L1:
    Push a;
    Call _PrintInt;
```



Basic blocks

- A basic block is a sequence of IR instructions where
  - There is exactly one spot where control enters the sequence, which must be at the start of the sequence
  - There is exactly one spot where control leaves the sequence, which must be at the end of the sequence
- Informally, a sequence of instructions that always execute as a group

Control-Flow Graphs

- A control-flow graph (CFG) is a graph of the basic blocks in a function
- The term CFG is overloaded – from here on out, we'll mean "control-flow graph" and not "context free grammar"
- Each edge from one basic block to another indicates that control can flow from the end of the first block to the start of the second block
- There is a dedicated node for the start and end of a function

Common **S**ubexpression **E**limination

- If we have two variable assignments
  v1 = a op b
  …
  v2 = a op b
- and the values of v1, a, and b have not changed between the assignments, rewrite the code as
  v1 = a op b
  …
  v2 = v1
- Eliminates useless recalculation
- Paves the way for later optimizations

Common Subexpression Elimination

- If we have two variable assignments
  v1 = a op b   [or: v1 = a]
  …
  v2 = a op b   [or: v2 = a]
- and the values of v1, a, and b have not changed between the assignments, rewrite the code as
  v1 = a op b   [or: v1 = a]
  …
  v2 = v1
- Eliminates useless recalculation
- Paves the way for later optimizations

Copy Propagation

- If we have a variable assignment
  v1 = v2
  then as long as v1 and v2 are not reassigned, we can rewrite expressions of the form
  a = … v1 …
  as
  a = … v2 …
  provided that such a rewrite is legal

Dead Code Elimination

- An assignment to a variable v is called dead if the value of that assignment is never read anywhere
- Dead code elimination removes dead assignments from IR
- Determining whether an assignment is dead depends on what variable is being assigned to and when it's being assigned

Live variables

- The analysis corresponding to dead code elimination is called liveness analysis
- A variable is live at a point in a program if later in the program its value will be read before it is written to again
- Dead code elimination works by computing liveness for each variable, then eliminating assignments to dead variables

Computing live variables

- To know if a variable will be used at some point, we iterate across the statements in a basic block in reverse order
- Initially, some small set of values are known to be live (which ones depends on the particular program)
- When we see the statement a = b op c:
  - Just before the statement, a is not alive, since its value is about to be overwritten
  - Just before the statement, both b and c are alive, since we're about to read their values
  - (what if we have a = a + b?)

Liveness analysis

```
 { b }
a = b;
 { a, b }
c = a;
 { a, b }
d = a + b;
 { a, b, d }
e = d;
 { a, b, e }
d = a;
 { b, d, e }
f = e;
 { b, d } - given
```

Which statements are dead?

Dead Code Elimination

```
 { b }
a = b;
 { a, b }
c = a;
 { a, b }
d = a + b;
 { a, b, d }
e = d;
 { a, b, e }
d = a;
 { b, d, e }
f = e;
 { b, d }
```

Which statements are dead?

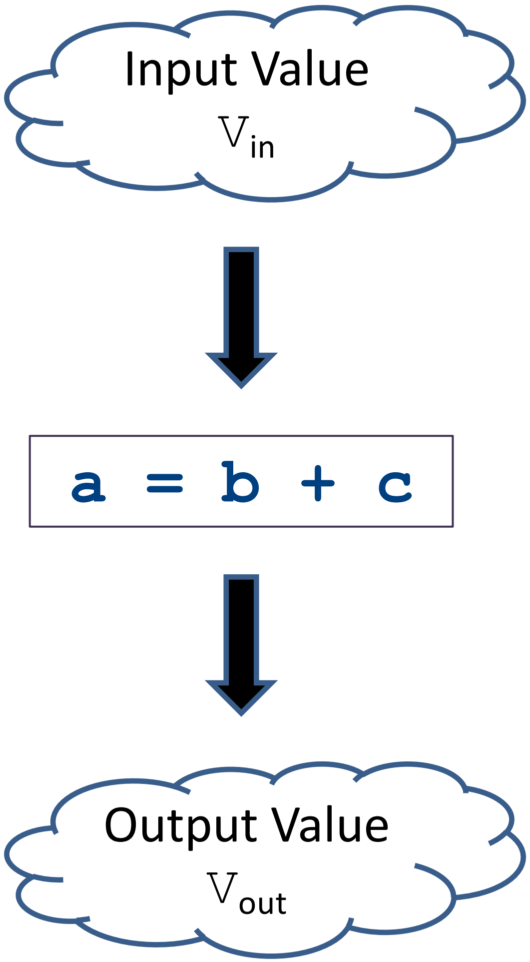
Dead Code Elimination

```
 { b }
a = b;
 { a, b }


 { a, b }
d = a + b;
 { a, b, d }
e = d;
 { a, b, e }
d = a;
 { b, d, e }


 { b, d }
```

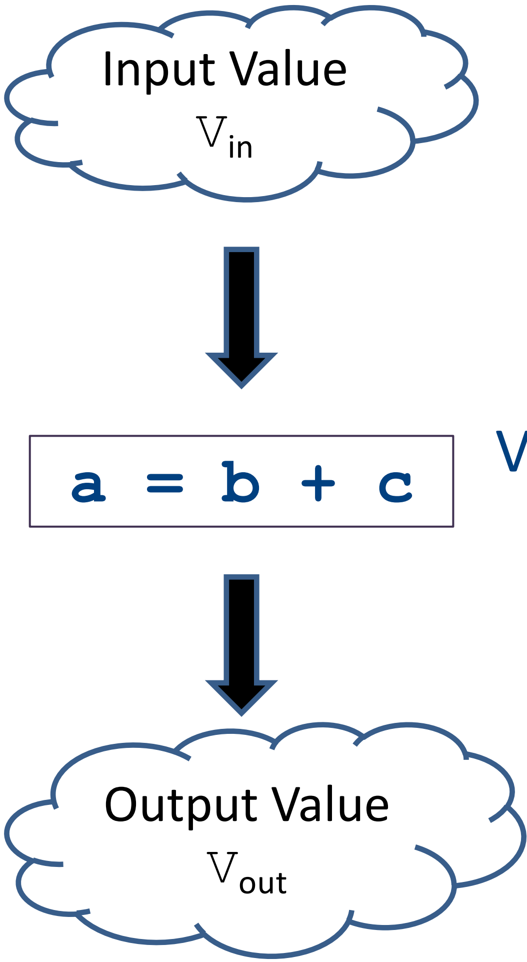
Formalizing local analyses



Transfer Function

$$V_{out} = f_{\mathtt{a=b+c}}(V_{in})$$

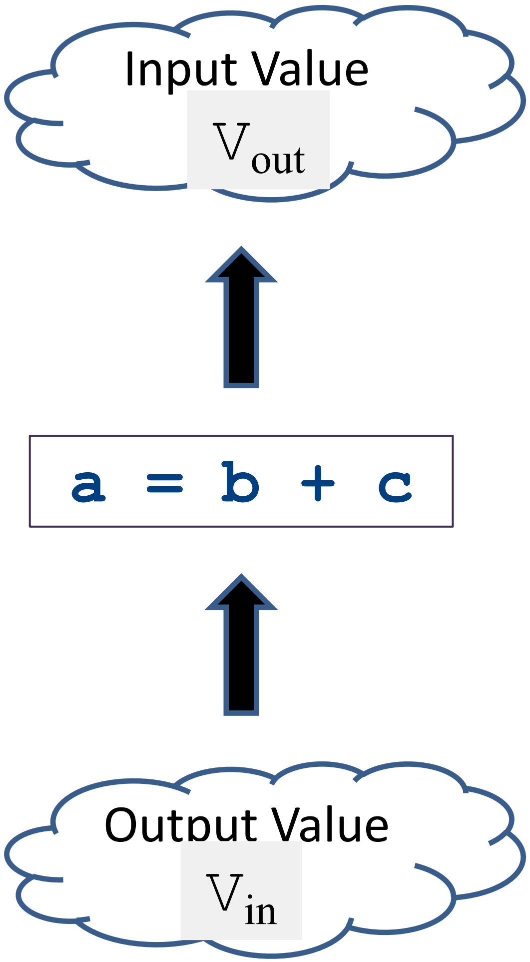
Available Expressions



$V_{out} = (V_{in} \setminus \{e \mid e \text{ contains } \mathbf{a}\}) \cup \{a=b+c\}$

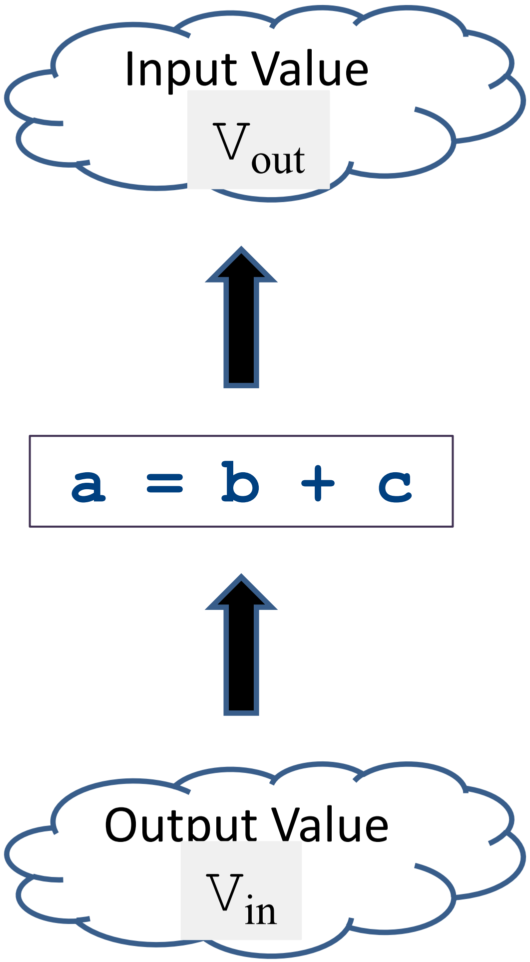
Expressions of the forms
`a`=… and x=…**`a`**…

Live Variables



$$V_{in} = (V_{out} \setminus \{\mathbf{a}\}) \cup \{\mathbf{b},\mathbf{c}\}$$

Live Variables



$$V_{in} = (V_{out} \setminus \{\mathbf{a}\}) \cup \{\mathbf{b},\mathbf{c}\}$$

Information for a local analysis

- What direction are we going?
  - Sometimes forward (available expressions)
  - Sometimes backward (liveness analysis)
- How do we update information after processing a statement?
  - What are the new semantics?
  - What information do we know initially?

Formalizing local analyses

- Define an analysis of a basic block as a quadruple (D, V, F, I) where
  - **D** is a direction (forwards or backwards)
  - **V** is a set of values the program can have at any point
  - **F** is a family of transfer functions defining the meaning of any expression as a function $f : \mathbf{V} \rightarrow \mathbf{V}$
  - **I** is the initial information at the top (or bottom) of a basic block

Liveness Analysis

- **Direction:** Backward
- **Values:** Sets of variables
- **Transfer functions:** Given a set of variable assignments V and statement a = b + c:
- Remove a from V (any previous value of a is now dead.)
- Add b and c to V (any previous value of b or c is now live.)
- Formally: $\mathbb{V}_{in} = (\mathbb{V}_{out} \setminus \{\mathtt{a}\}) \cup \{\mathtt{b}, \mathtt{c}\}$
- **Initial value:** Depends on semantics of language
  - E.g., function arguments and return values (pushes)
  - Result of local analysis of other blocks as part of a global analysis

Running local analyses

- Given an analysis **(D, V, F, I)** for a basic block
- Assume that **D** is “forward;” analogous for the reverse case
- Initially, set OUT[**entry**] to **I**
- For each statement **s**, in order:
  - Set IN[**s**] to OUT[**prev**], where **prev** is the previous statement
  - Set OUT[**s**] to f_s(IN[**s**]), where f_s is the transfer function for statement **s**

Global analysis

- A global analysis is an analysis that works on a control-flow graph as a whole
- Substantially more powerful than a local analysis
  - (Why?)
- Substantially more complicated than a local analysis
  - (Why?)

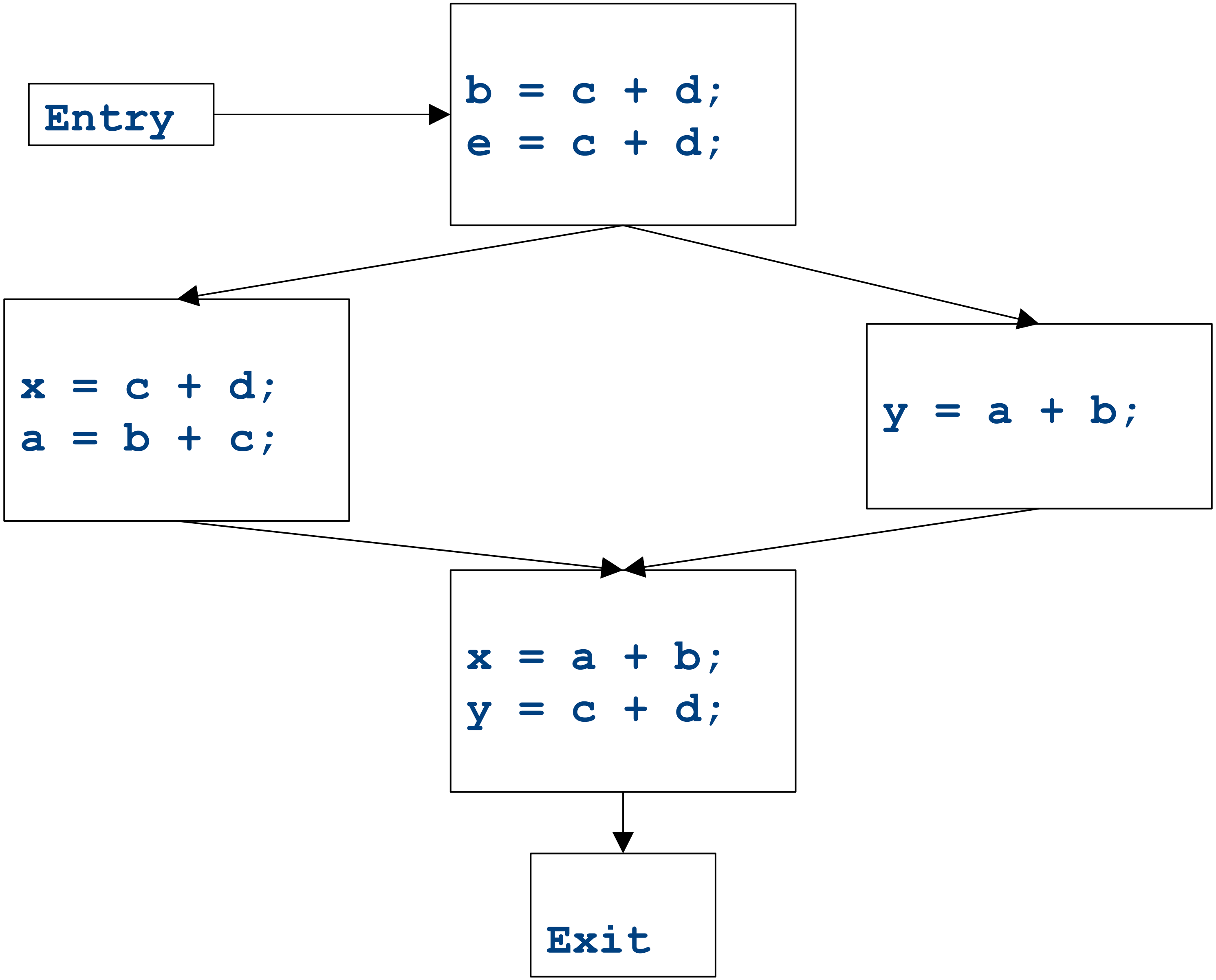
Why global analysis is hard

- Need to be able to handle multiple predecessors/successors for a basic block
- Need to be able to handle multiple paths through the control-flow graph and may need to iterate multiple times to compute the final value (but the analysis still needs to terminate!)
- Need to be able to assign each basic block a reasonable default value for before we've analyzed it

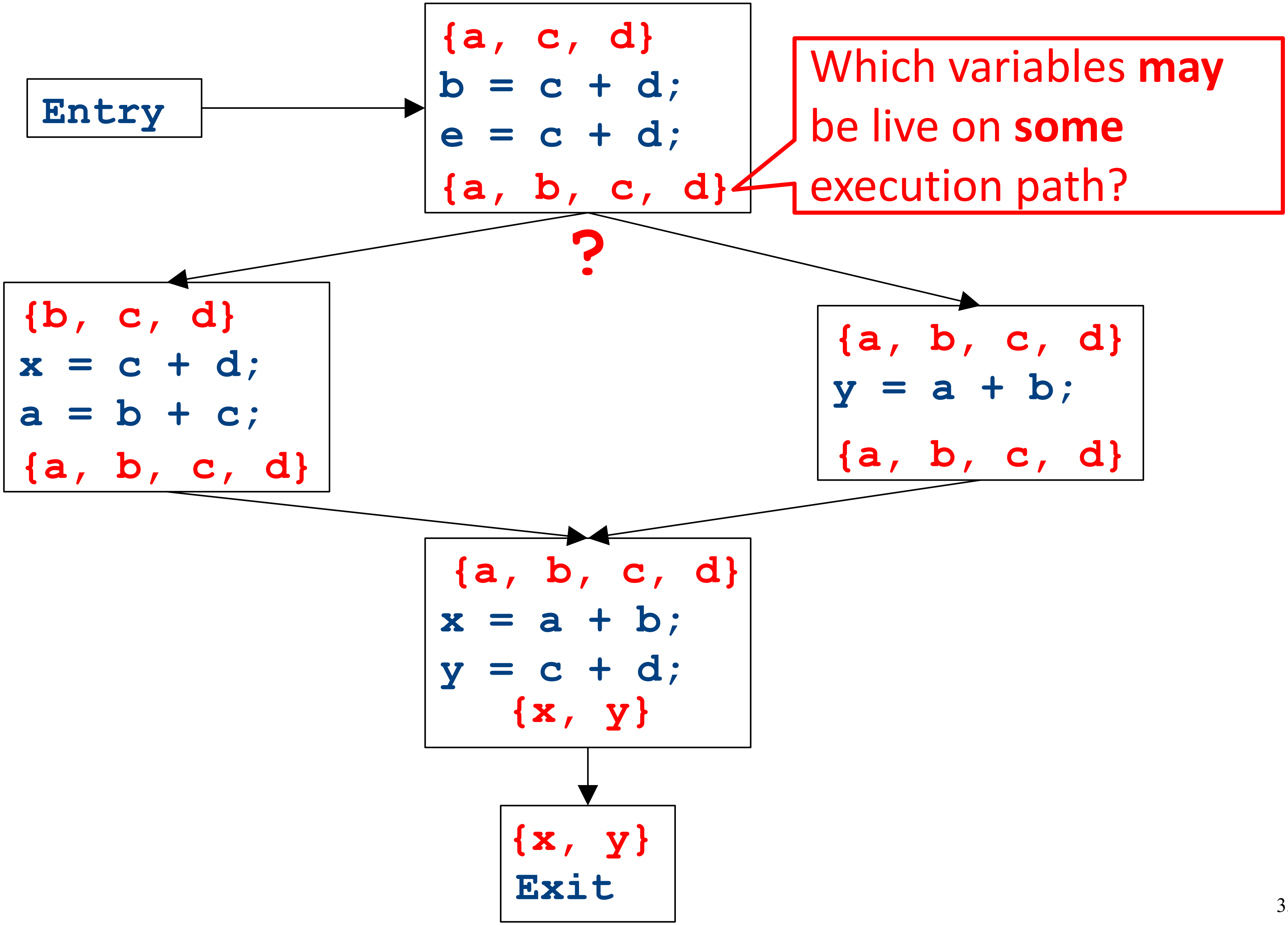
Global dead code elimination

- Local dead code elimination needed to know what variables were live on exit from a basic block
- This information can only be computed as part of a global analysis
- How do we modify our liveness analysis to handle a CFG?

CFGs without loops



CFGs without loops



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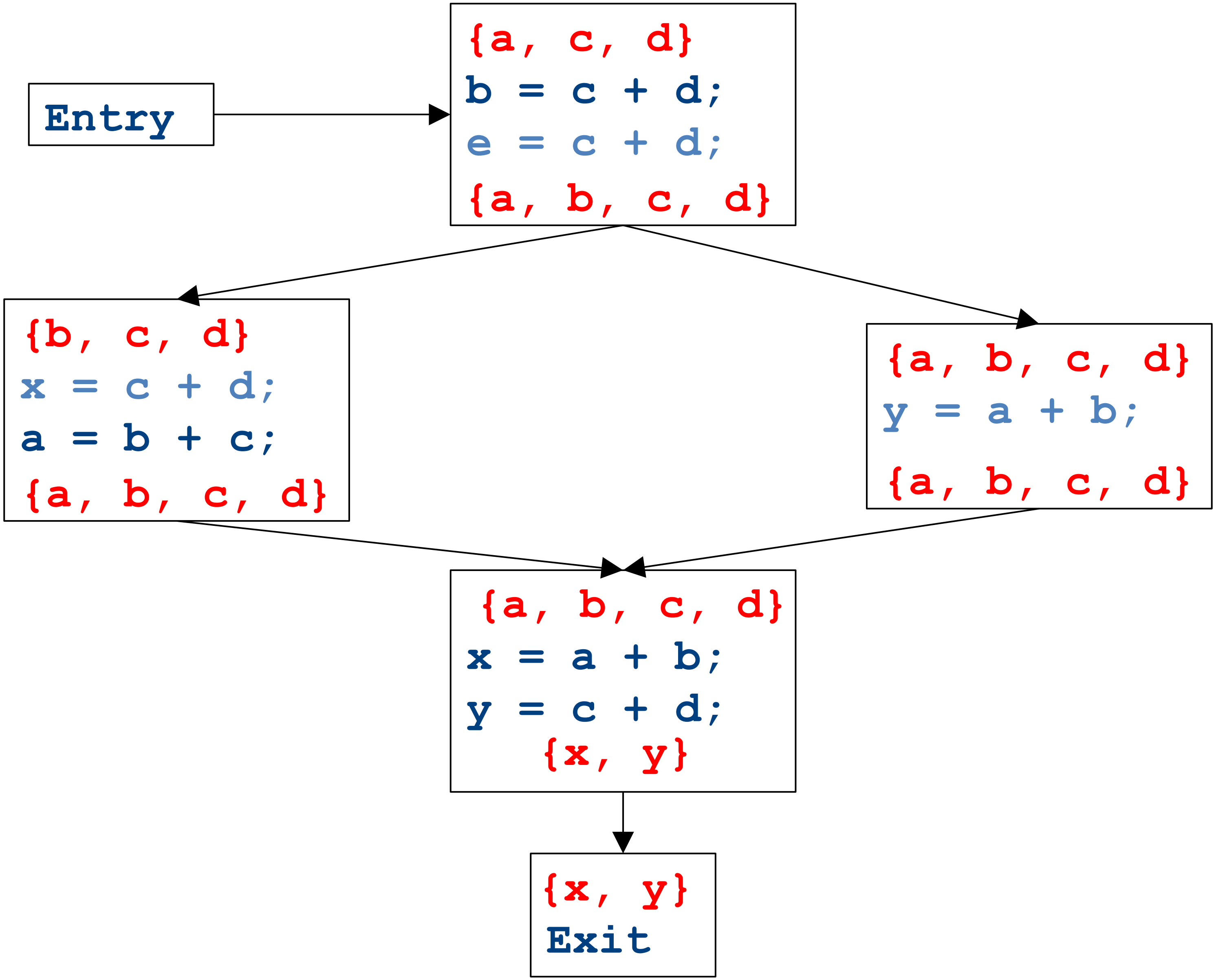

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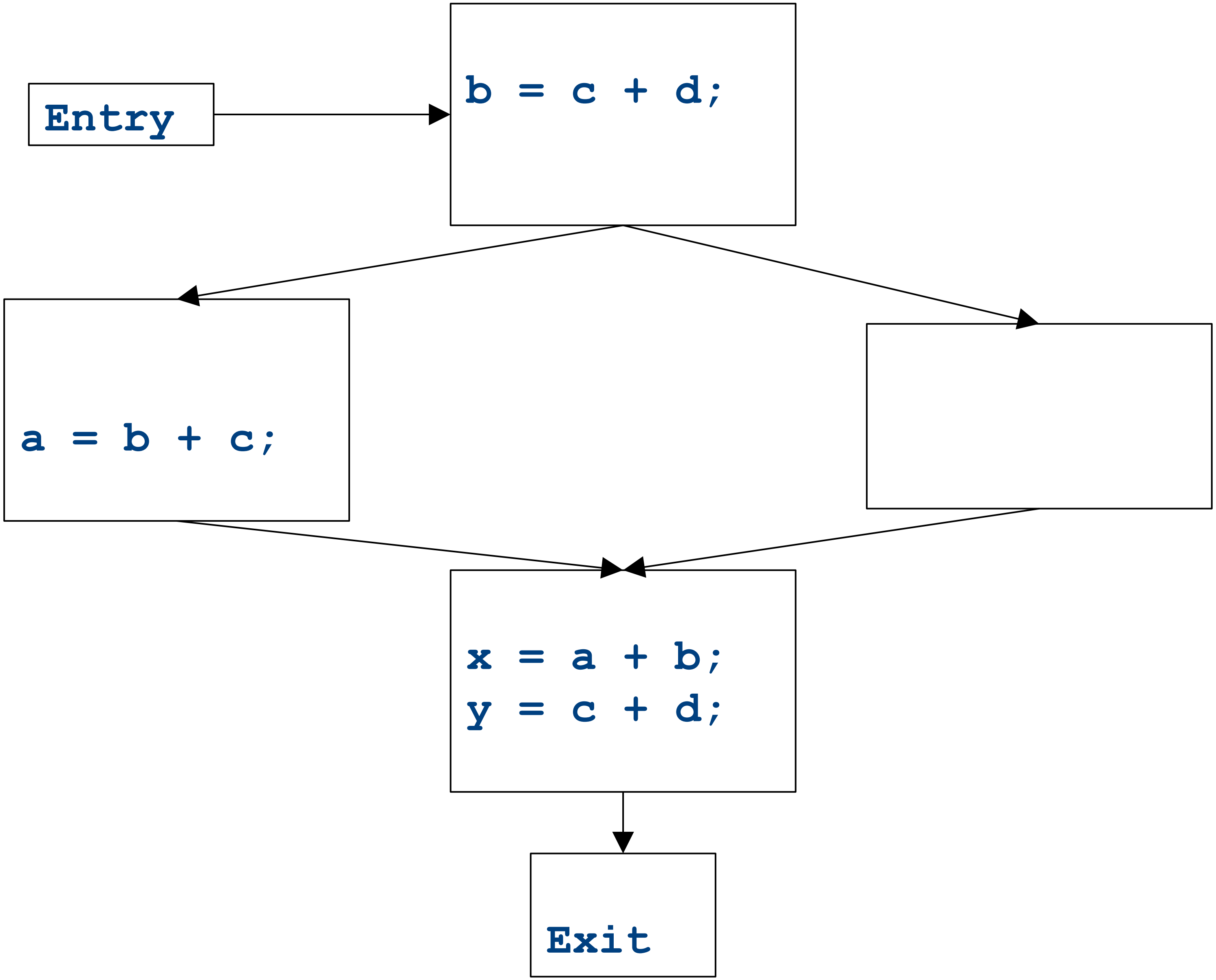

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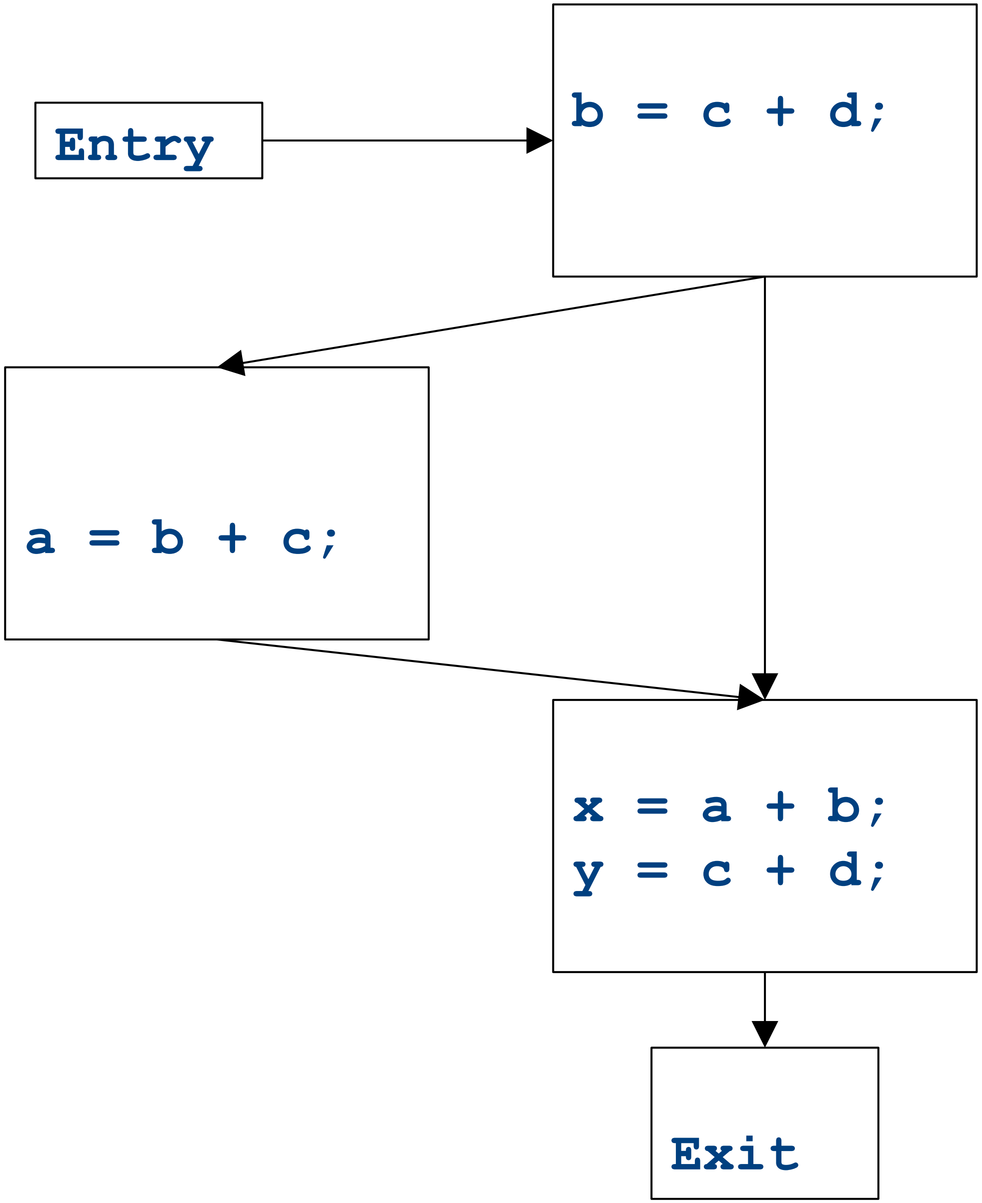
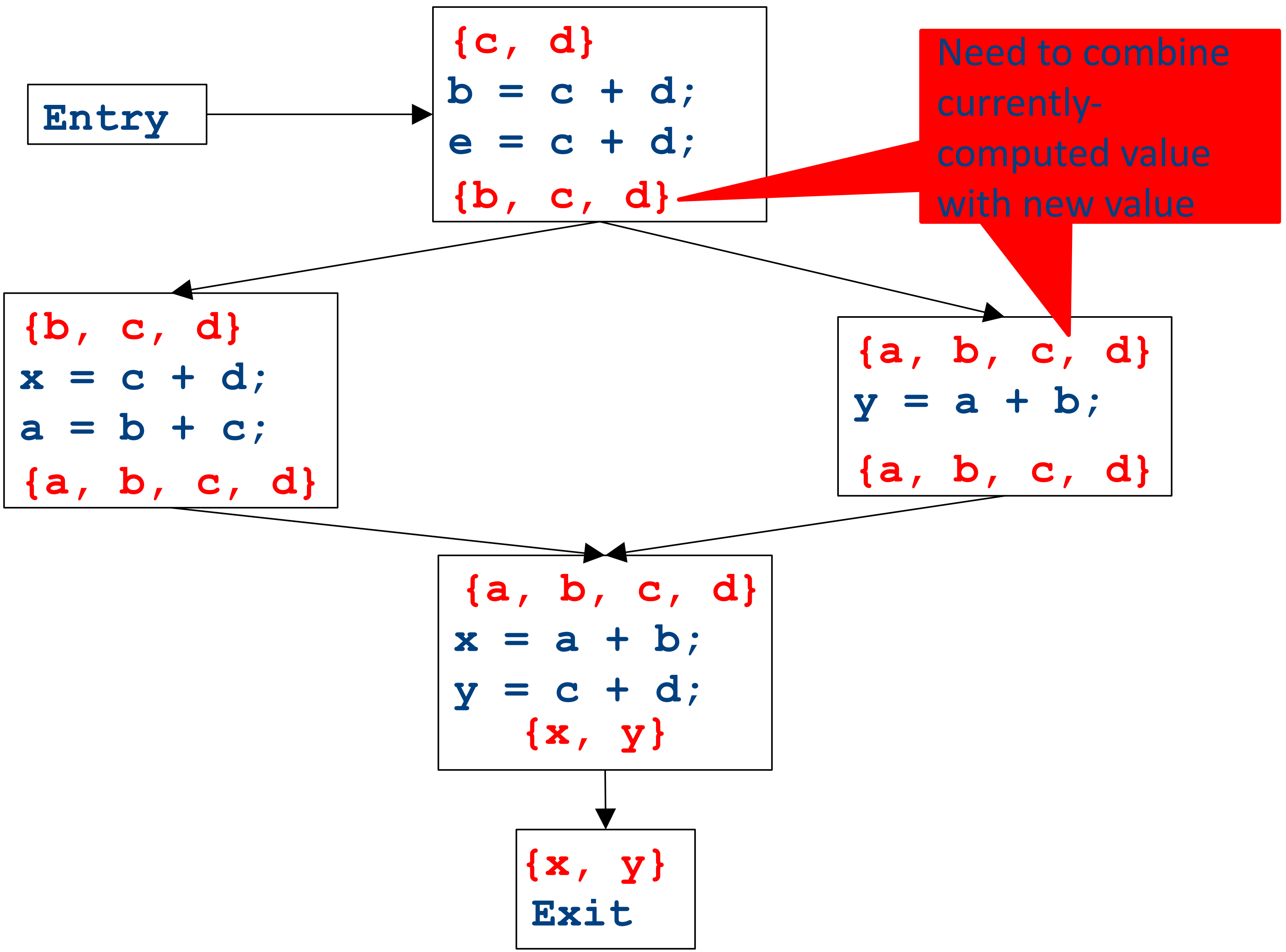


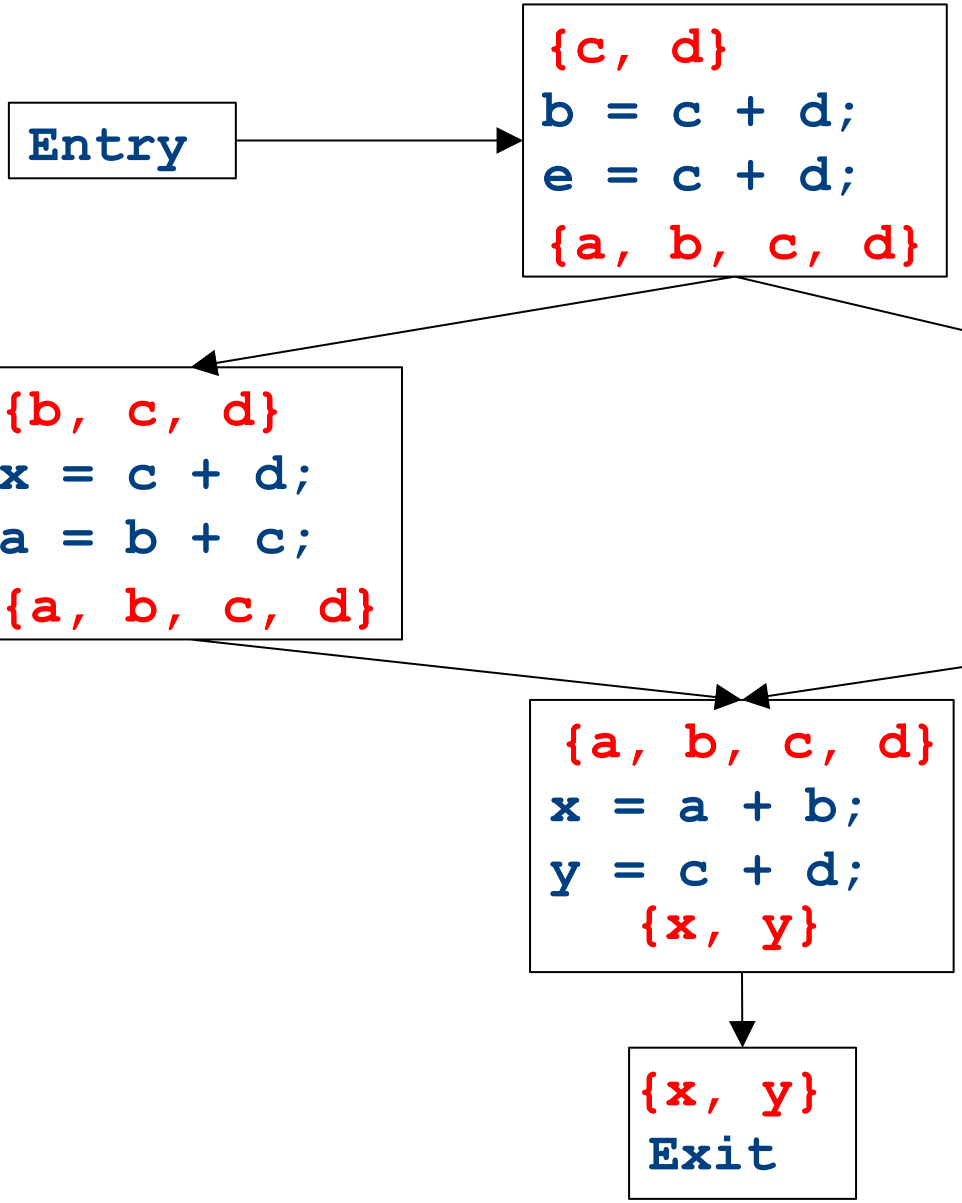
Major changes – part 1

- In a local analysis, each statement has exactly one predecessor
- In a global analysis, each statement may have **multiple** predecessors
- A global analysis must have some means of **combining information** from all predecessors of a basic block

CFGs without loops



CFGs without loops



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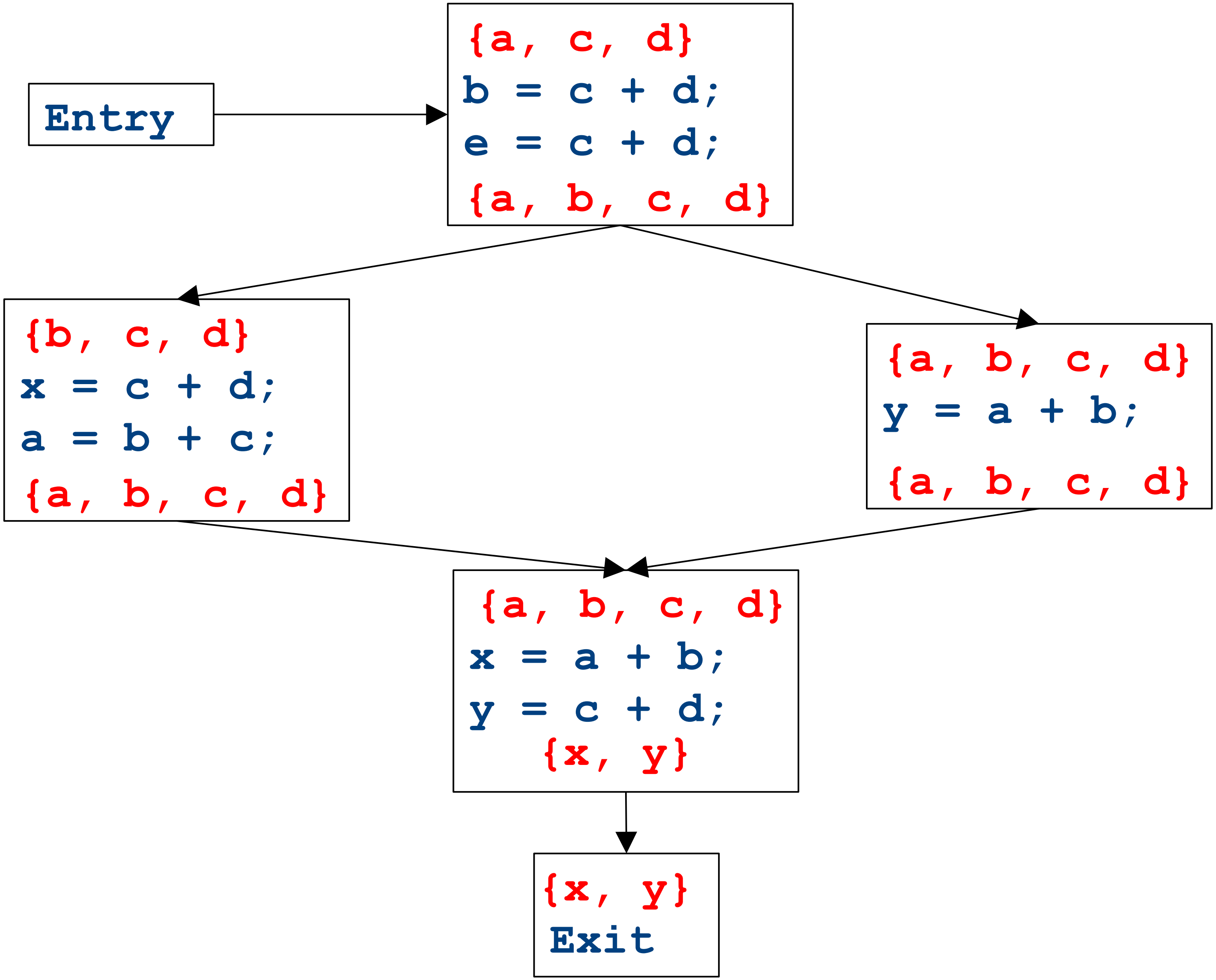
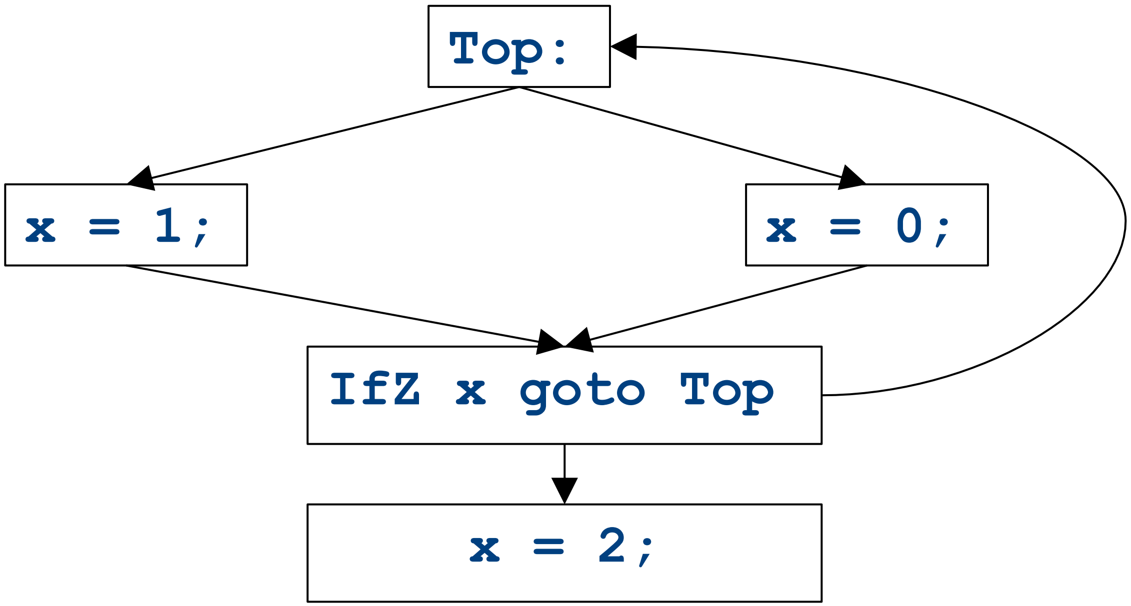


Major changes – part 2

- In a local analysis, there is only one possible path through a basic block
- In a global analysis, there may be **many** paths through a CFG
- May need to recompute values multiple times as more information becomes available
- Need to be careful when doing this not to loop infinitely!
  - (More on that later)
- Can order of computation affect result?

CFGs with loops

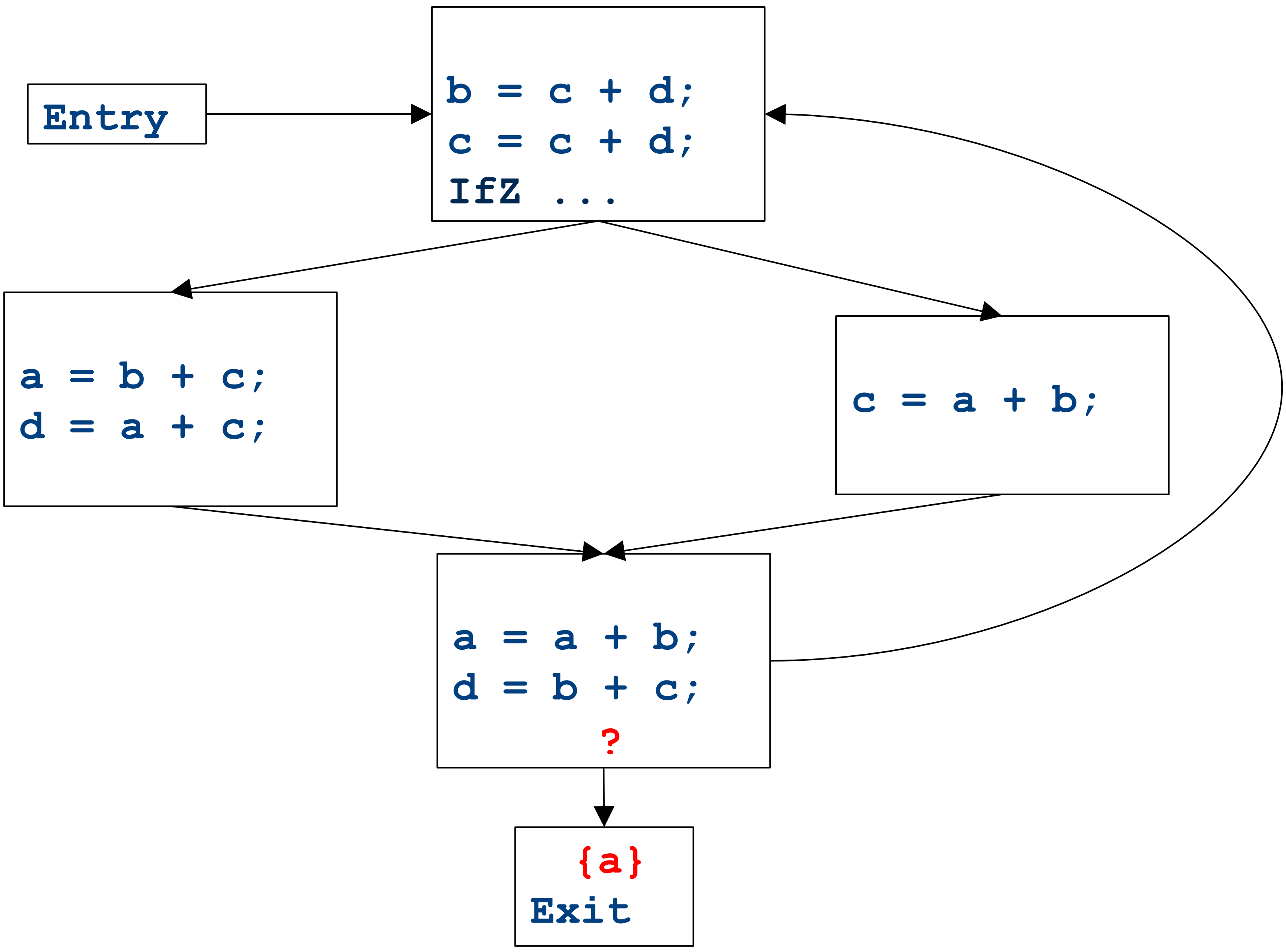
- Up to this point, we've considered loop-free CFGs, which have only finitely many possible paths
- When we add loops into the picture, this is no longer true
- Not all possible loops in a CFG can be realized in the actual program



CFGs with loops

- Up to this point, we've considered loop-free CFGs, which have only finitely many possible paths
- When we add loops into the picture, this is no longer true
- Not all possible loops in a CFG can be realized in the actual program
- **Sound approximation:** Assume that every possible path through the CFG corresponds to a valid execution
  - Includes all realizable paths, but some additional paths as well
  - May make our analysis less precise (but still sound)
  - Makes the analysis feasible; we'll see how later

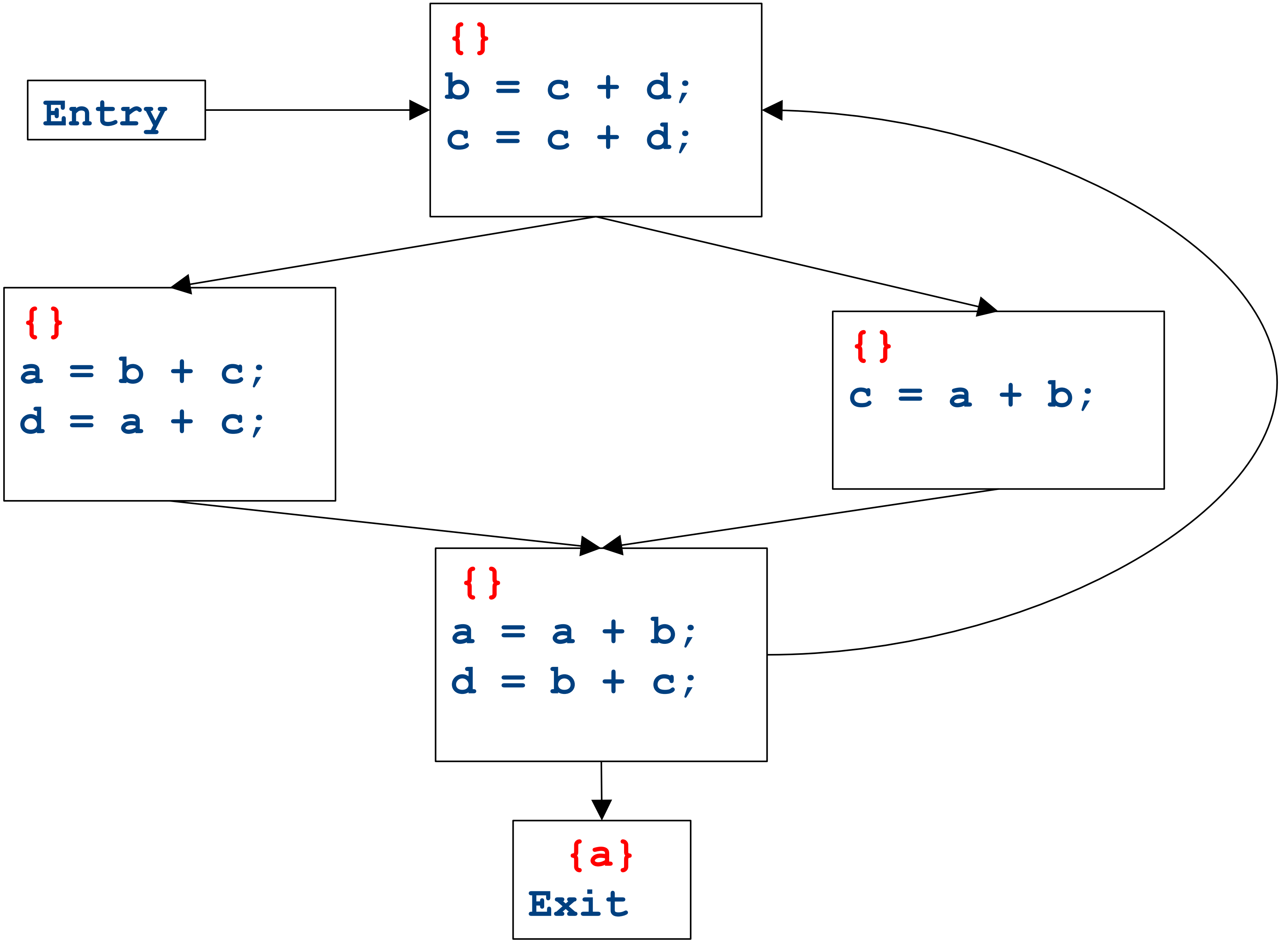
CFGs with loops



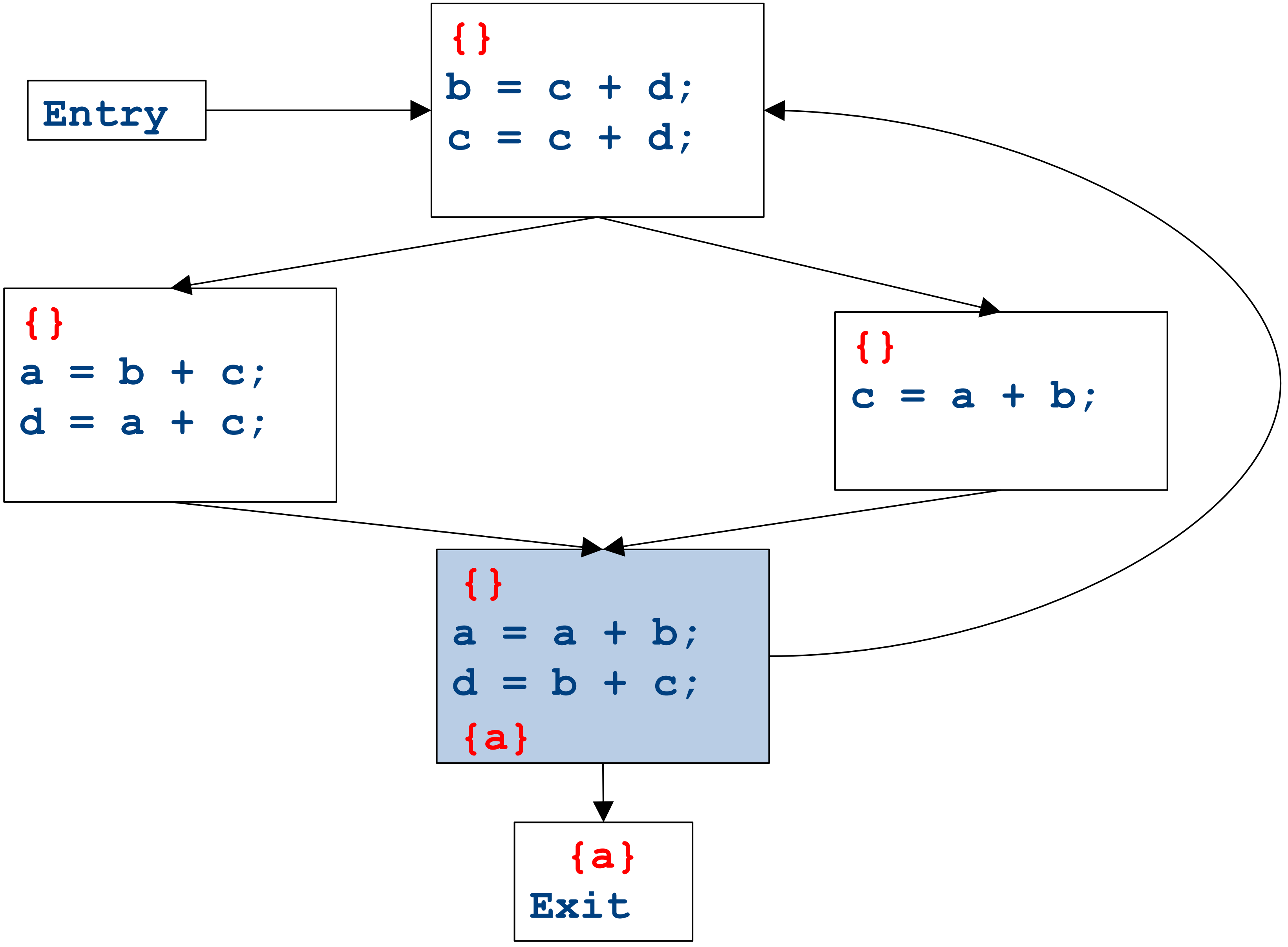
Major changes – part 3

- In a local analysis, there is always a well defined "first" statement to begin processing
- In a global analysis with loops, every basic block might depend on every other basic block
- To fix this, we need to assign initial values to all of the blocks in the CFG

CFGs with loops - initialization



CFGs with loops - iteration



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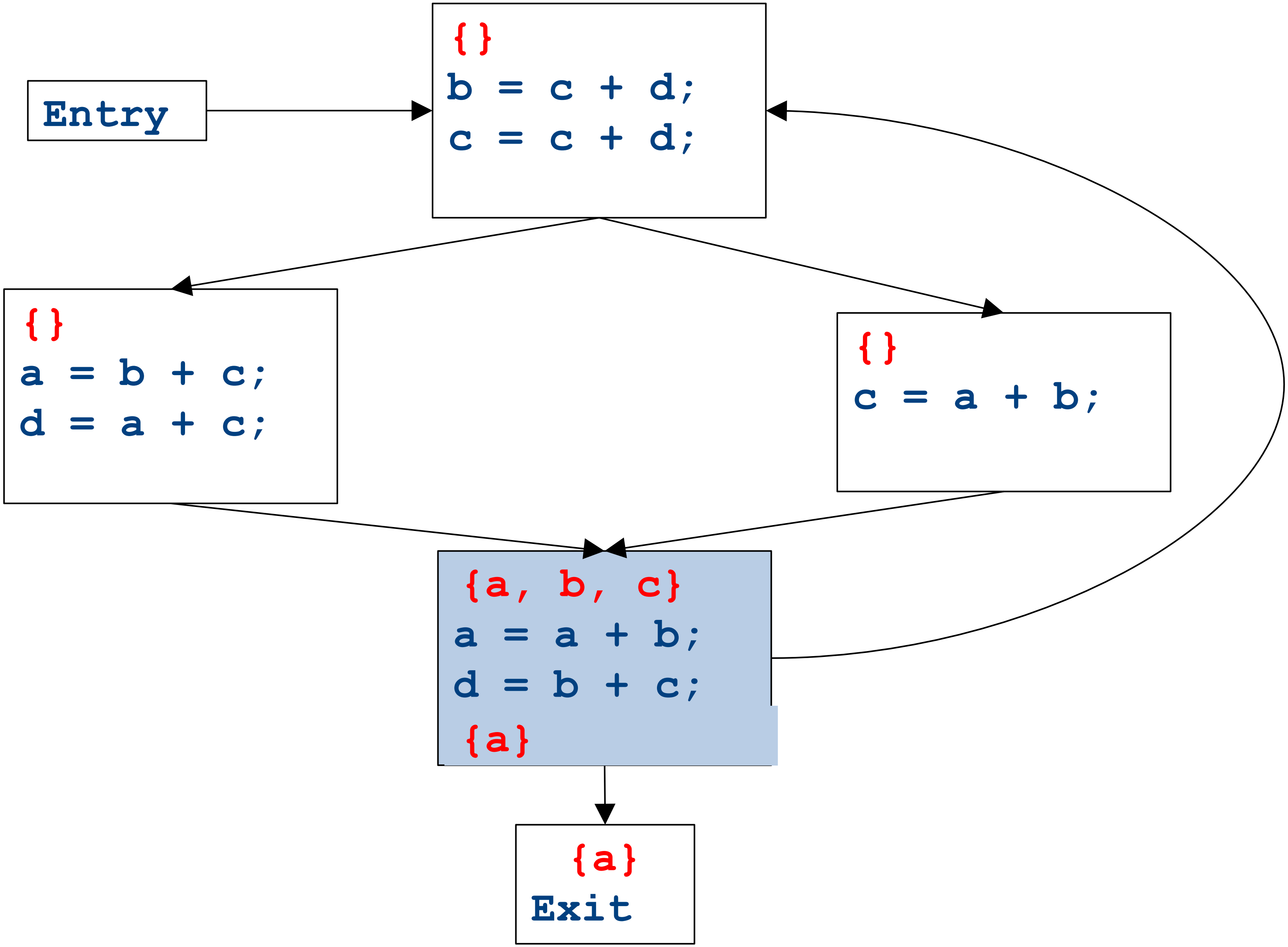
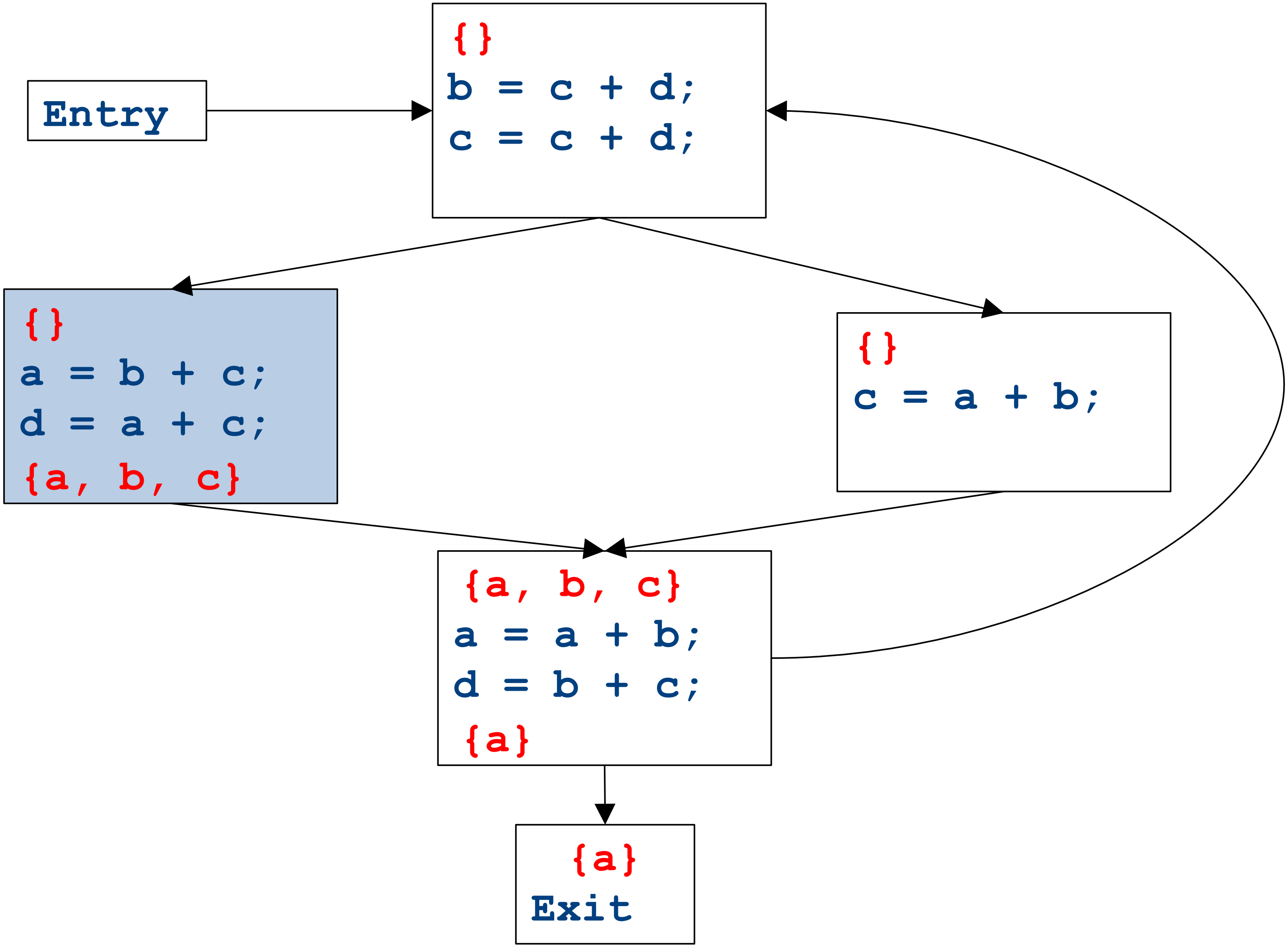


CFGs with loops - iteration



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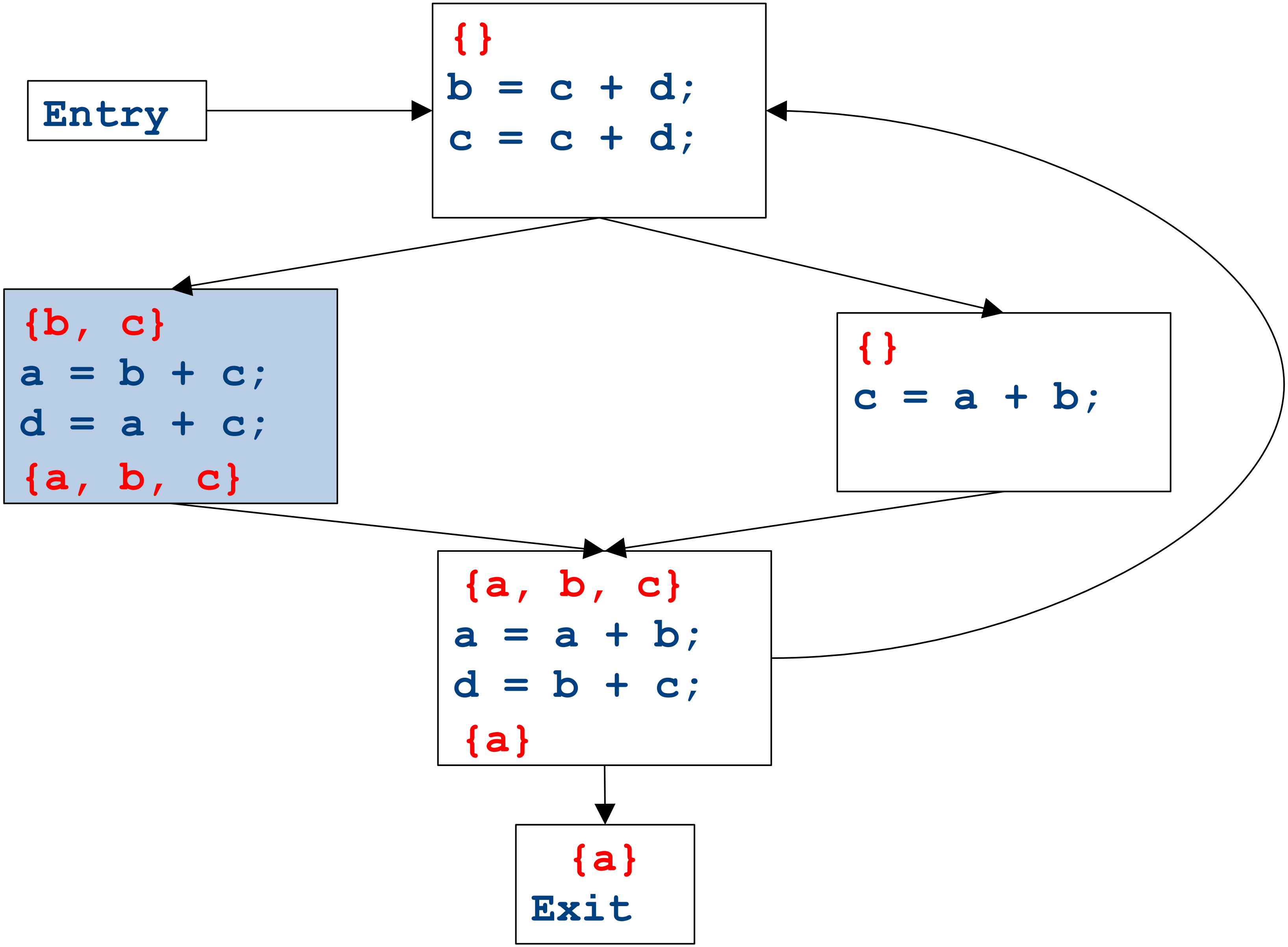

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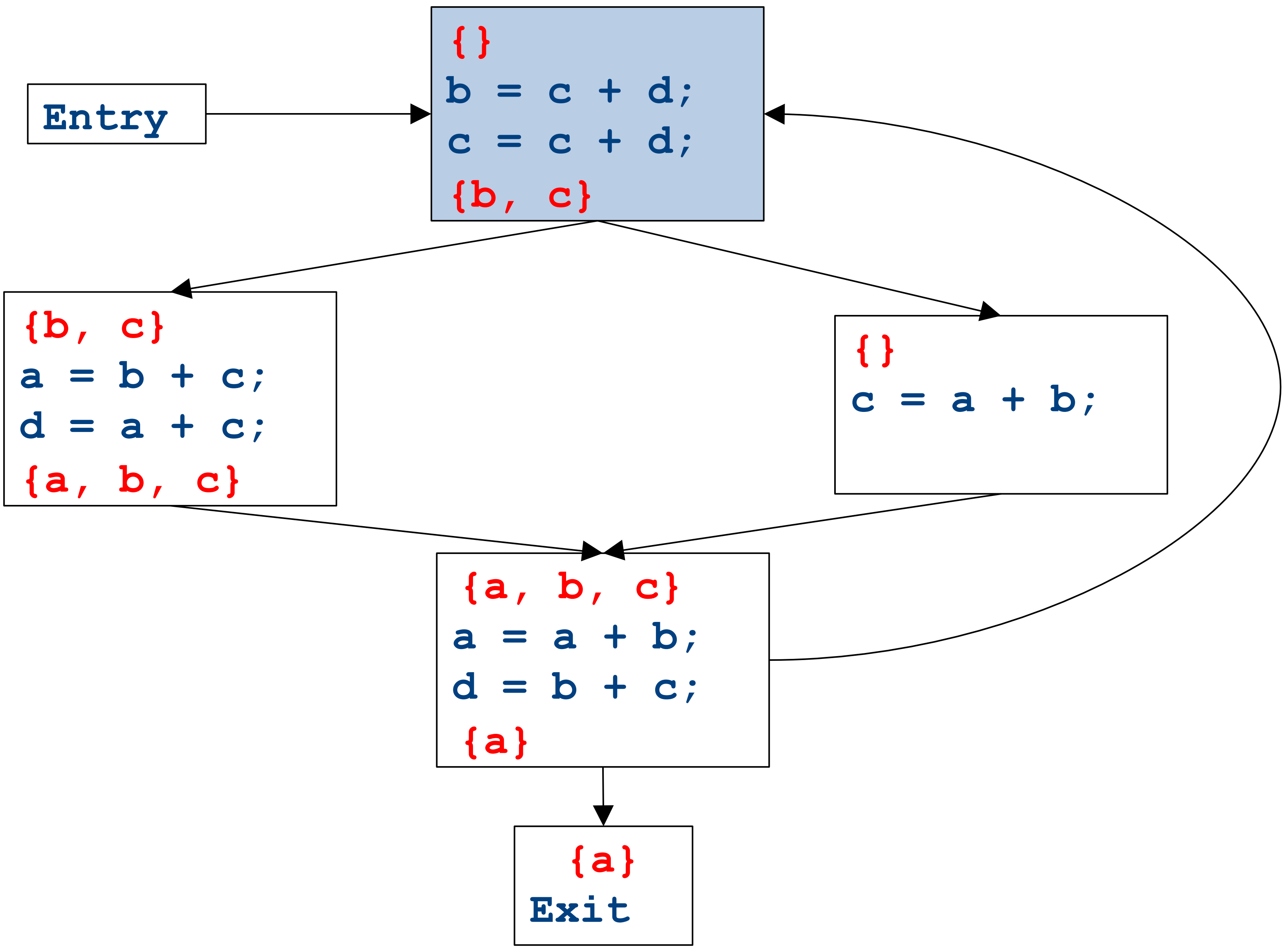

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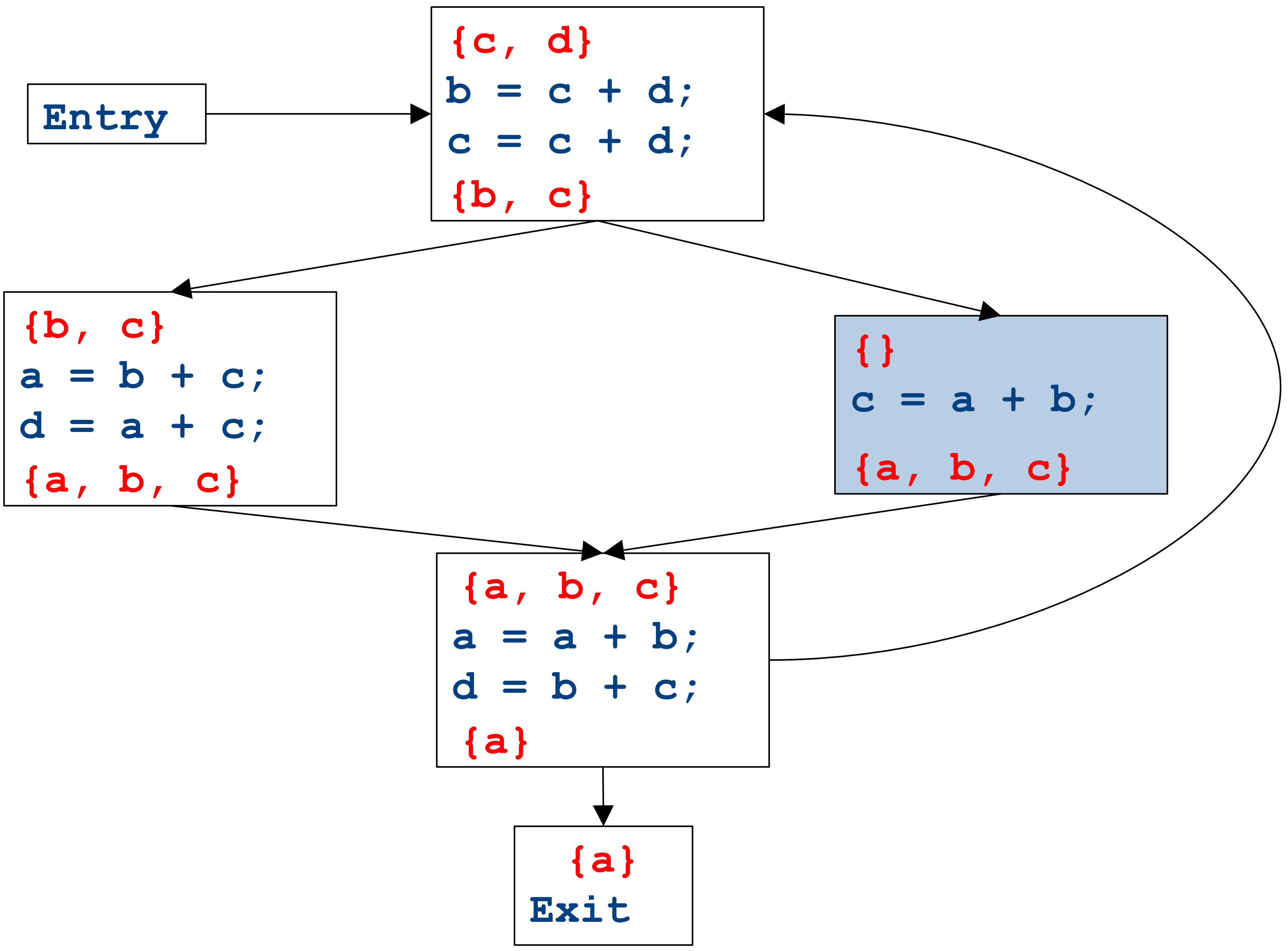

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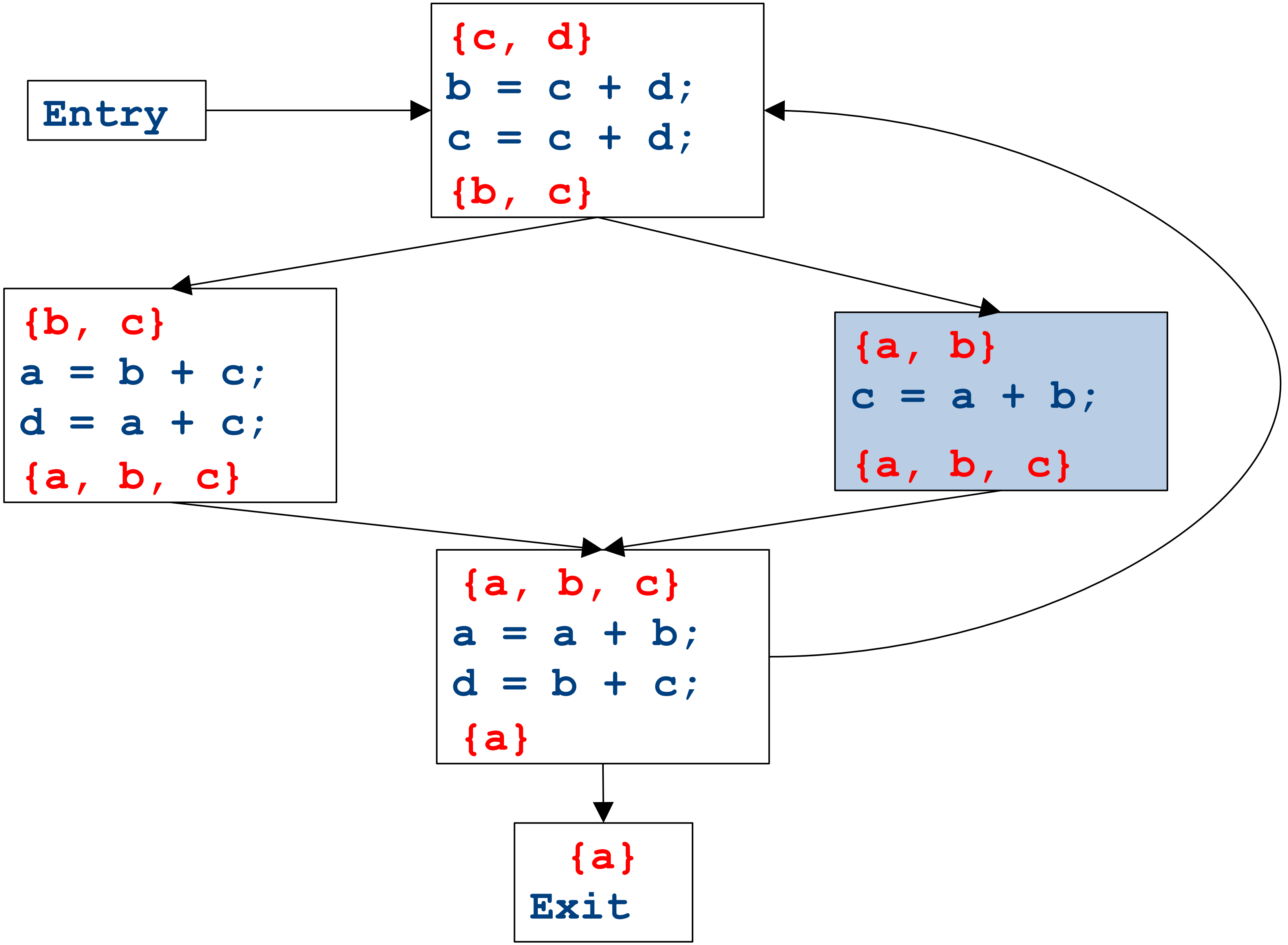

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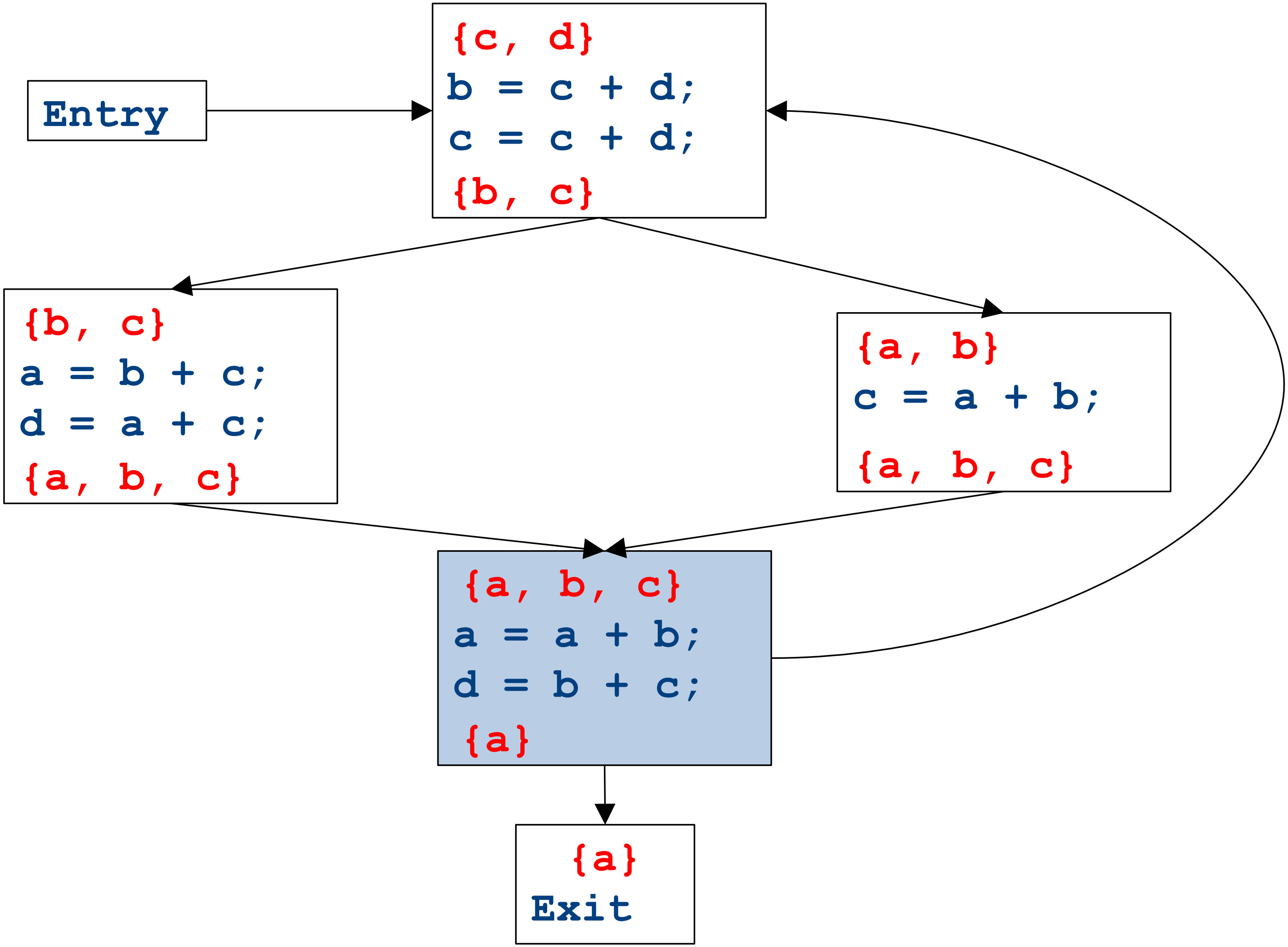

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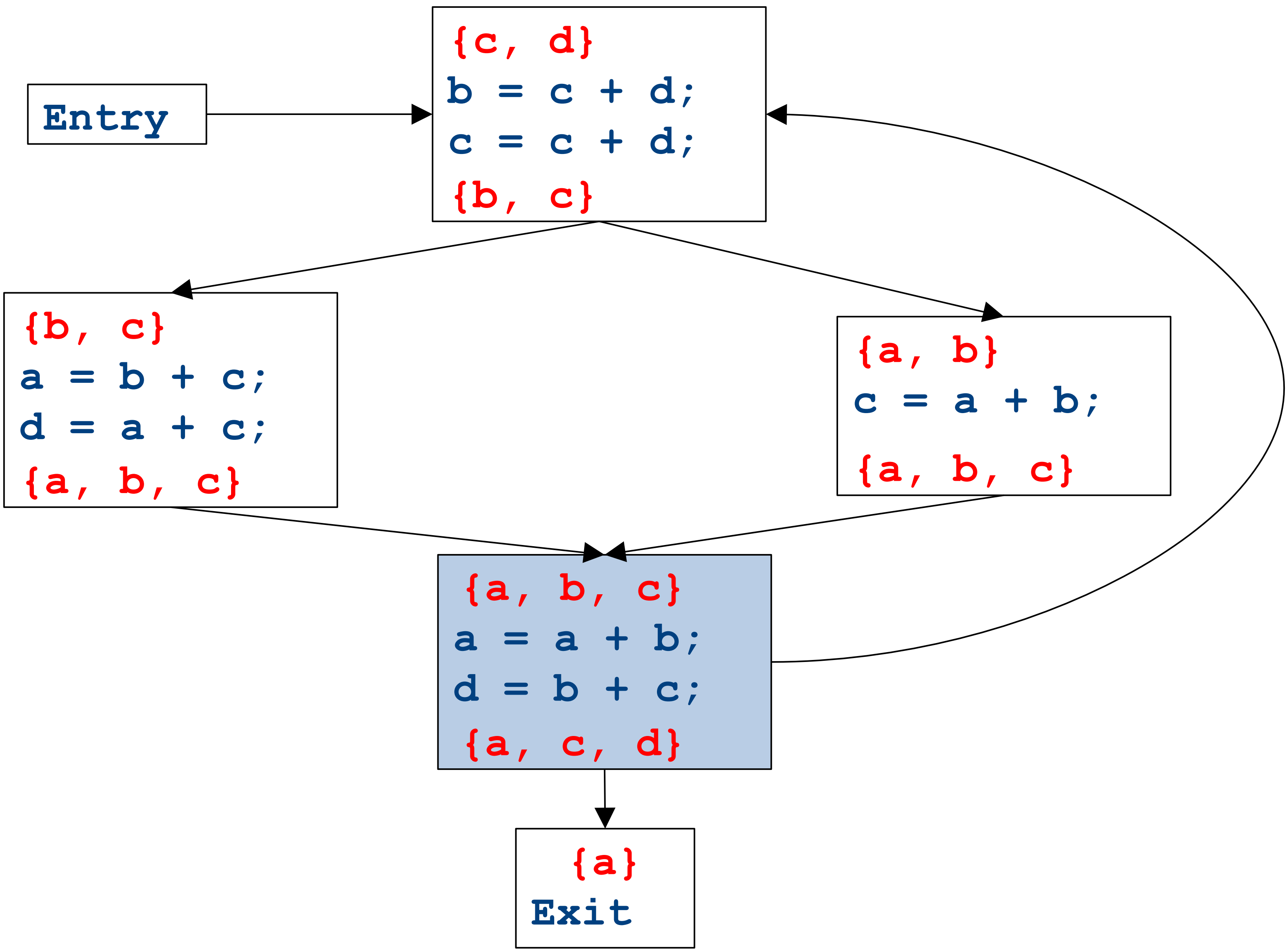

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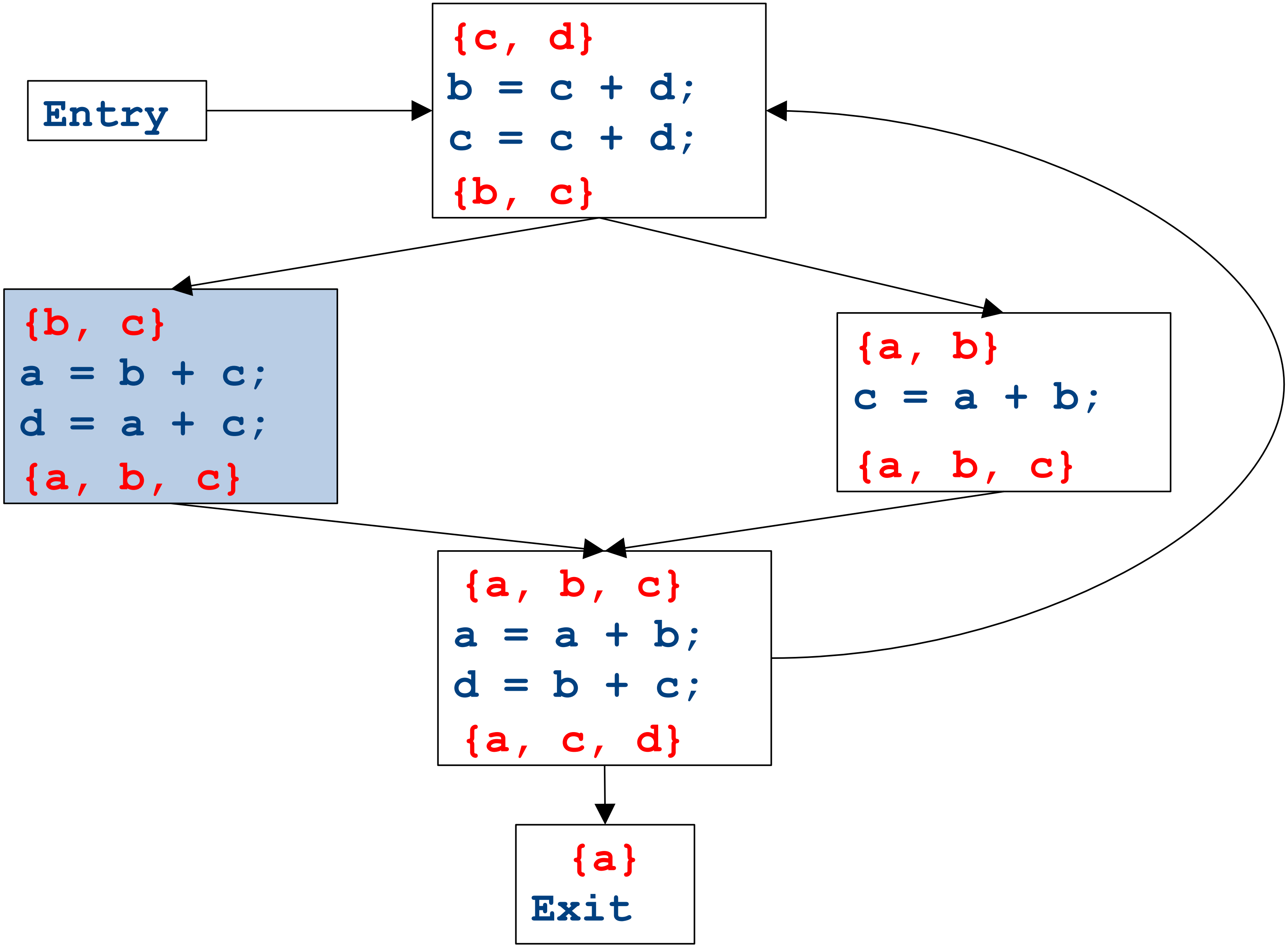
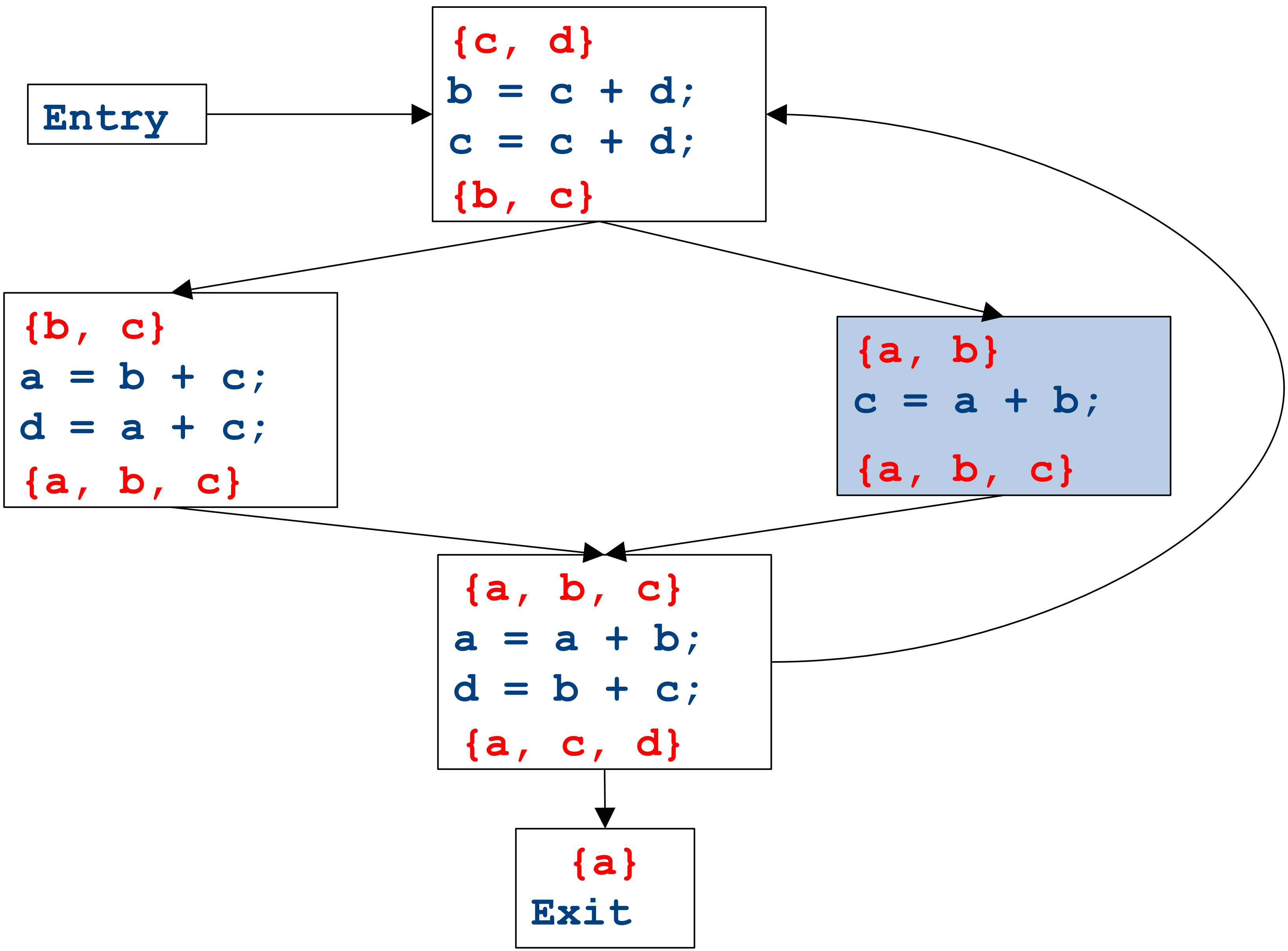


CFGs with loops - iteration



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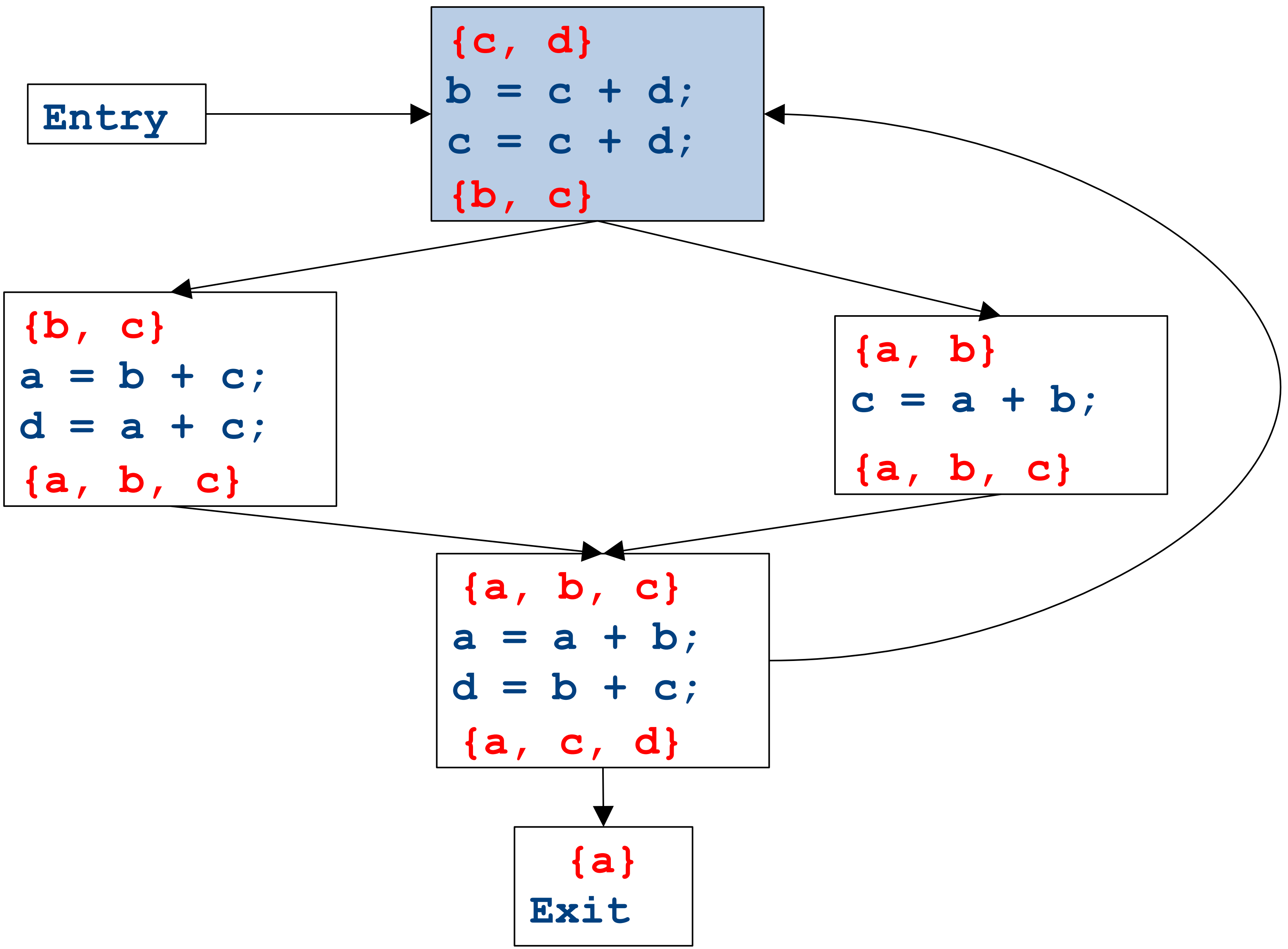

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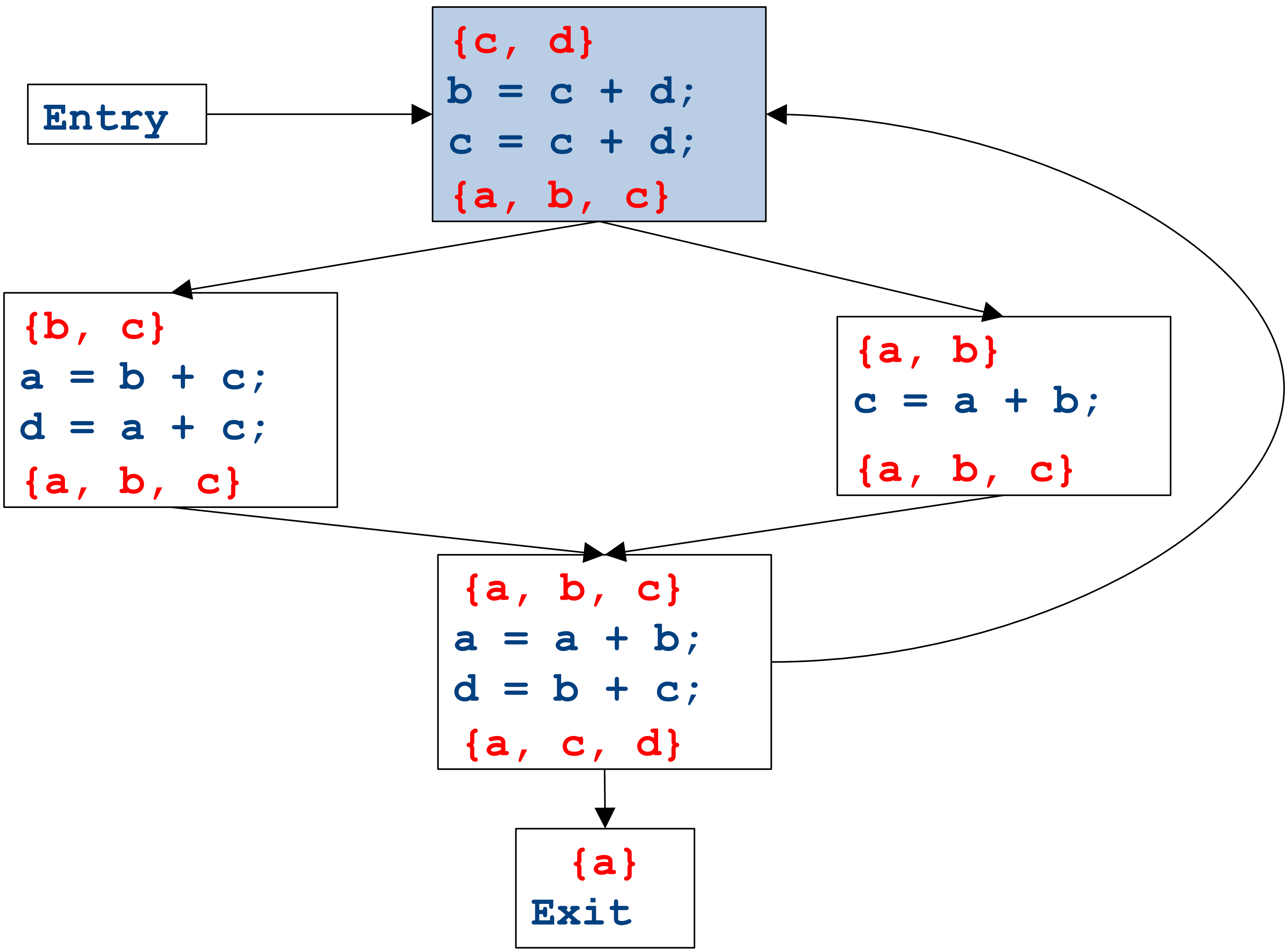

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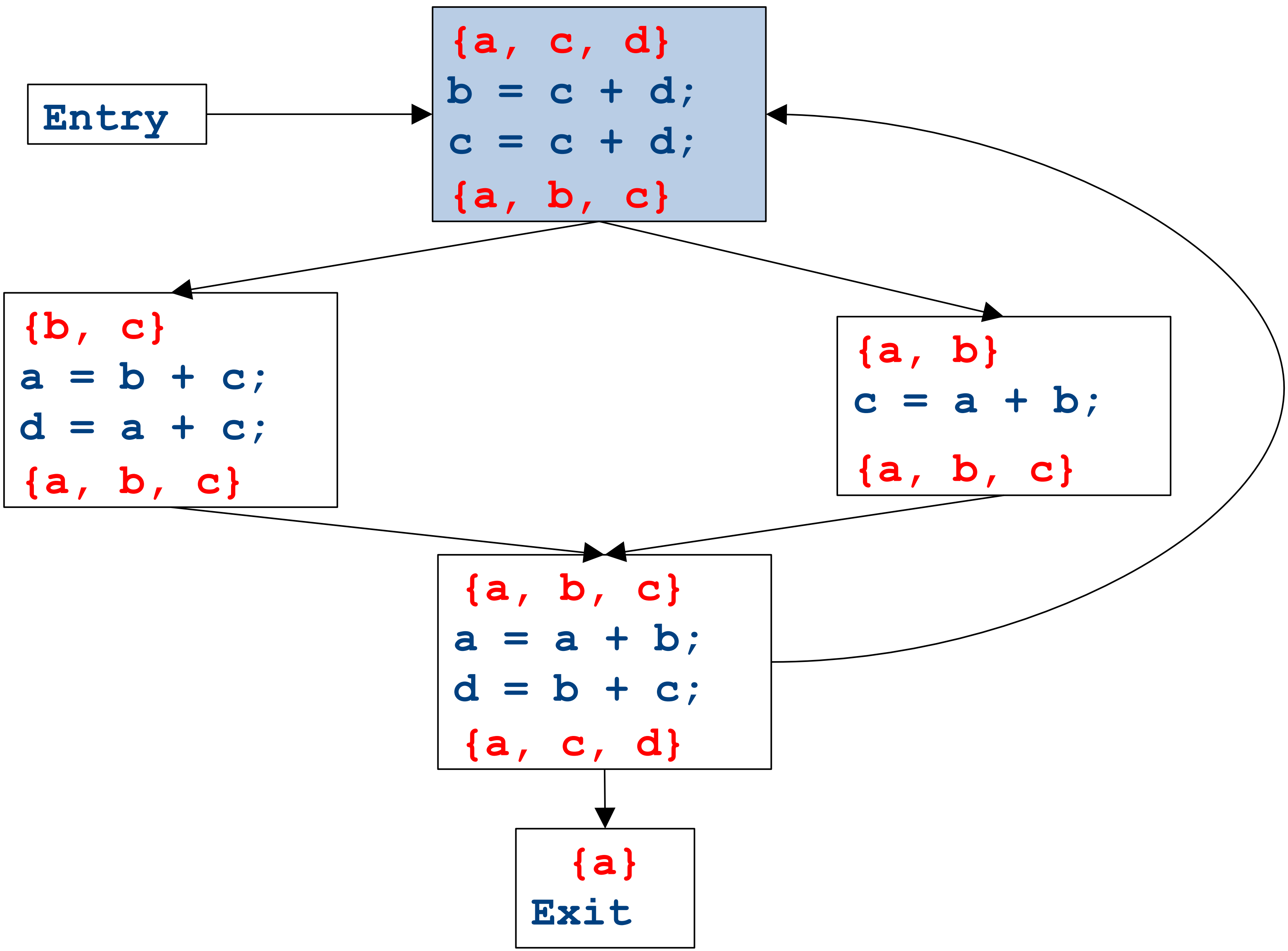

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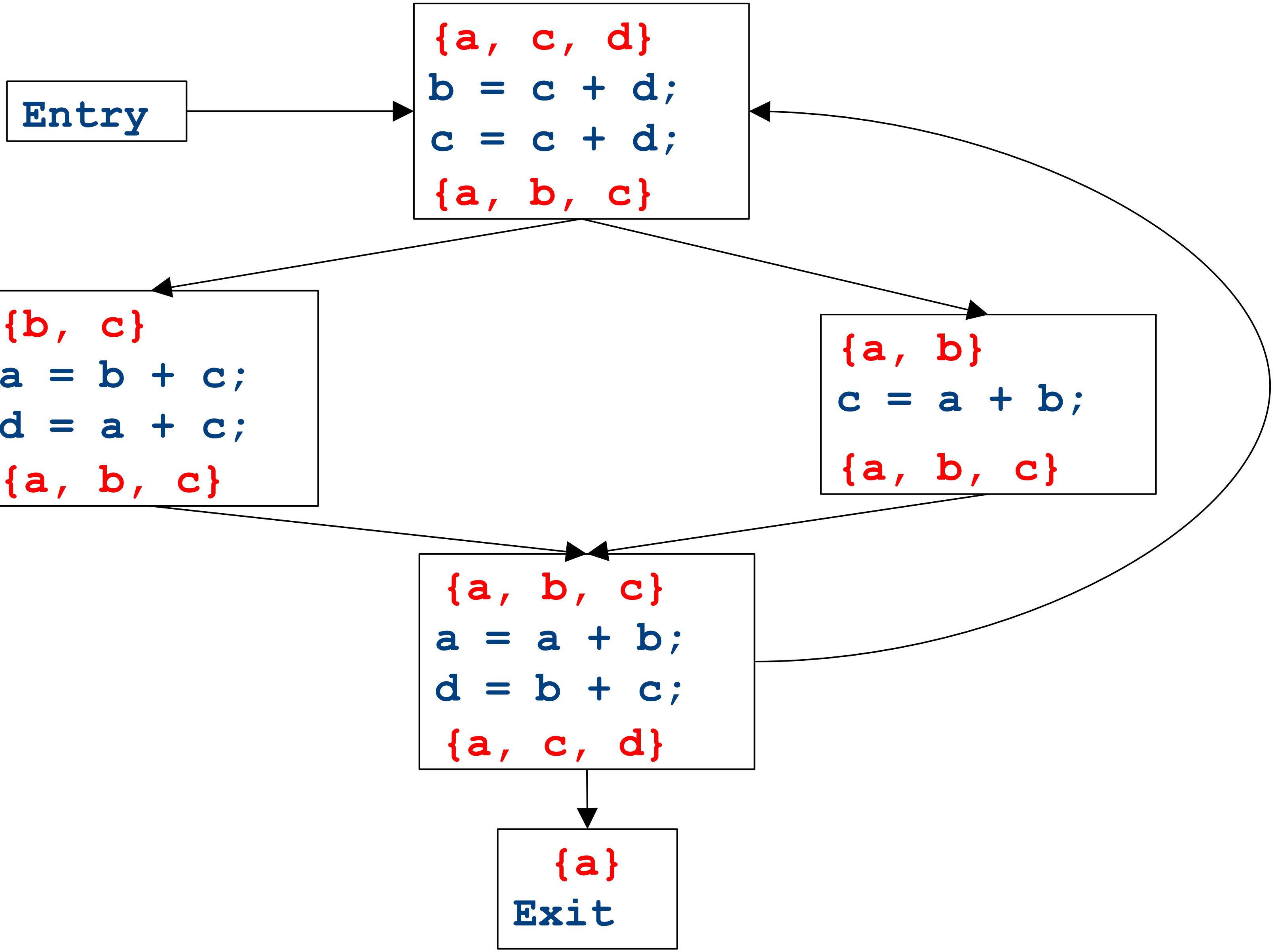


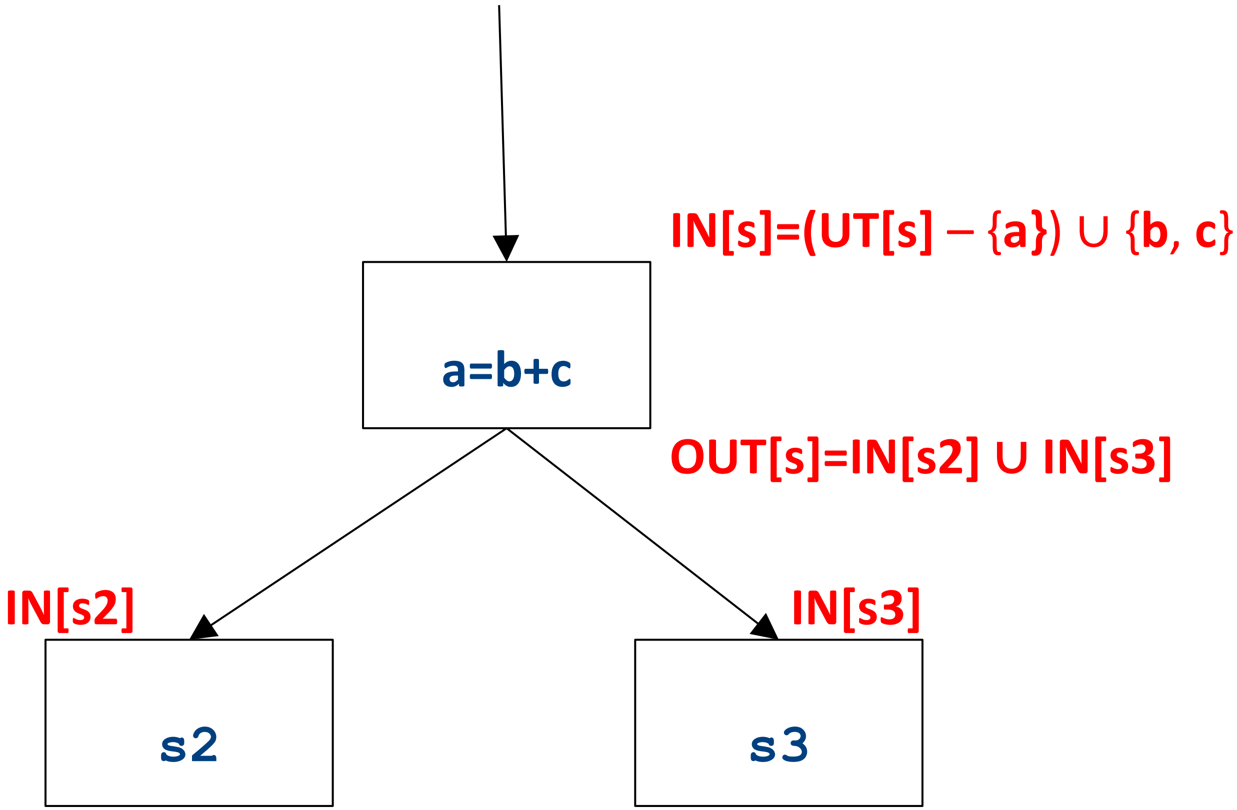
Summary of differences

- Need to be able to handle multiple predecessors/successors for a basic block
- Need to be able to handle multiple paths through the control-flow graph, and may need to iterate multiple times to compute the final value
  - But the analysis still needs to terminate!
- Need to be able to assign each basic block a reasonable default value for before we've analyzed it

Global liveness analysis

- Initially, set IN[**s**] = { } for each statement **s**
- Set IN[**exit**] to the set of variables known to be live on exit (language-specific knowledge)
- Repeat until no changes occur:
  - For each statement **s** of the form **a = b + c**, in any order you'd like:
    - Set OUT[**s**] to set union of IN[**p**] for each successor **p** of **s**
    - Set IN[**s**] to (OUT[**s**] – **a**) ∪ {**b**, **c**}.
- Yet another fixed-point iteration!

Global liveness analysis



Why does this work?

- To show correctness, we need to show that
  - The algorithm eventually terminates, and
  - When it terminates, it has a sound answer
- Termination argument:
  - Once a variable is discovered to be live during some point of the analysis, it always stays live
  - Only finitely many variables and finitely many places where a variable can become live
- Soundness argument (sketch):
  - Each individual rule, applied to some set, correctly updates liveness in that set
  - When computing the union of the set of live variables, a variable is only live if it was live on some path leaving the statement

Abstract Interpretation

- Theoretical foundations of program analysis
- Cousot and Cousot 1977
- Abstract meaning of programs
  - Executed at compile time

Another view of local optimization

- In local optimization, we want to reason about some property of the runtime behavior of the program
- Could we run the program and just watch what happens?
- **Idea:** Redefine the semantics of our programming language to give us information about our analysis

Properties of local analysis

- The only way to find out what a program will actually do is to run it
- Problems:
  - The program might not terminate
  - The program might have some behavior we didn't see when we ran it on a particular input
- However, this is not a problem inside a basic block
  - Basic blocks contain no loops
  - There is only one path through the basic block

Assigning new semantics

- Example: Available Expressions
- Redefine the statement **a = b + c to** mean "**a now holds the value of b + c,** and any variable holding the value **a is** now invalid"
- Run the program assuming these new semantics
- Treat the optimizer as an interpreter for these new semantics

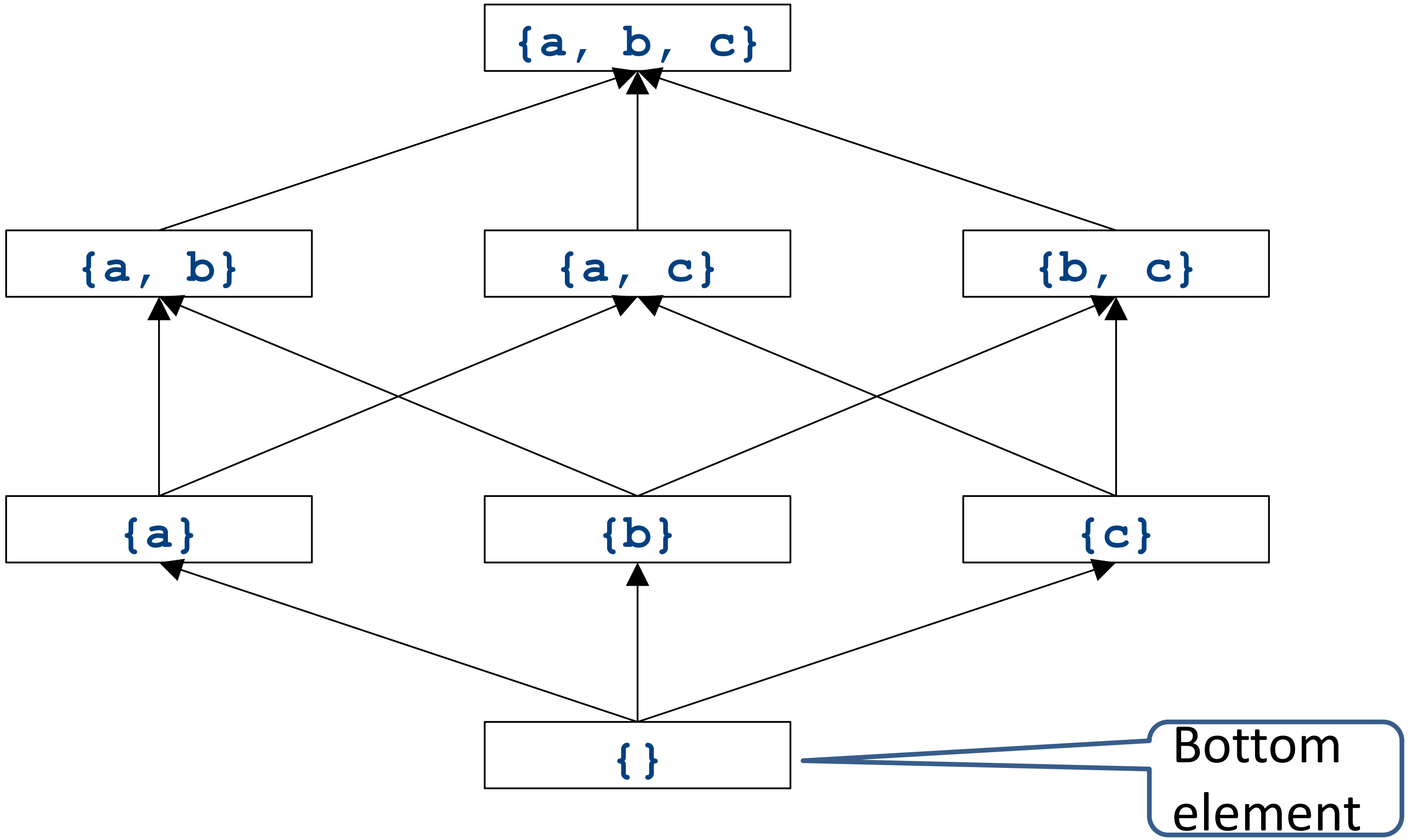
Theory to the rescue

- Building up all of the machinery to design this analysis was tricky
- The key ideas, however, are mostly independent of the analysis:
  - We need to be able to compute functions describing the behavior of each statement
  - We need to be able to merge several subcomputations together
  - We need an initial value for all of the basic blocks
- There is a beautiful formalism that captures many of these properties

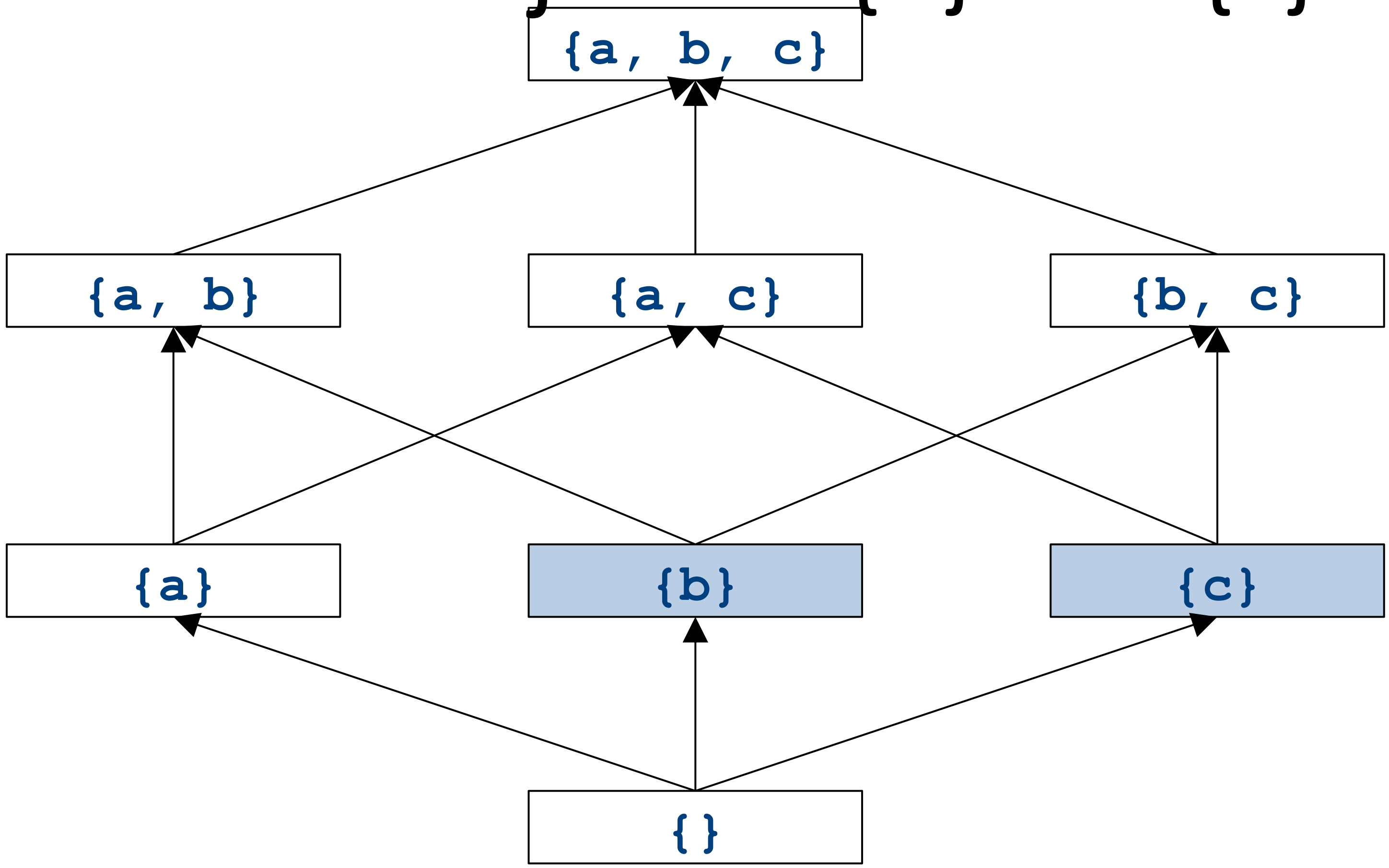
Join semilattices

- A join semilattice is an ordering defined on a set of elements
- Any two elements have some join that is the smallest element larger than both elements
- There is a unique bottom element, which is smaller than all other elements
- Intuitively:
  - The join of two elements represents combining information from two elements by an overapproximation
- The bottom element represents “no information yet” or “the least conservative possible answer”

Join semilattice for liveness



What is the join of {b} and {c}?



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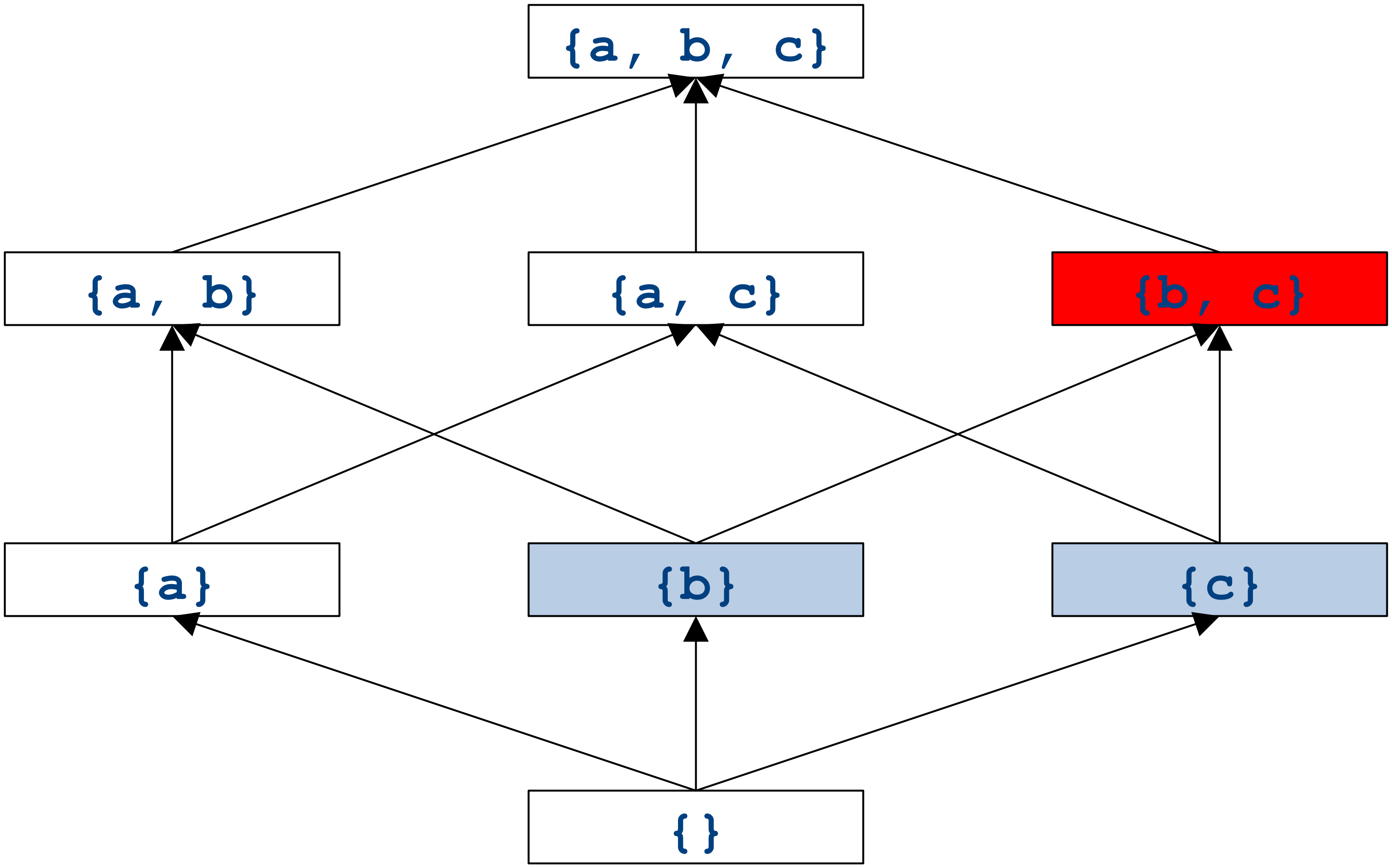

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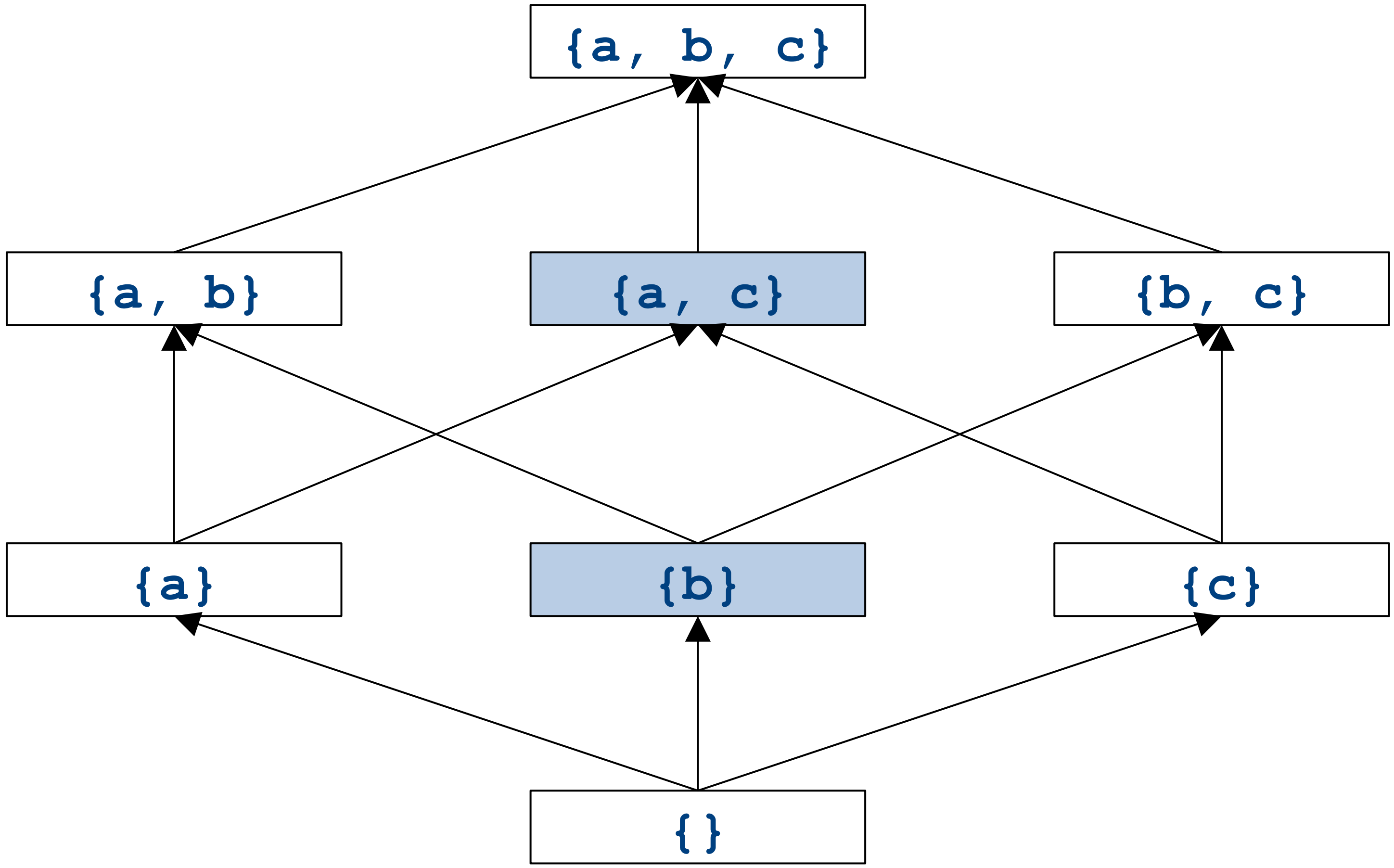

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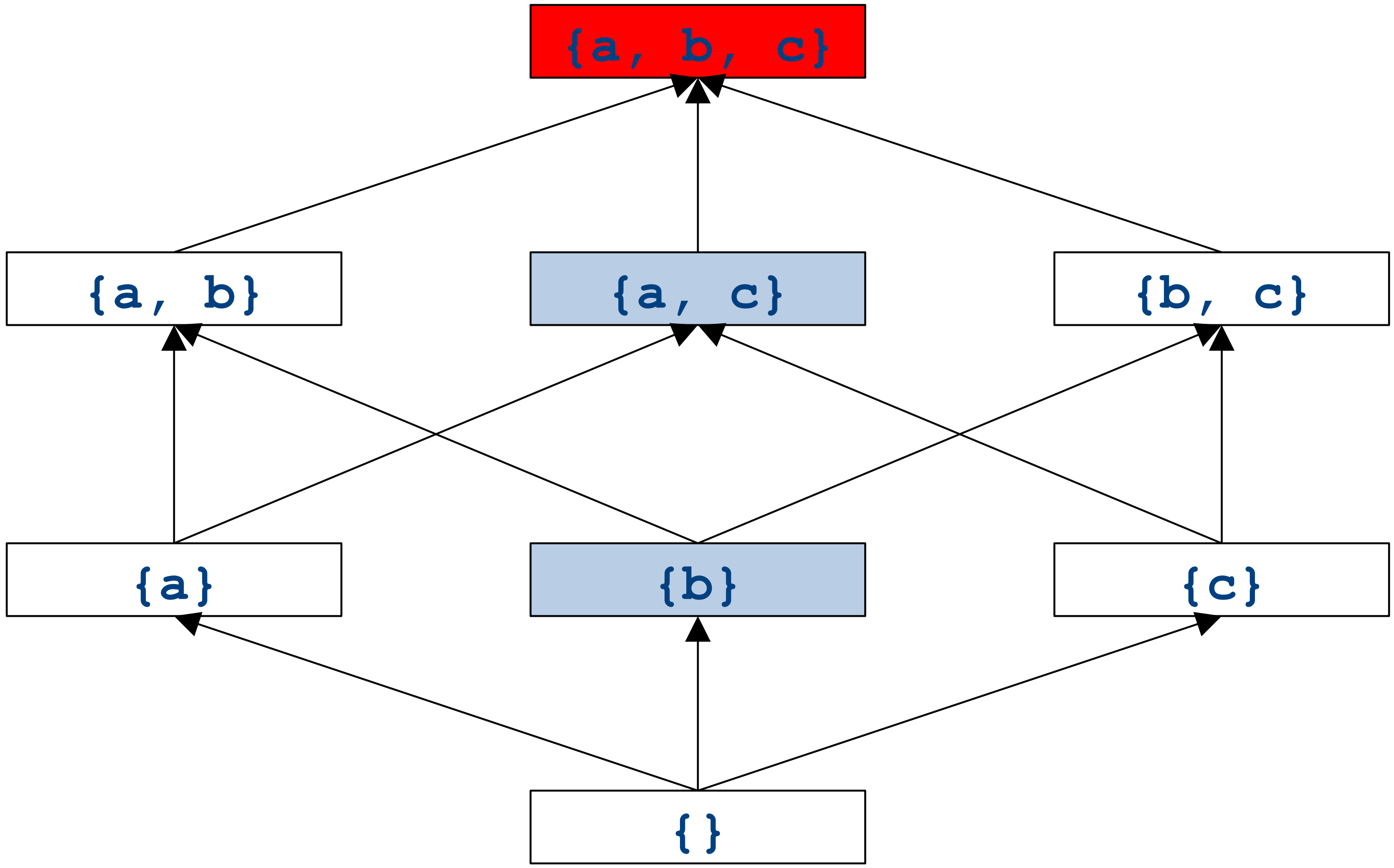
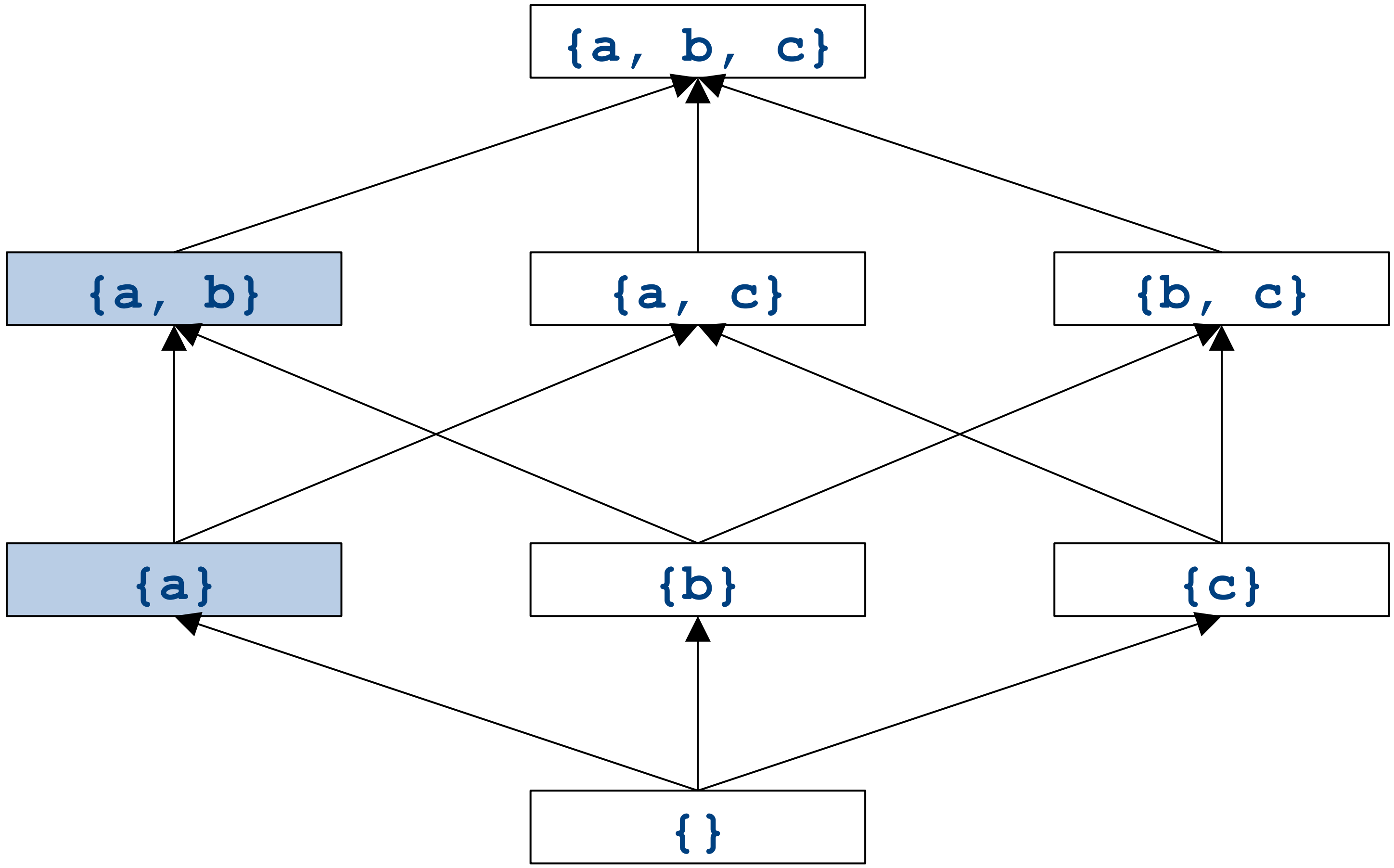


What is the join of {a} and {a,b}?



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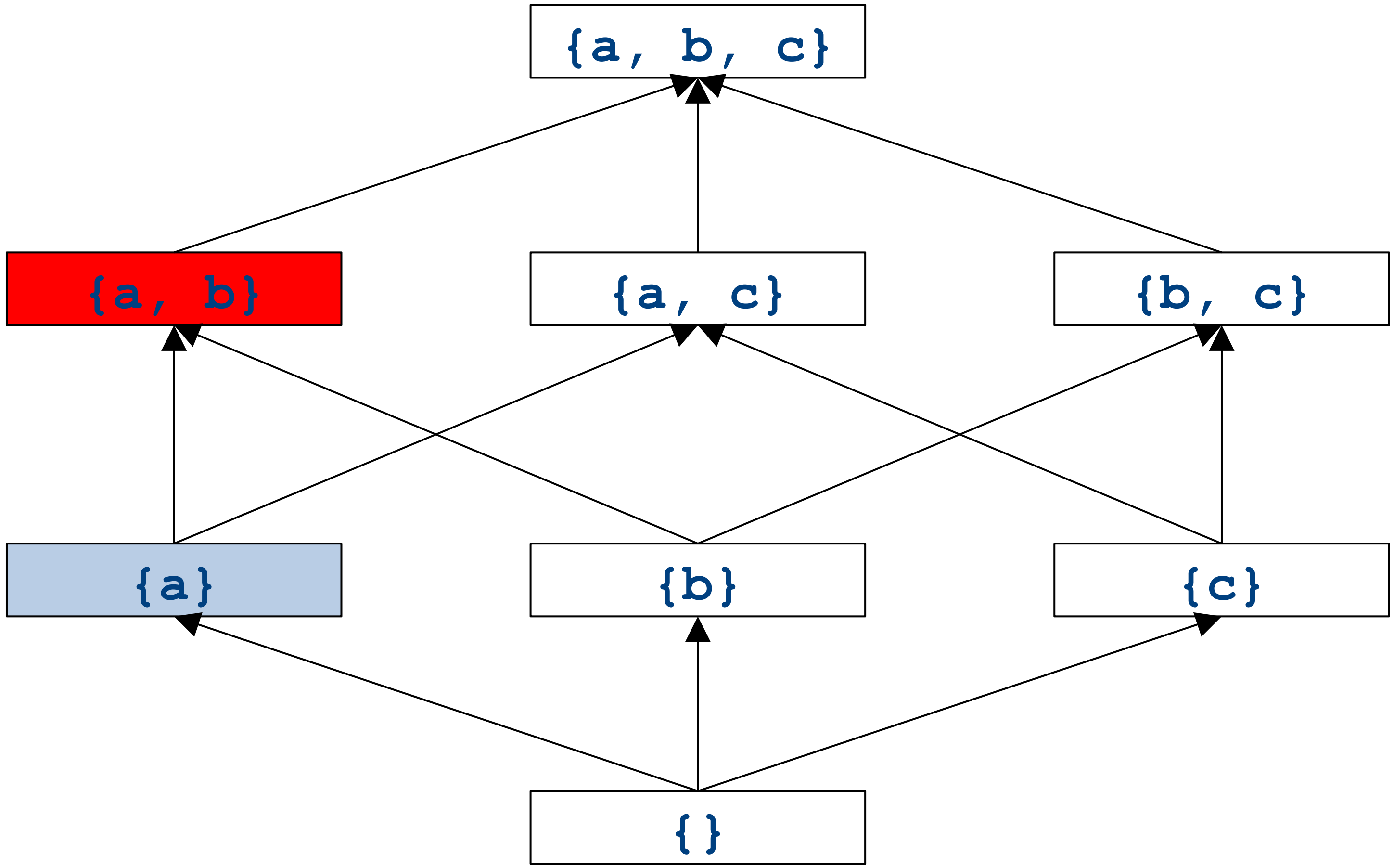
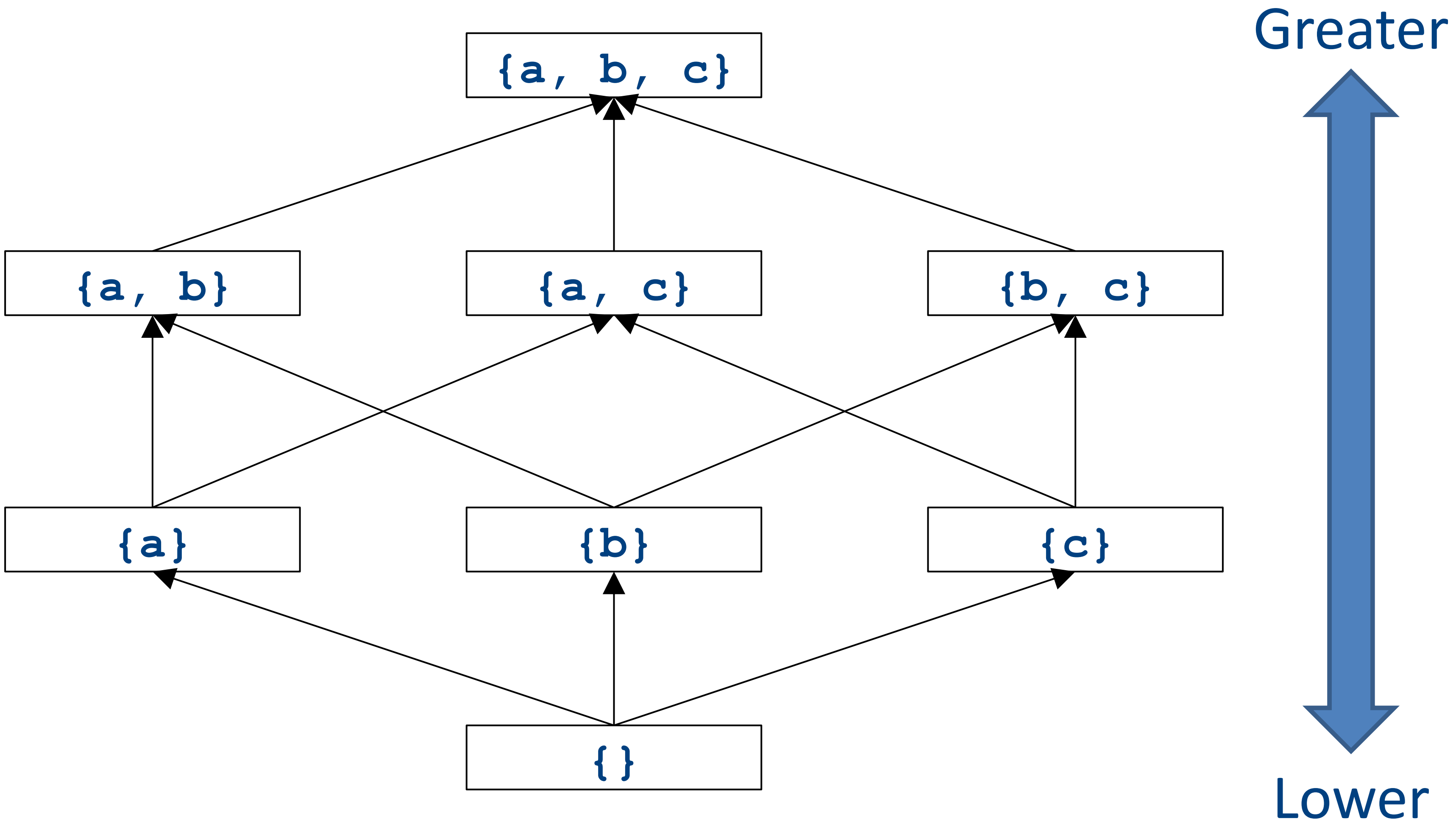


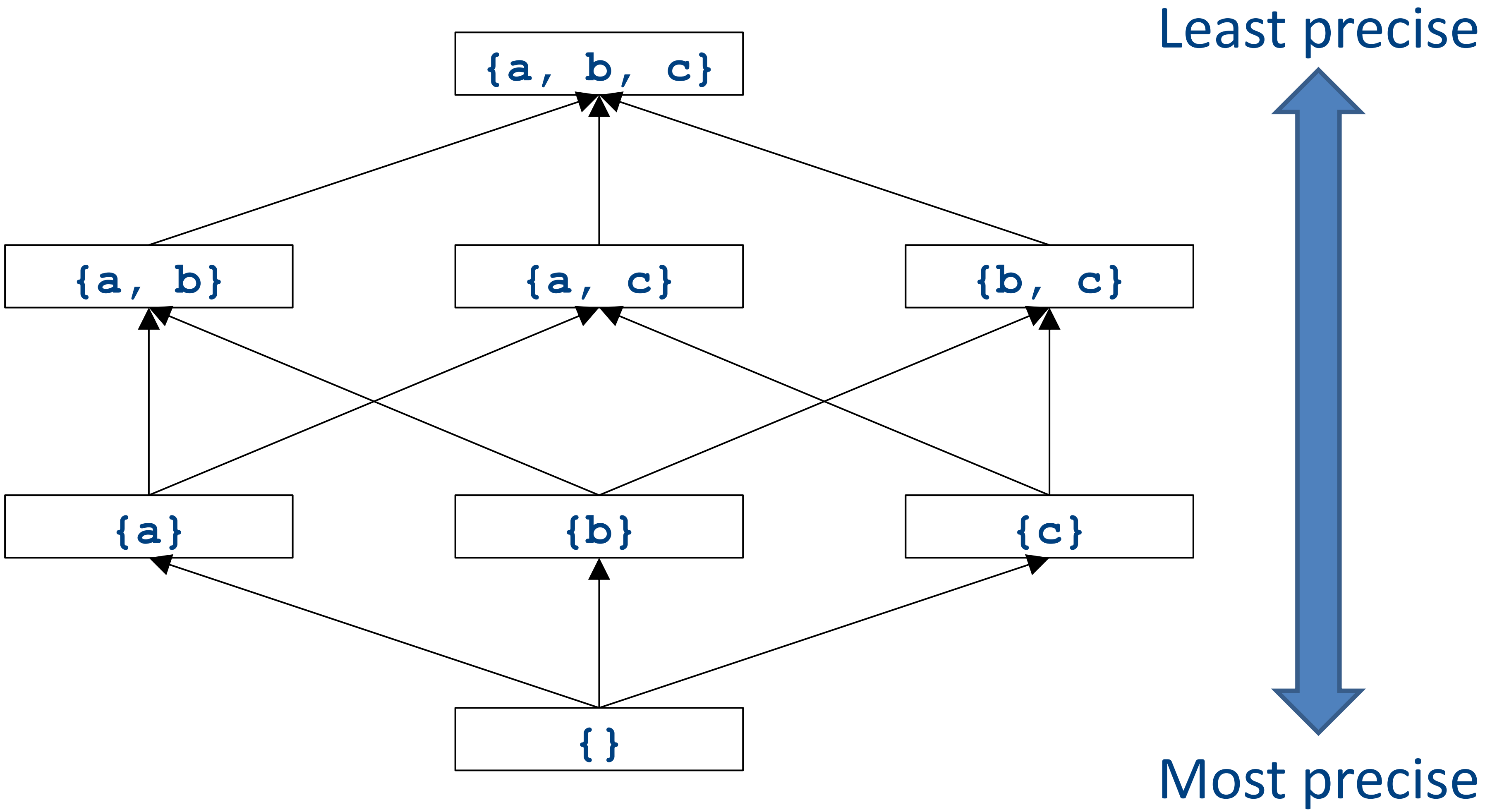
Formal definitions

- A join semilattice is a pair (V, $\sqcup$), where
- V is a domain of elements
- $\sqcup$ is a join operator that is
  - commutative: $x \sqcup y = y \sqcup x$
  - associative: $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
  - idempotent: $x \sqcup x = x$
- If $x \sqcup y = z$, we say that z is the join or (**l**east **u**pper **b**ound) of x and y
- Every join semilattice has a bottom element denoted $\perp$ such that $\perp \sqcup x = x$ for all x

Join semilattices and ordering



Join semilattices and ordering



Join semilattices and orderings

- Every join semilattice $(V, \sqcup)$ induces an ordering relationship $\sqsubseteq$ over its elements
- Define $x \sqsubseteq y$ iff $x \sqcup y = y$
- Need to prove
  - Reflexivity: $x \sqsubseteq x$
  - Antisymmetry: If $x \sqsubseteq y$ and $y \sqsubseteq x$, then $x = y$
  - Transitivity: If $x \sqsubseteq y$ and $y \sqsubseteq z$, then $x \sqsubseteq z$

An example join semilattice

- The set of natural numbers and the **max** function
- Idempotent
  - **max**{a, a} = a
- Commutative
  - **max**{a, b} = **max**{b, a}
- Associative
  - **max**{a, **max**{b, c}} = **max**{**max**{a, b}, c}
- Bottom element is 0:
  - **max**{0, a} = a
- What is the ordering over these elements?

A join semilattice for liveness

- Sets of live variables and the set union operation
- Idempotent:
  - $x \cup x = x$
- Commutative:
  - $x \cup y = y \cup x$
- Associative:
  - $(x \cup y) \cup z = x \cup (y \cup z)$
- Bottom element:
  - The empty set: $\emptyset \cup x = x$
- What is the ordering over these elements?

Semilattices and program analysis

- Semilattices naturally solve many of the problems we encounter in global analysis
- How do we combine information from multiple basic blocks?
- What value do we give to basic blocks we haven't seen yet?
- How do we know that the algorithm always terminates?

Semilattices and program analysis

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  - Use the bottom element
- How do we know that the algorithm always terminates?
  - Actually, we still don't! More on that later

A general framework

- A global analysis is a tuple (D, V, $\sqsubseteq$, F, I), where
  - D is a direction (forward or backward)
    - The order to visit statements within a basic block, not the order in which to visit the basic blocks
  - V is a set of values
  - $\sqcup$ is a join operator over those values
  - F is a set of transfer functions f : **V** → **V**
  - I is an initial value
- The only difference from local analysis is the introduction of the join operator

Running global analyses

- Assume that $(\mathbb{D}, \mathbb{V}, \sqcup, \mathbb{F}, \mathbb{I})$ is a forward analysis
- Set OUT[**s**] = $\perp$ for all statements **s**
- Set OUT[**entry**] = $\mathbb{I}$
- Repeat until no values change:
  - For each statement **s** with predecessors $\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}$:
    - Set IN[**s**] = OUT[$\mathbf{p_1}$] $\sqcup$ OUT[$\mathbf{p_2}$] $\sqcup$ … $\sqcup$ OUT[$\mathbf{p_n}$]
    - Set OUT[**s**] = f_s (IN[**s**])
- The order of this iteration does not matter
  - This is sometimes called chaotic iteration

For comparison

- Set OUT[**s**] = ⊥ for all statements **s**
- Set OUT[**entry**] = **I**
- Repeat until no values change:
  - For each statement **s** with predecessors $\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}$:
    - Set IN[**s**] = OUT[$\mathbf{p_1}$] ⊔ OUT[$\mathbf{p_2}$] ⊔ … ⊔ OUT[$\mathbf{p_n}$]
    - Set OUT[**s**] = f_s (IN[**s**])

- Set IN[**s**] = {} for all statements **s**
- Set OUT[**exit**] = the set of variables known to be live on exit
- Repeat until no values change:
  - For each statement **s** of the form **a=b+c**:
    - Set OUT[**s**] = set union of IN[**x**] for each successor **x** of **s**
    - Set IN[**s**] = (OUT[**s**]-{a}) ∪ {b,c}

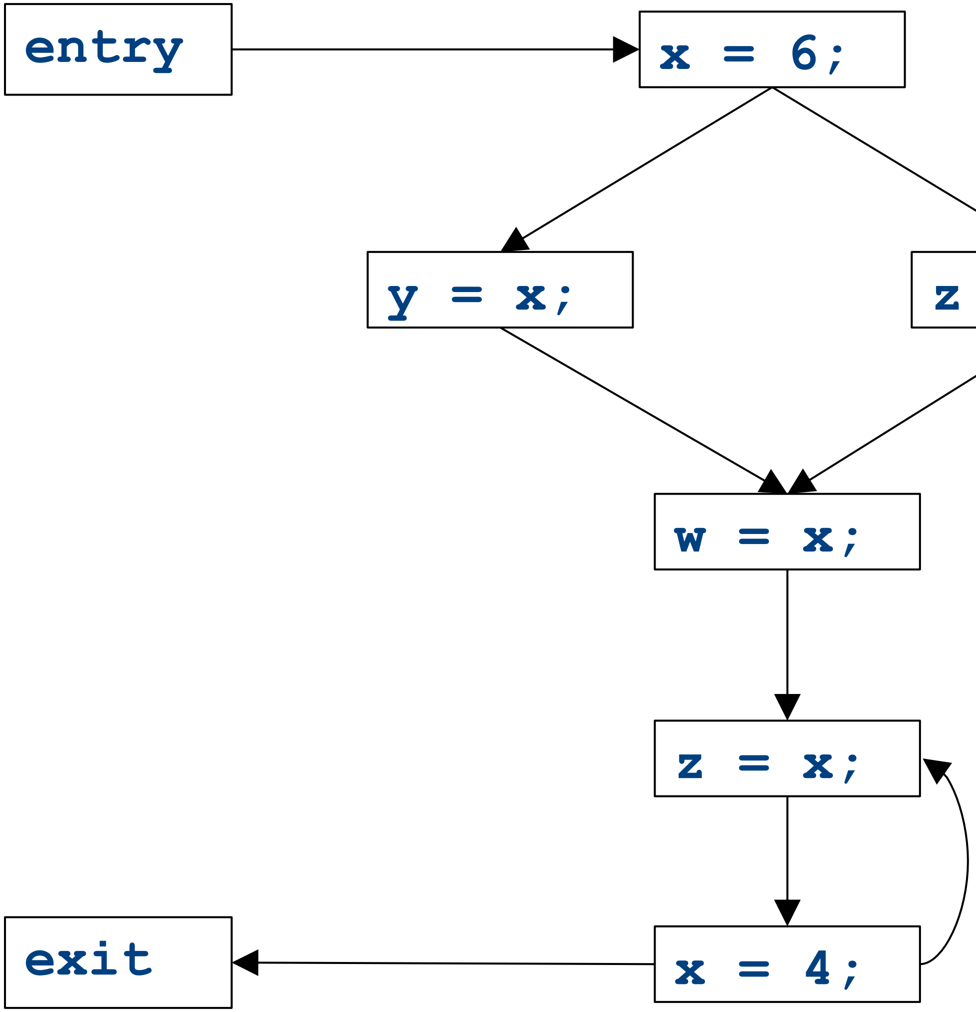
The dataflow framework

- This form of analysis is called the dataflow framework
- Can be used to easily prove an analysis is sound
- With certain restrictions, can be used to prove that an analysis eventually terminates
  - Again, more on that later

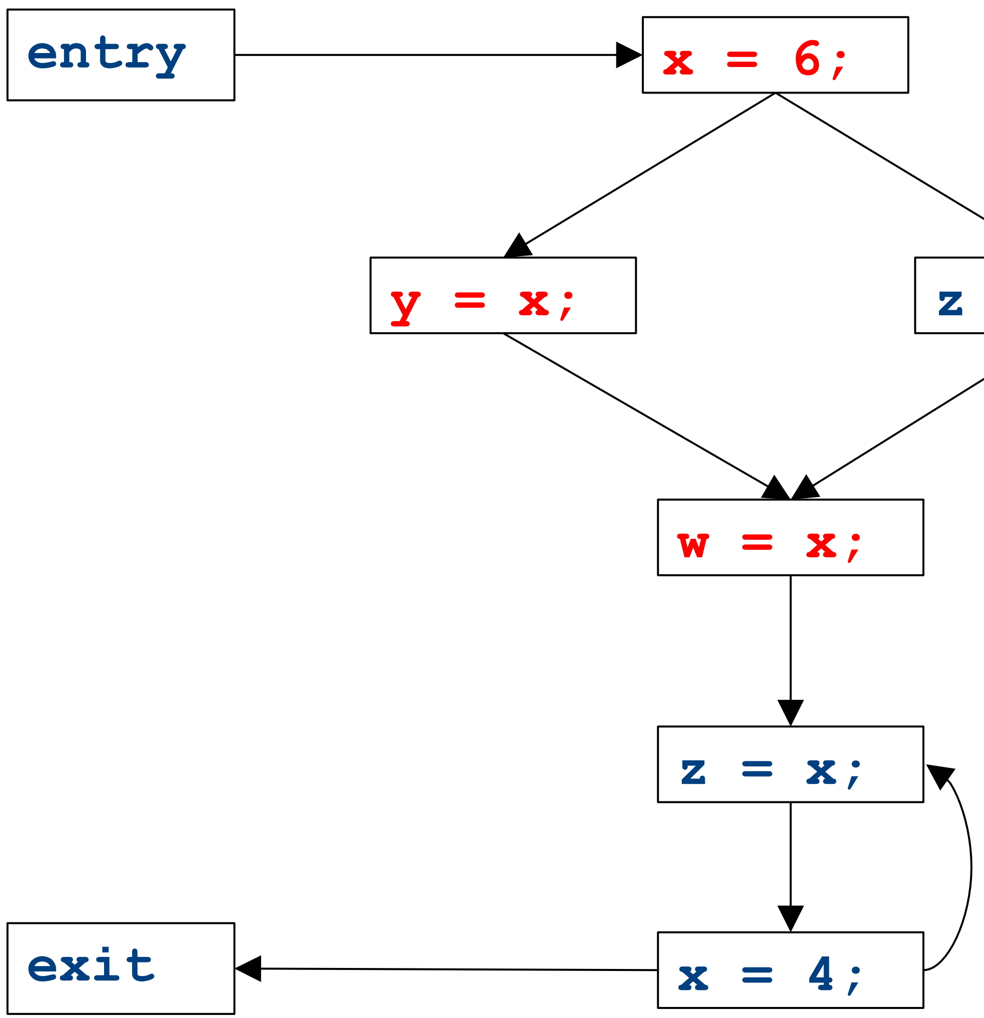
Global constant propagation

- Constant propagation is an optimization that replaces each variable that is known to be a constant value with that constant
- An elegant example of the dataflow framework

Global constant propagation



Global constant propagation



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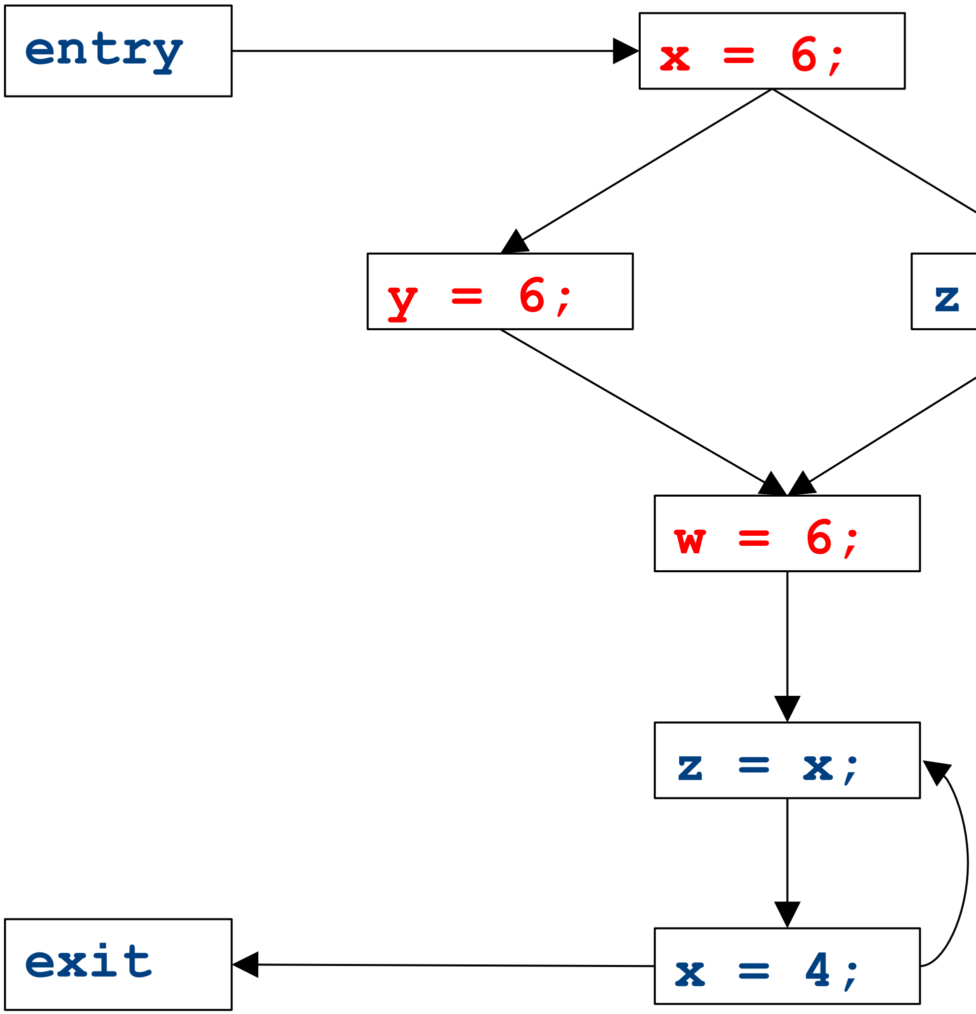


Constant propagation analysis

- In order to do a constant propagation, we need to track what values might be assigned to a variable at each program point
- Every variable will either
  - Never have a value assigned to it,
  - Have a single constant value assigned to it,
  - Have two or more constant values assigned to it, or
  - Have a known non-constant value.
  - Our analysis will propagate this information throughout a CFG to identify locations where a value is constant

Properties of constant propagation

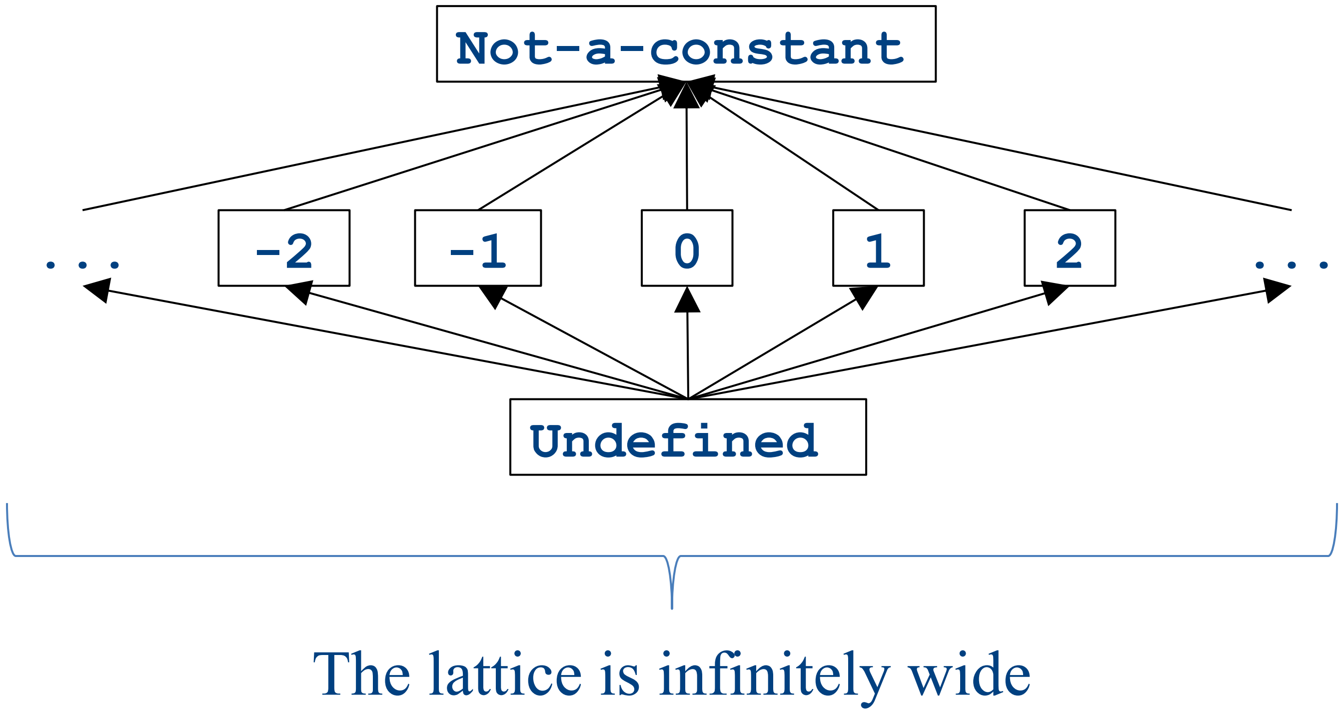
- For now, consider just some single variable **x**
- At each point in the program, we know one of three things about the value of **x**:
  - **x** is definitely not a constant, since it's been assigned two values or assigned a value that we know isn't a constant
  - **x** is definitely a constant and has value **k**
  - We have never seen a value for **x**
- Note that the first and last of these are **not** the same!
  - The first one means that there may be a way for **x** to have multiple values
  - The last one means that **x** never had a value at all

Defining a join operator

- The join of any two different constants is **Not-a-Constant**
  - (If the variable might have two different values on entry to a statement, it cannot be a constant)
- The join of **Not a Constant** and any other value is **Not-a-Constant**
  - (If on some path the value is known not to be a constant, then on entry to a statement its value can't possibly be a constant)
- The join of **Undefined** and any other value is that other value
  - (If **x** has no value on some path and does have a value on some other path, we can just pretend it always had the assigned value)

A semilattice for constant propagation

- One possible semilattice for this analysis is shown here (for each variable):



A semilattice for constant propagation

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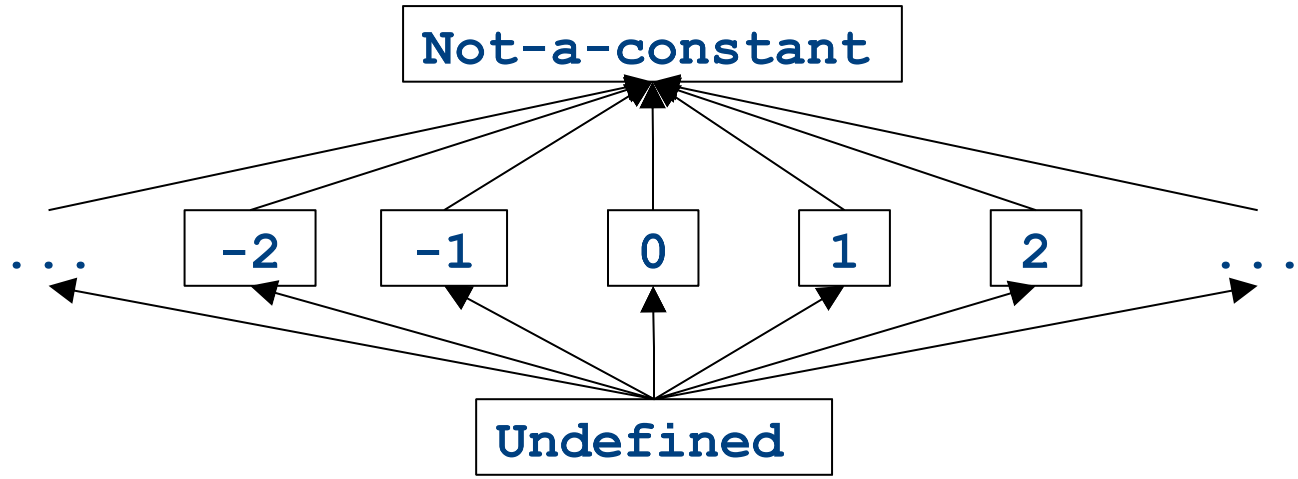
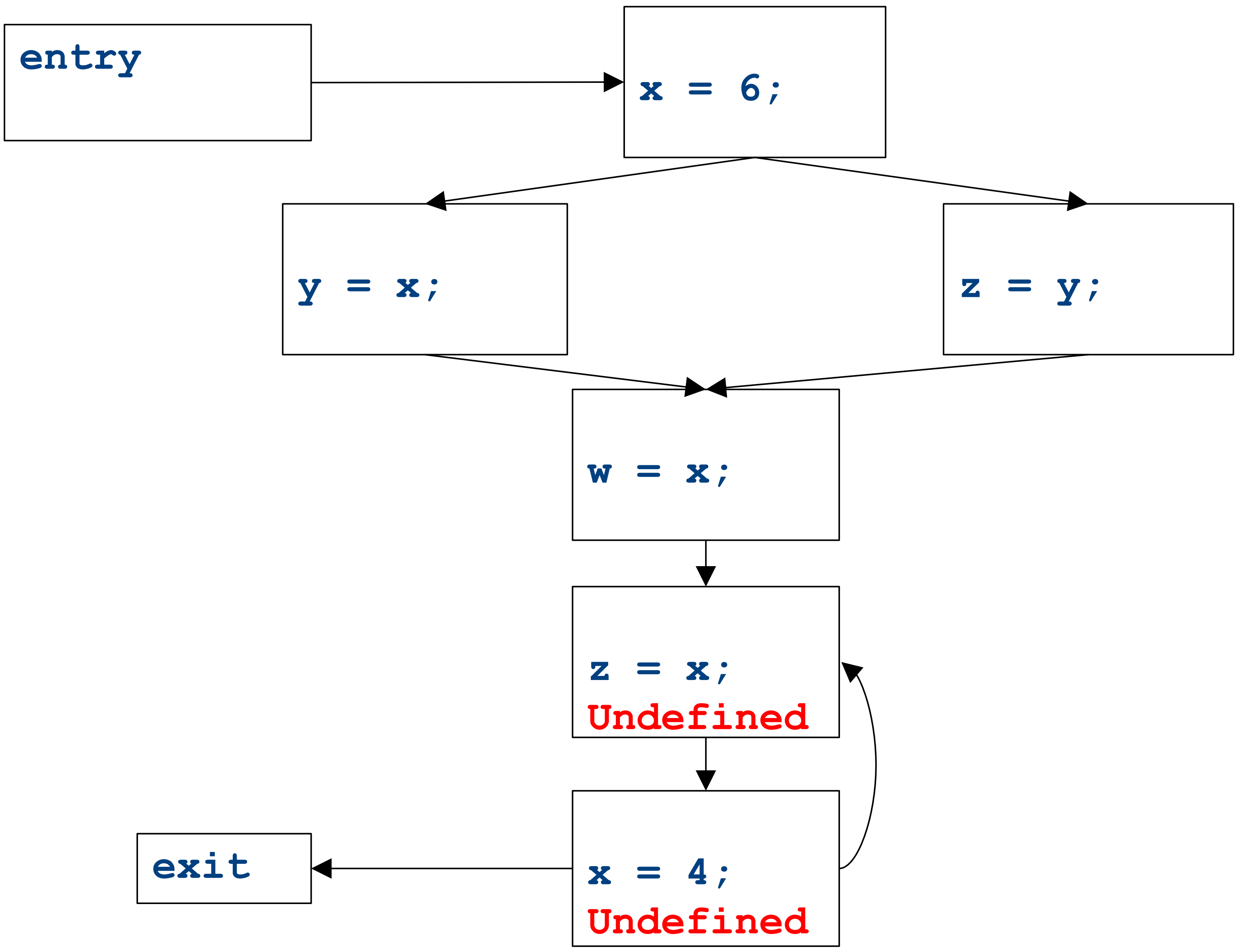


- Note:
  - The join of any two different constants is **Not-a-Constant**
  - The join of **Not a Constant** and any other value is **Not-a-Constant**
  - The join of **Undefined** and any other value is that other value

Global constant propagation



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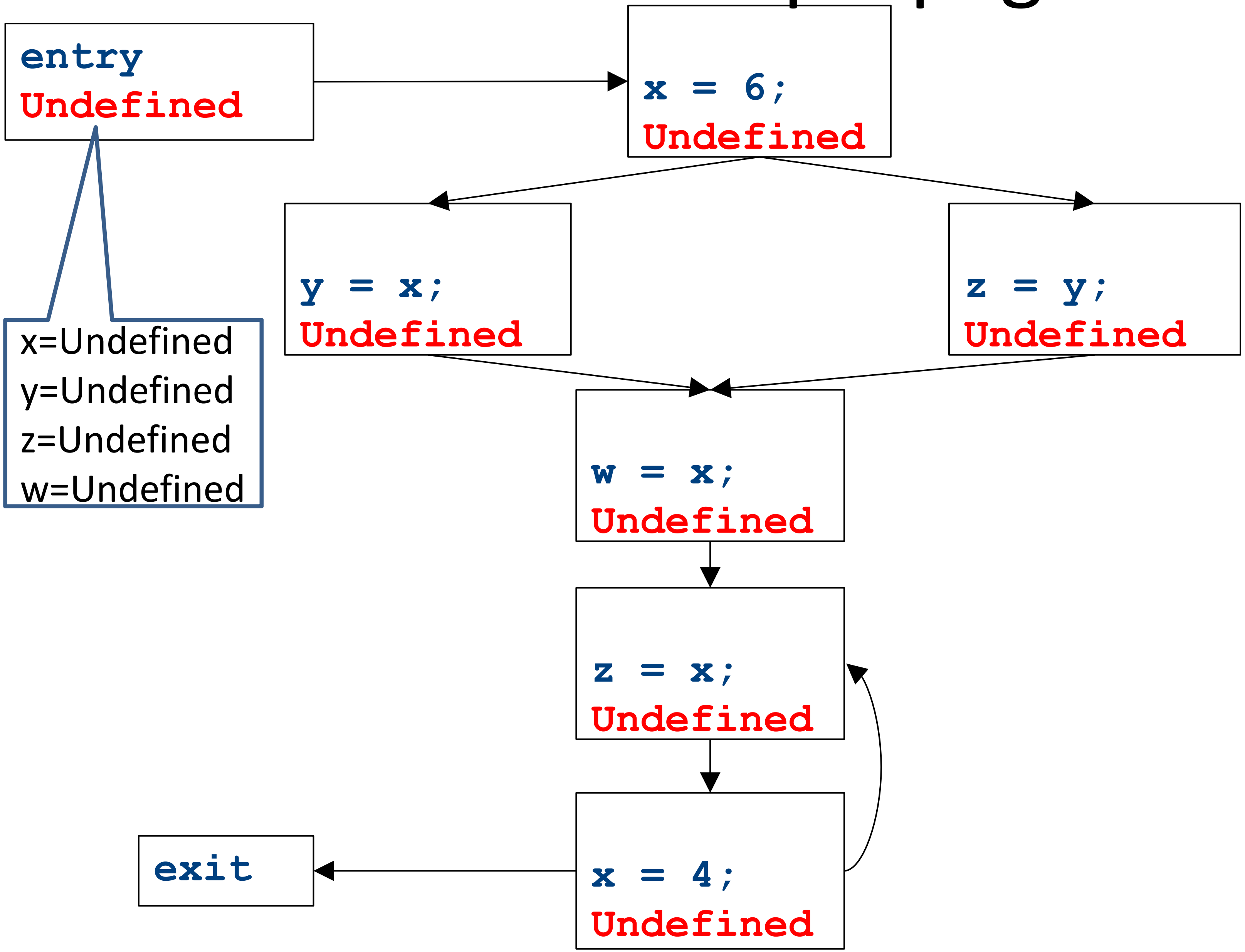

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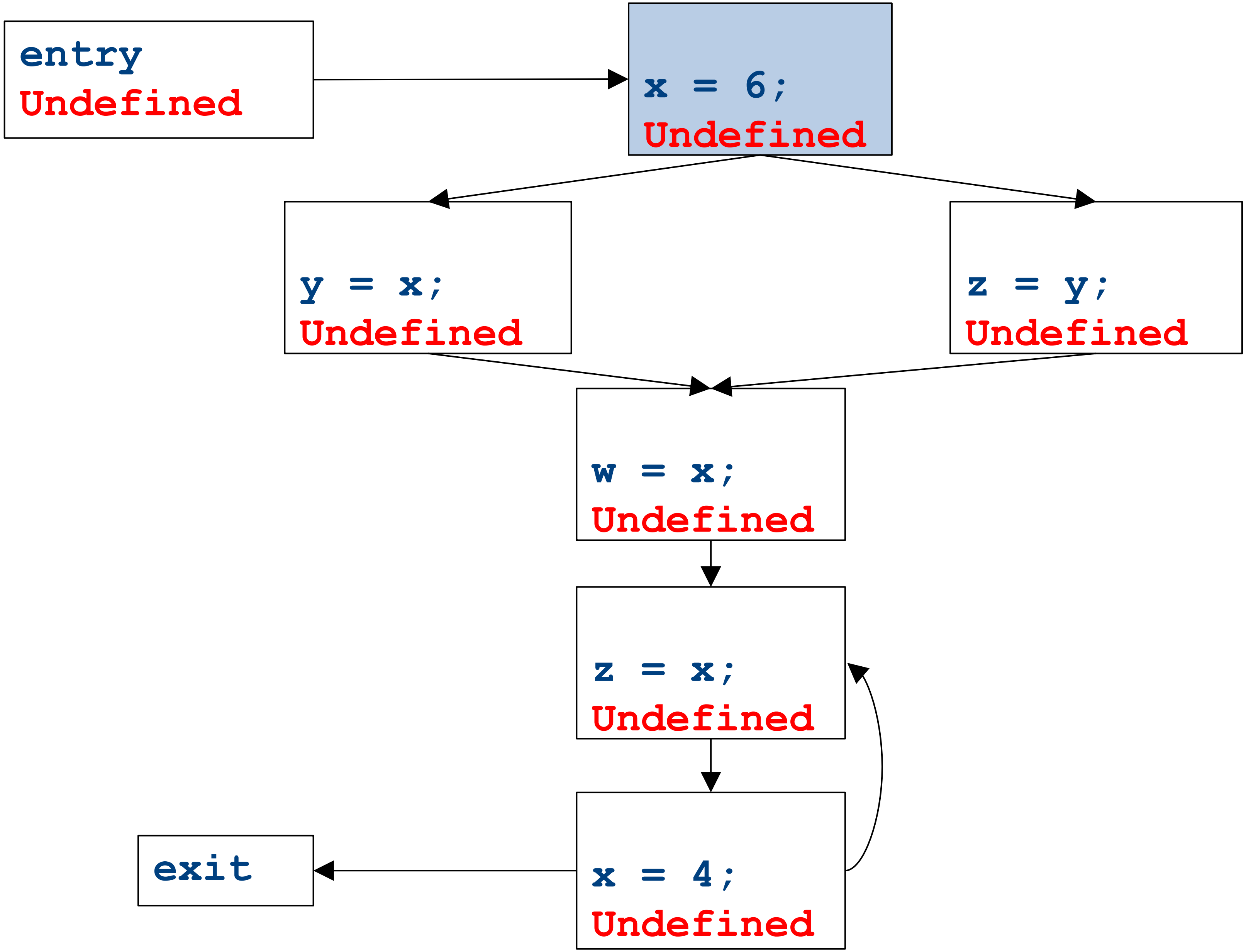

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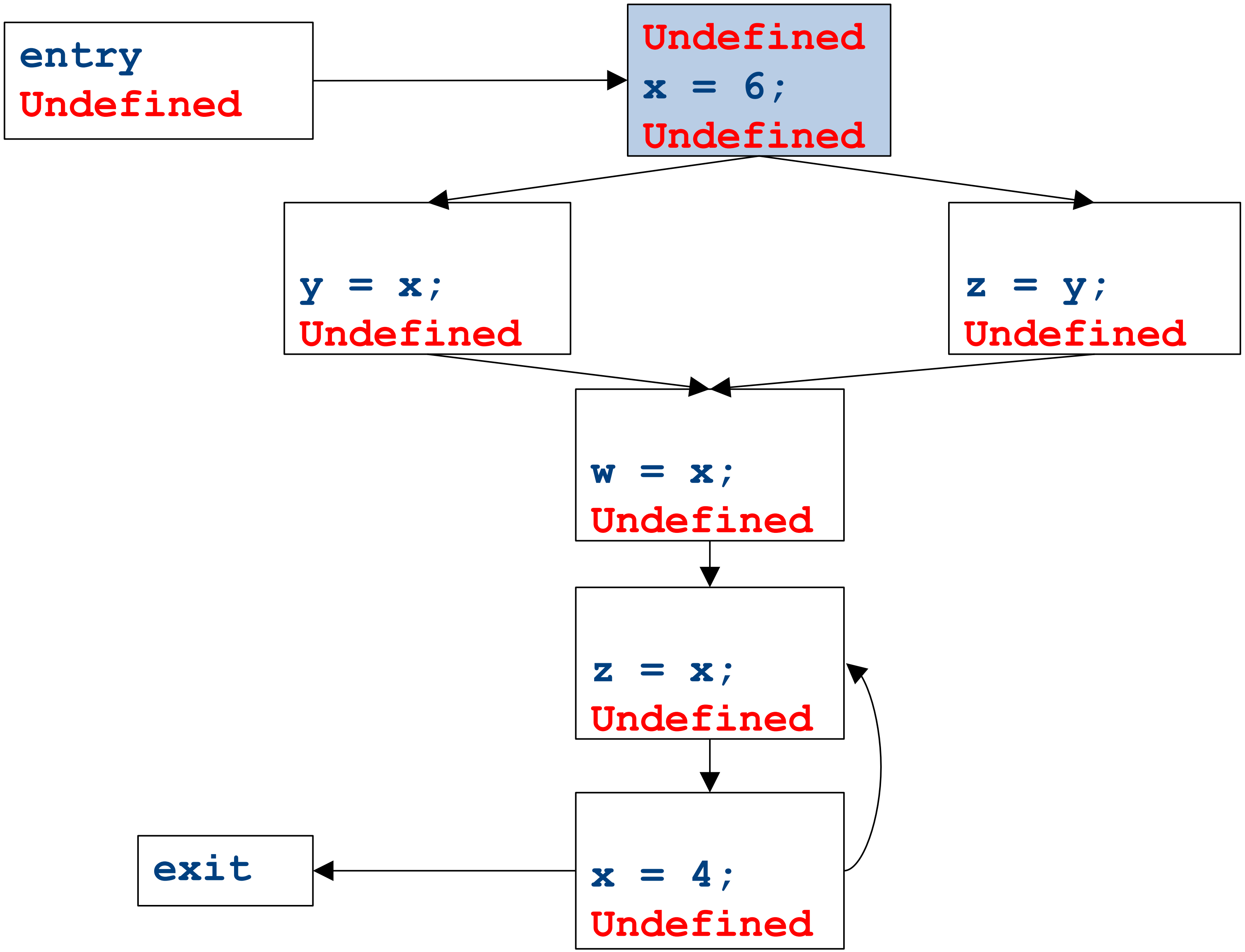

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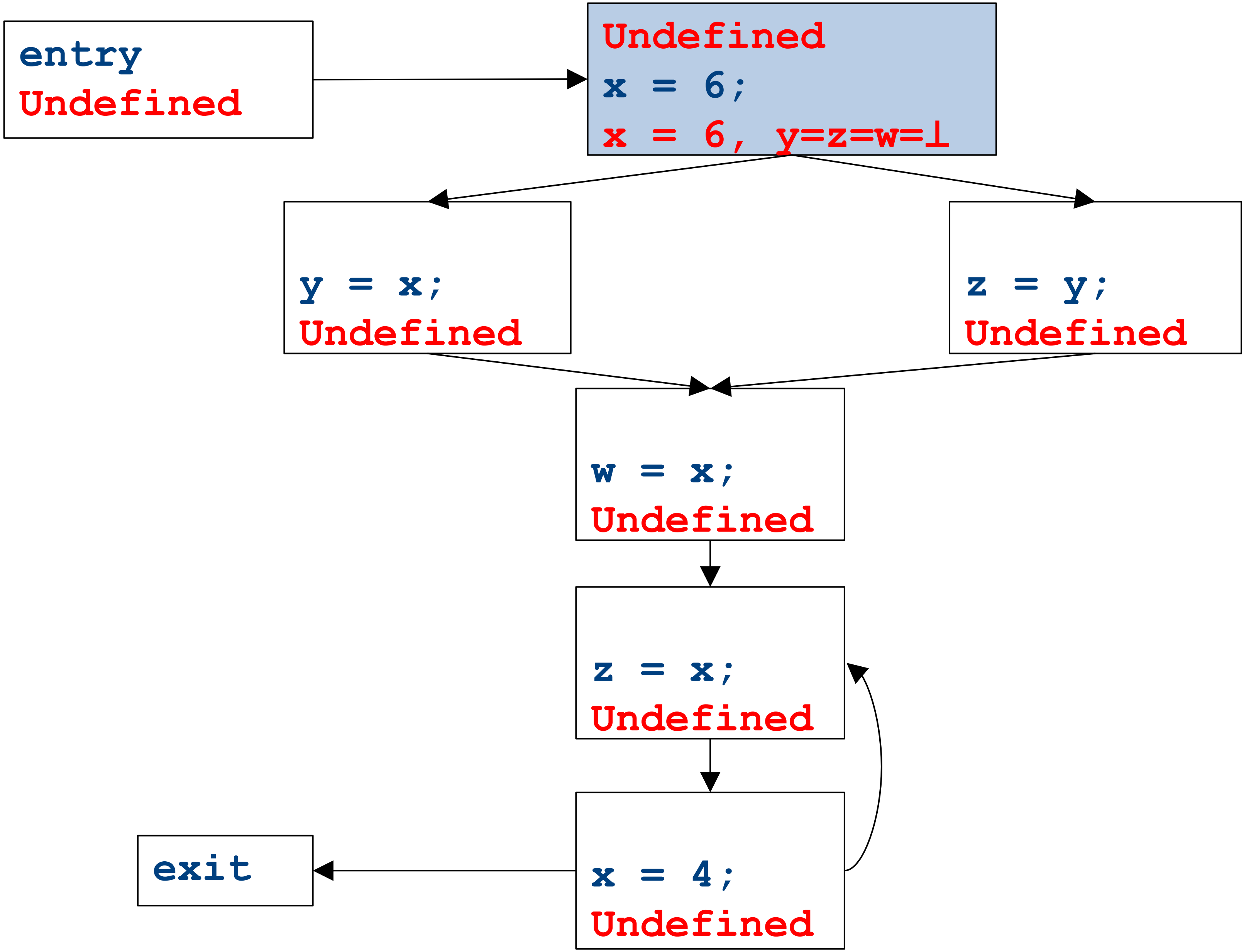

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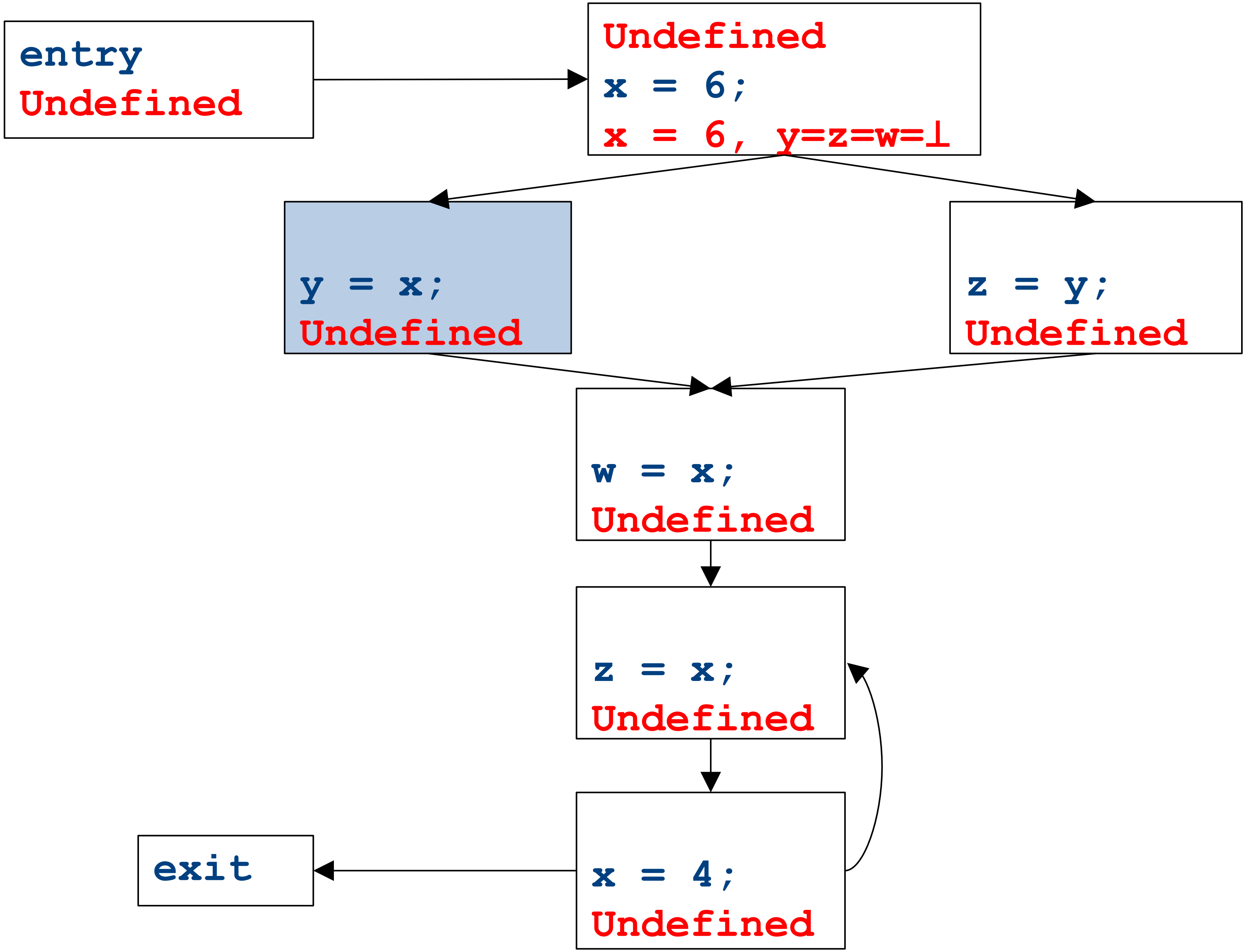

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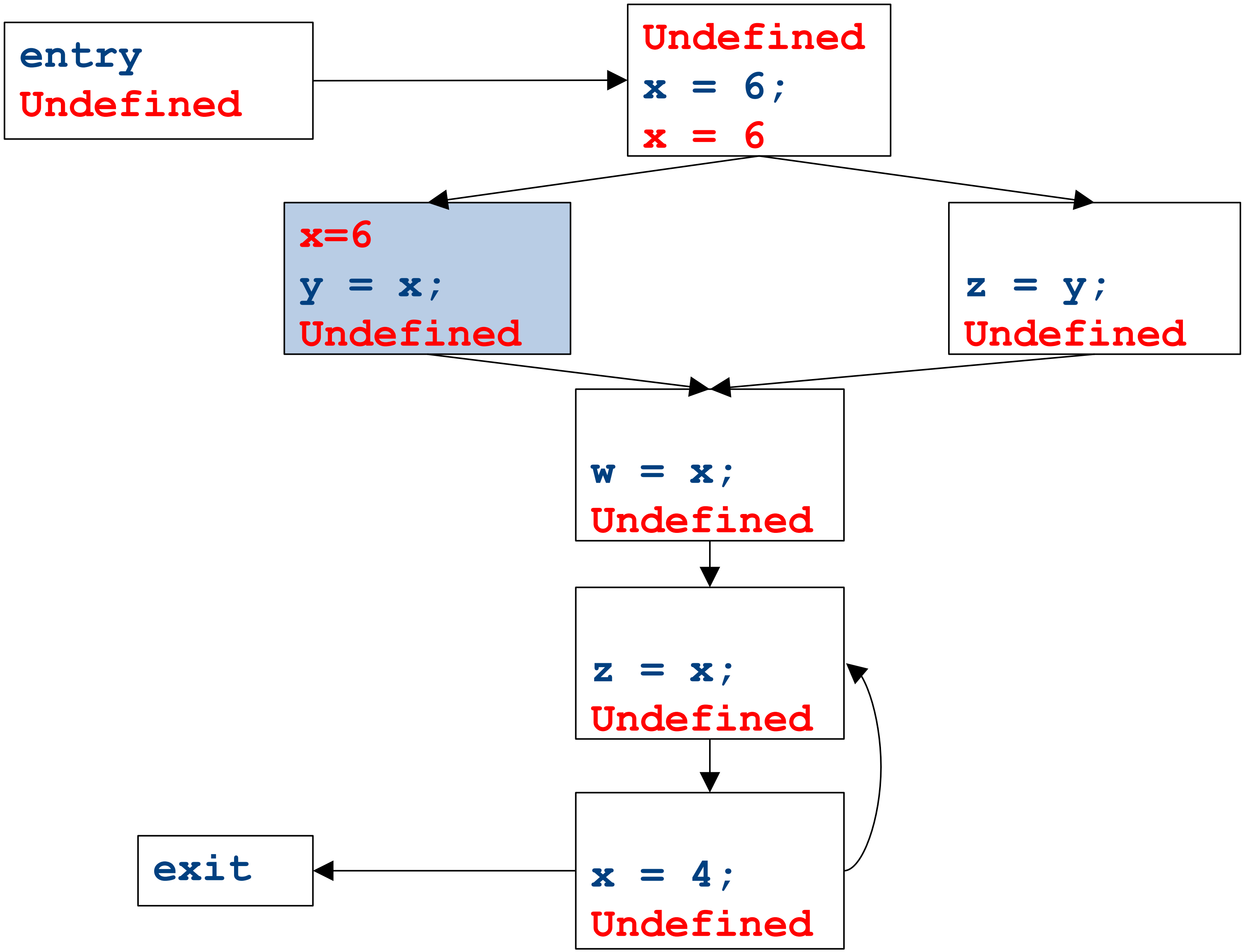

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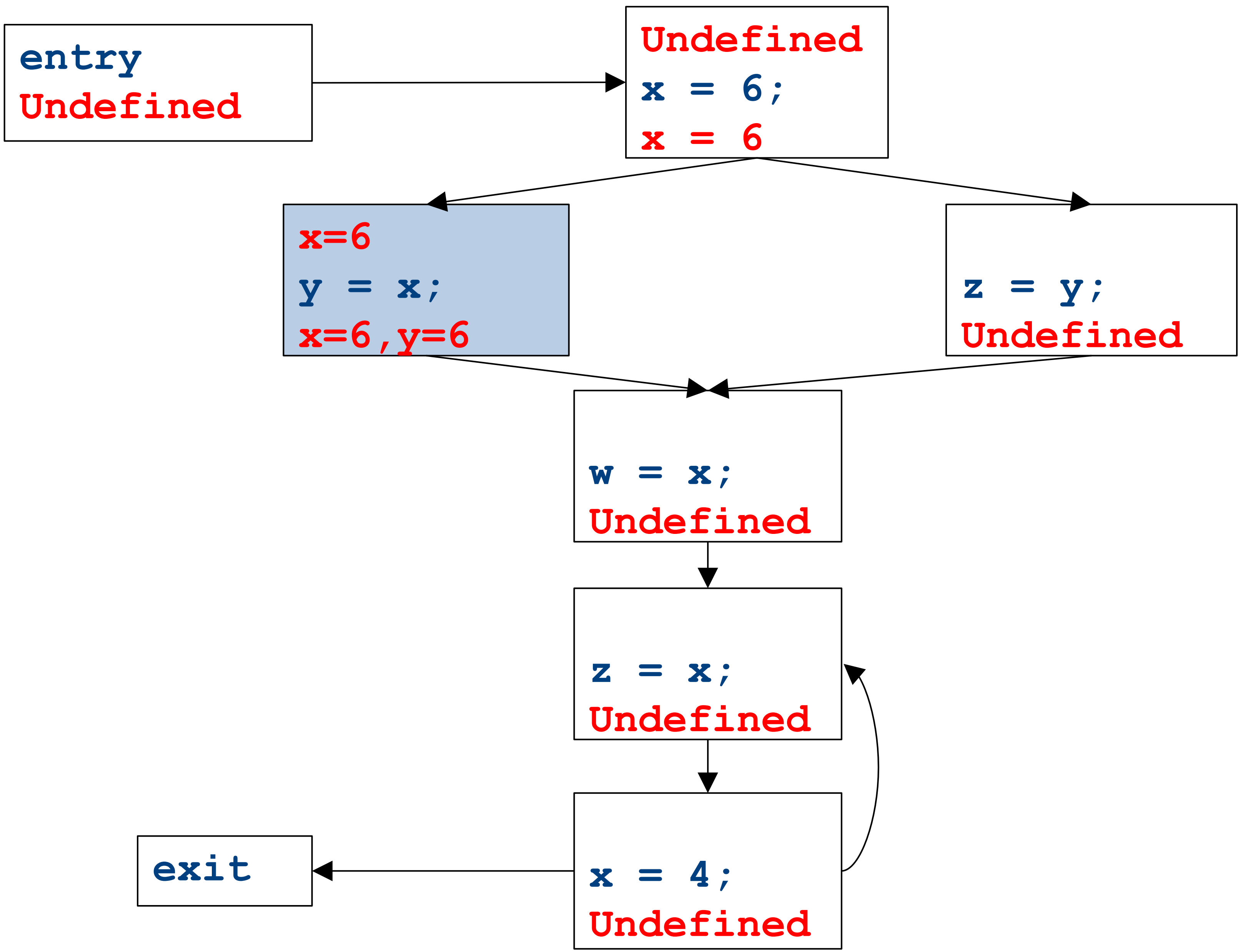

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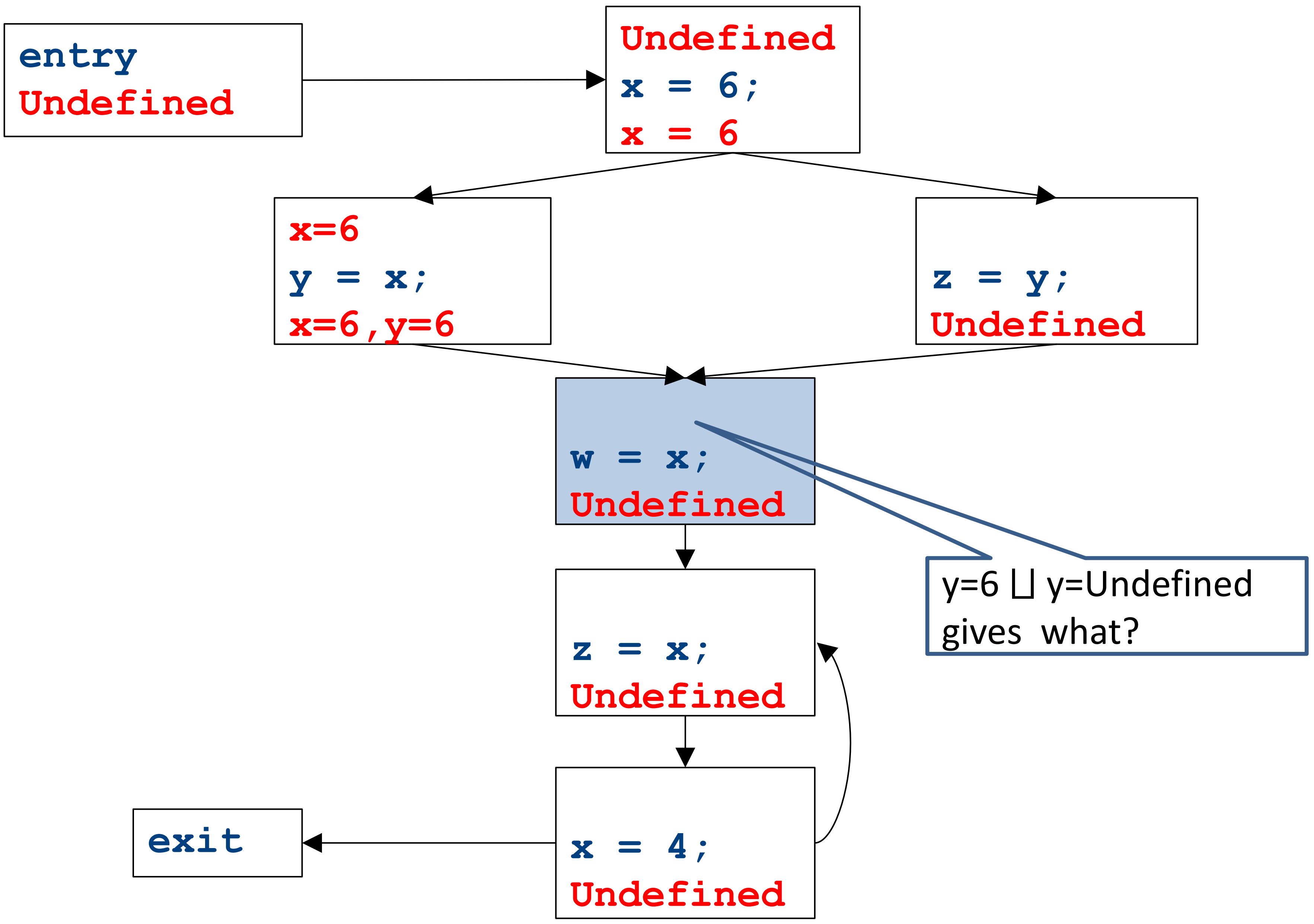

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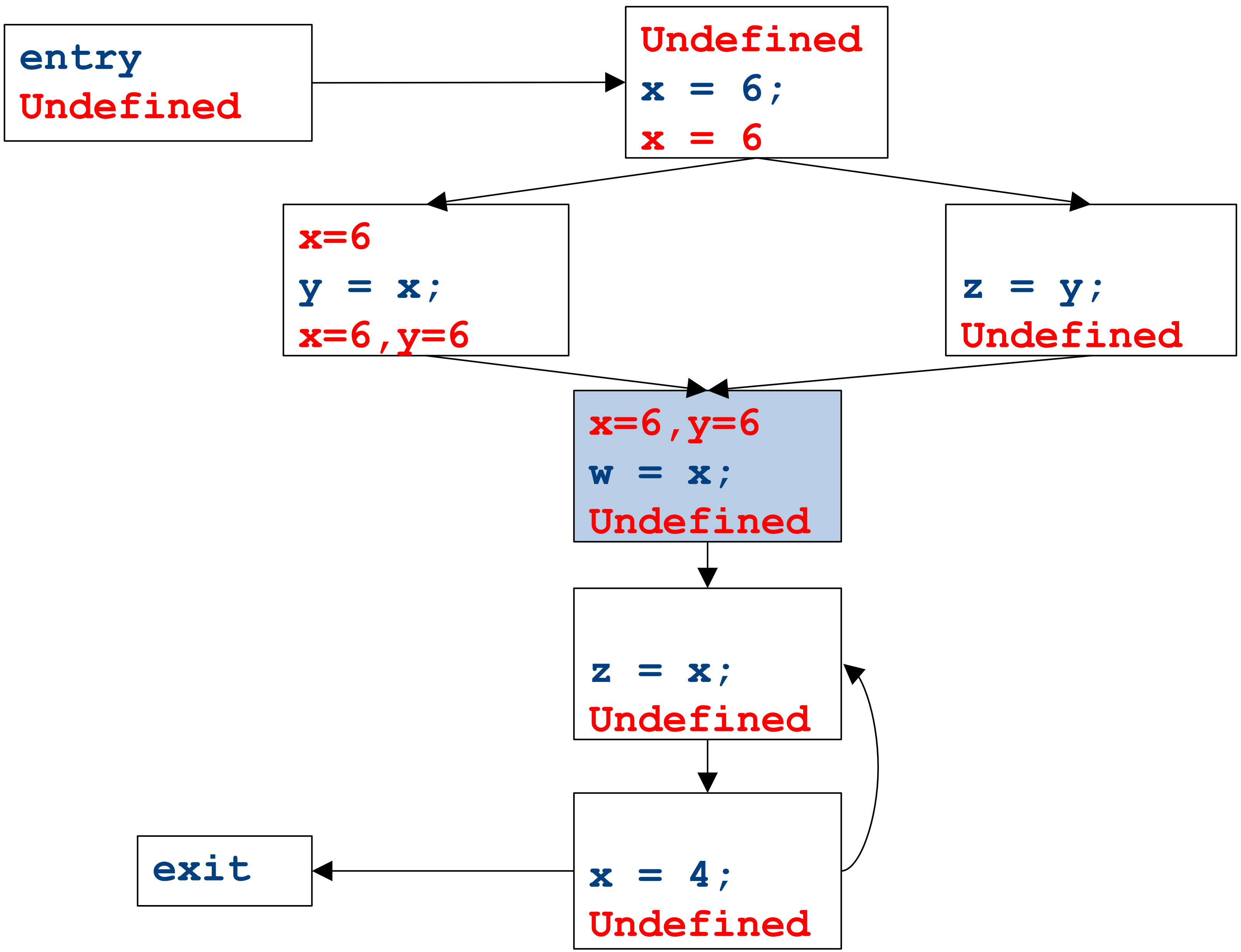

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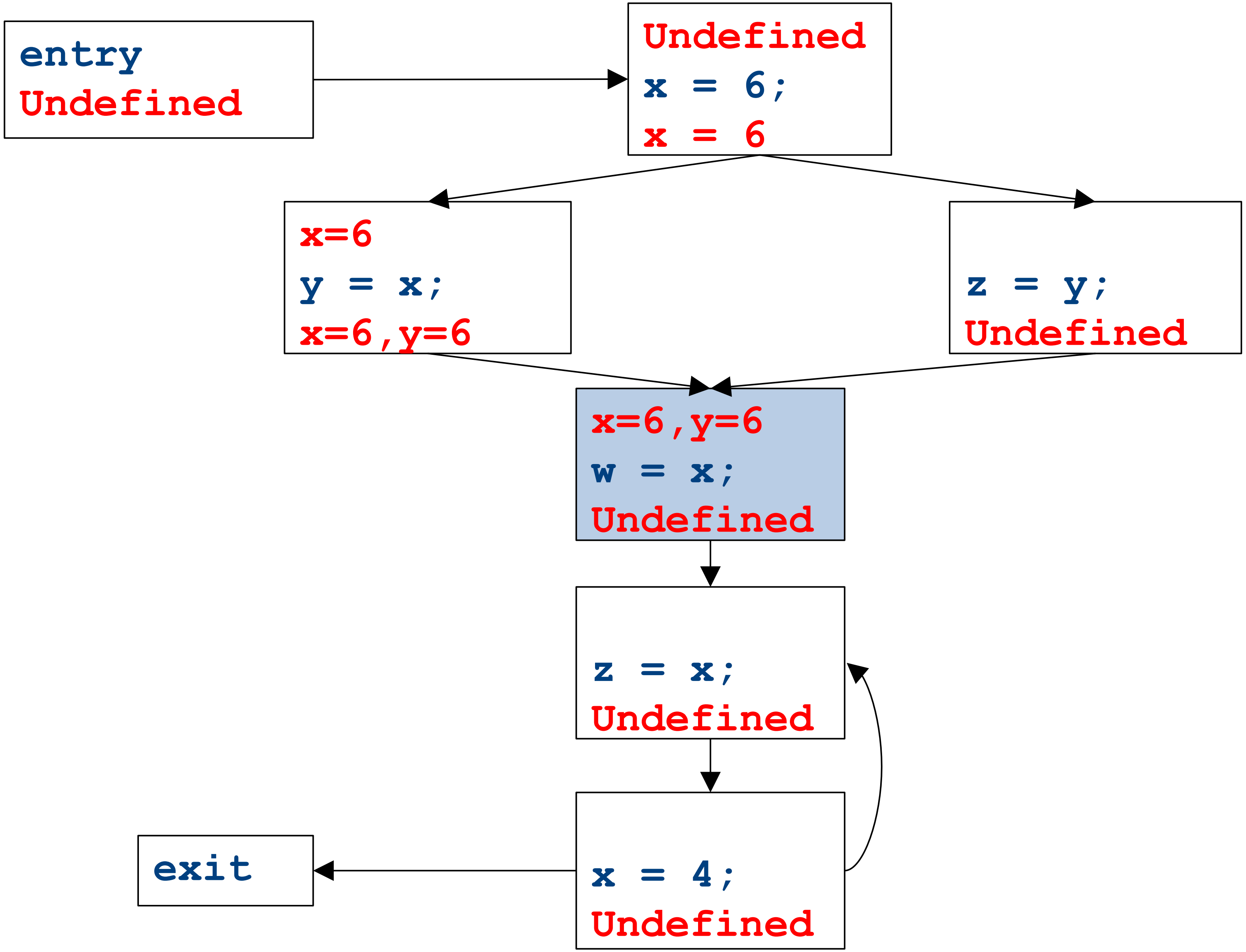

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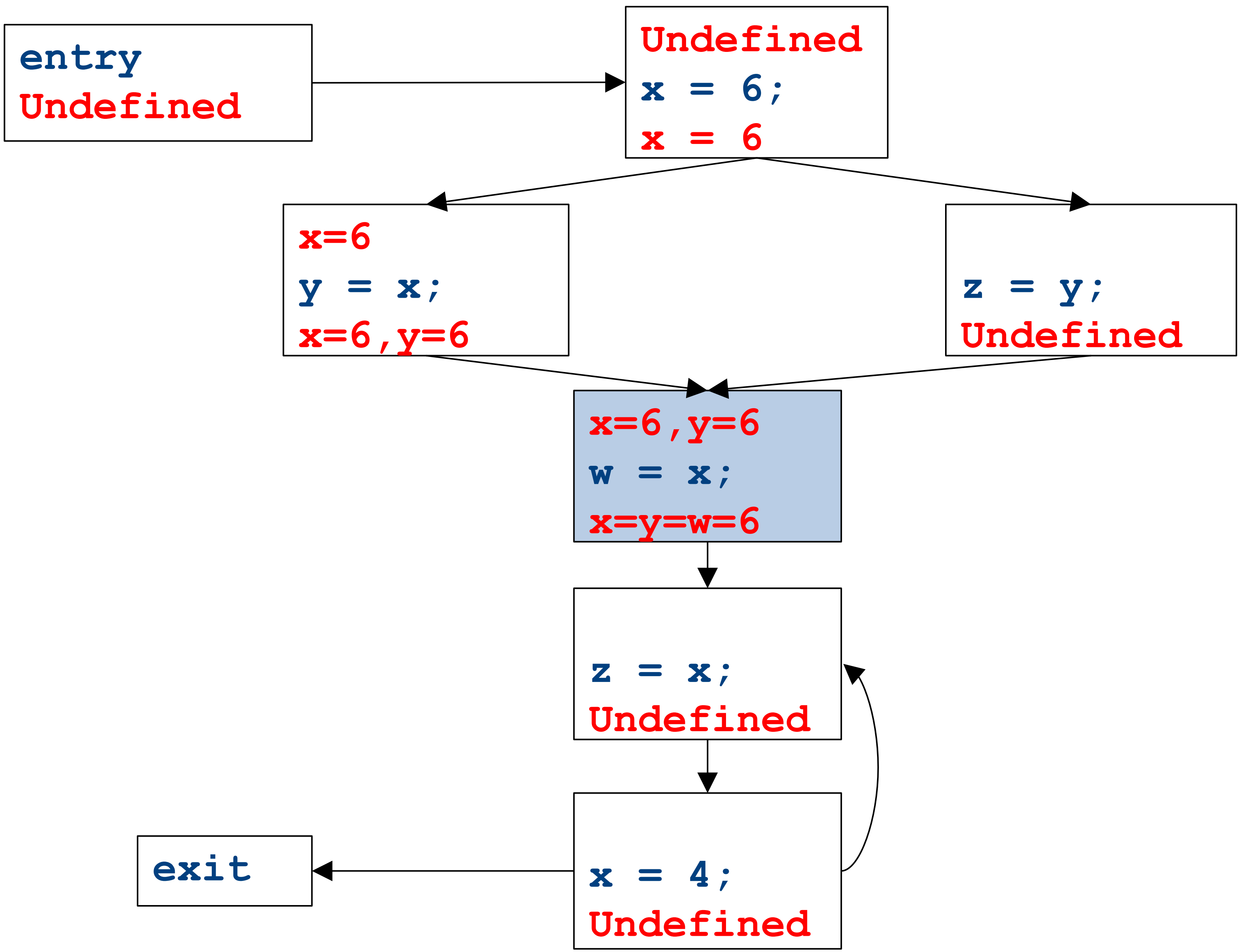

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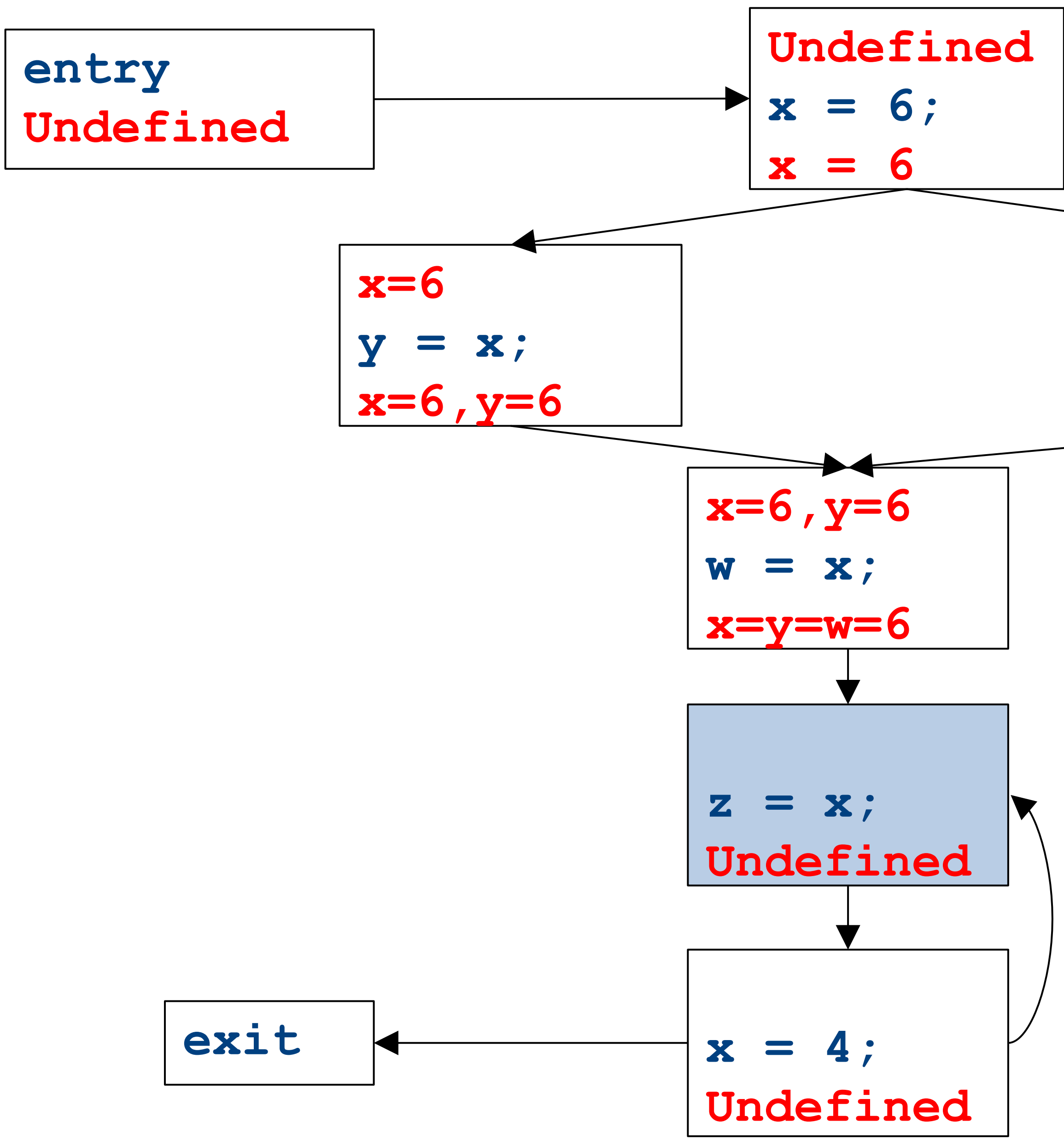

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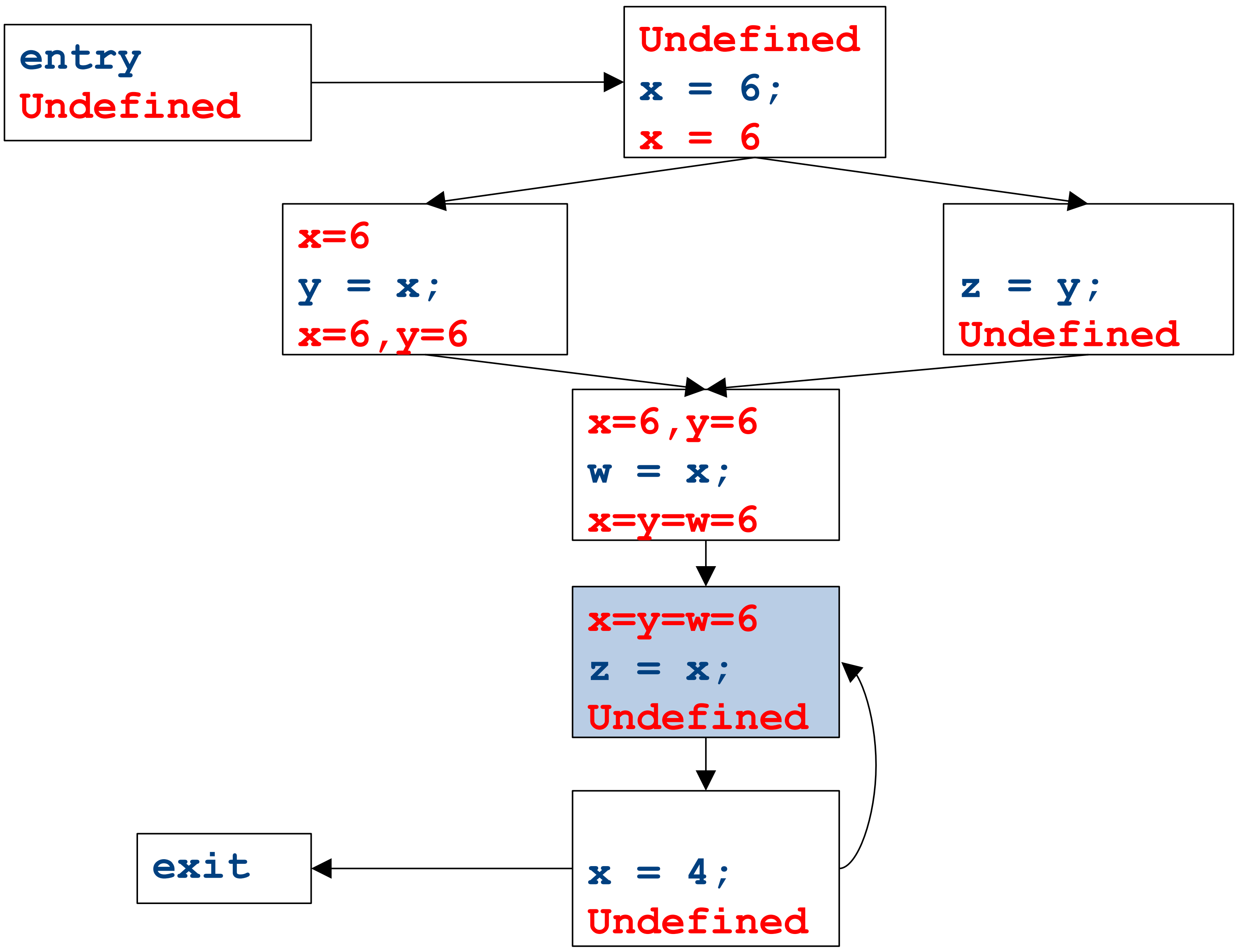

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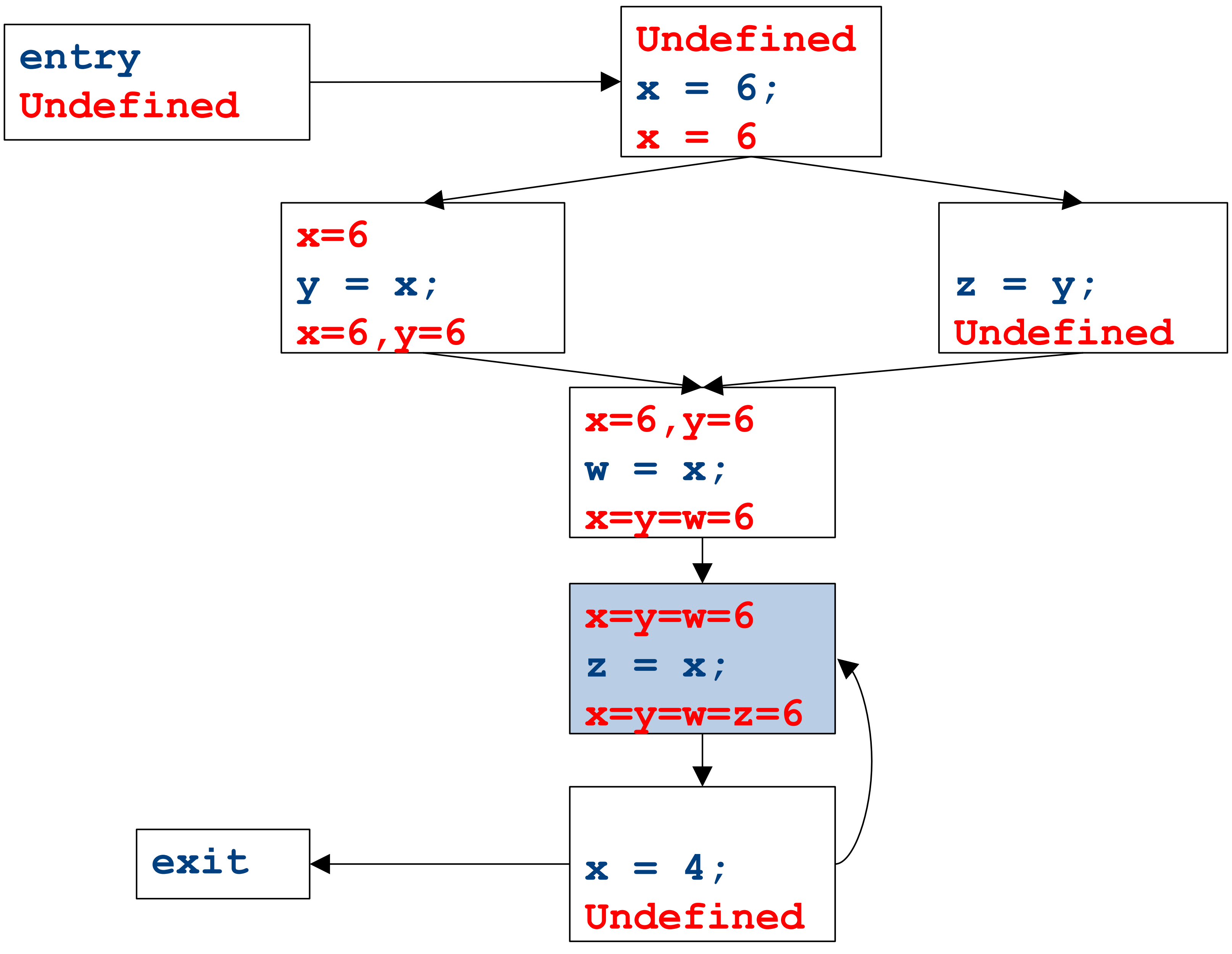

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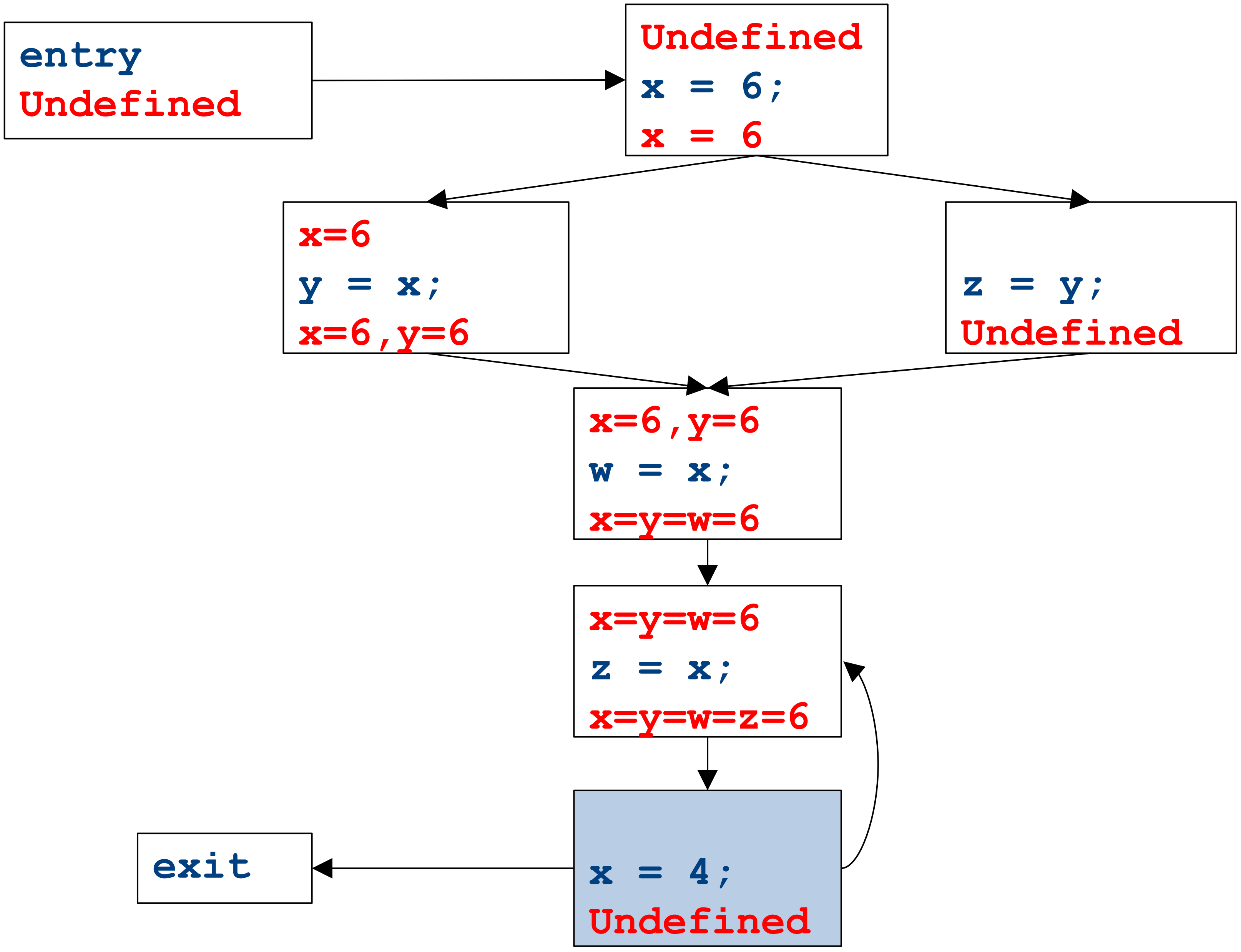

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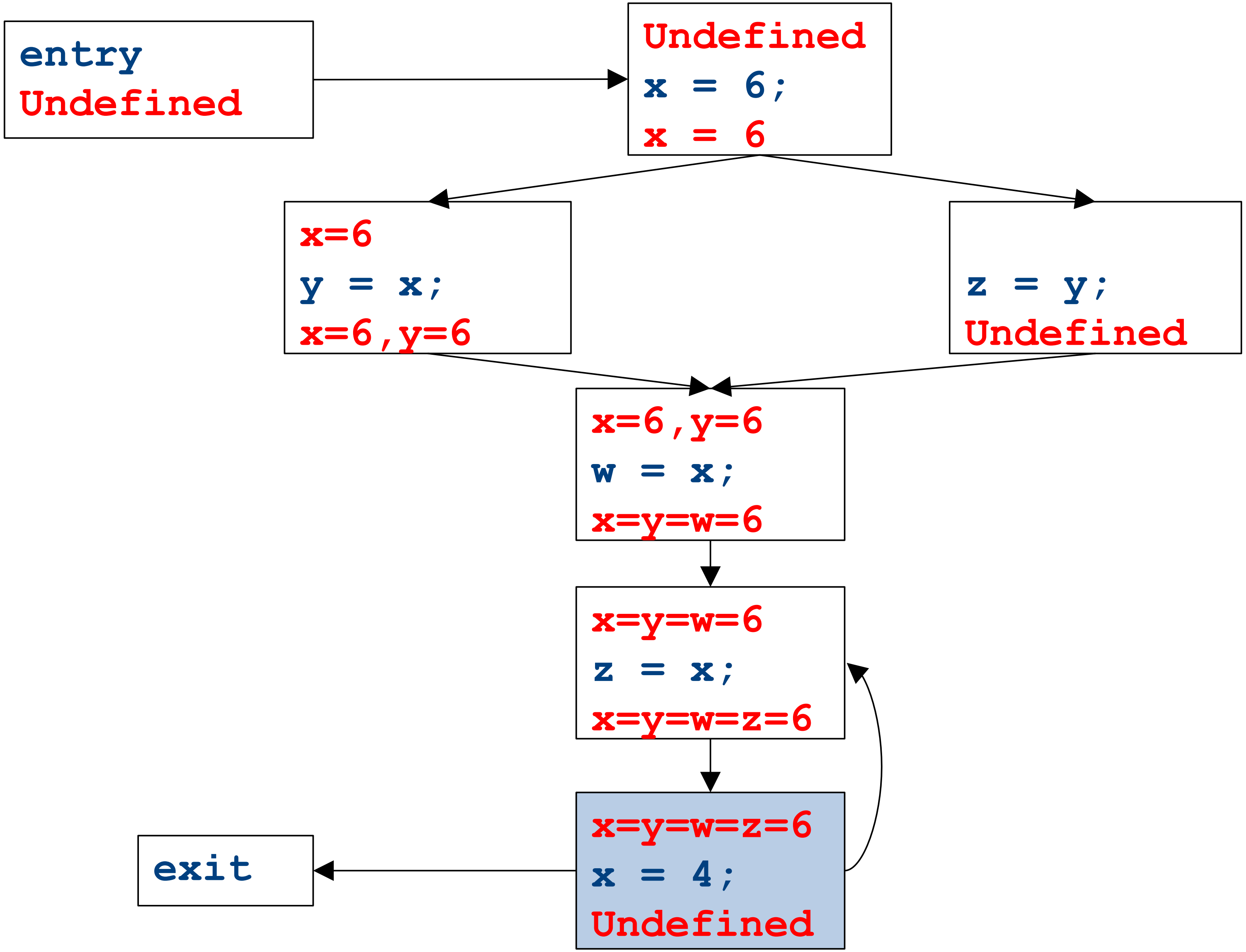

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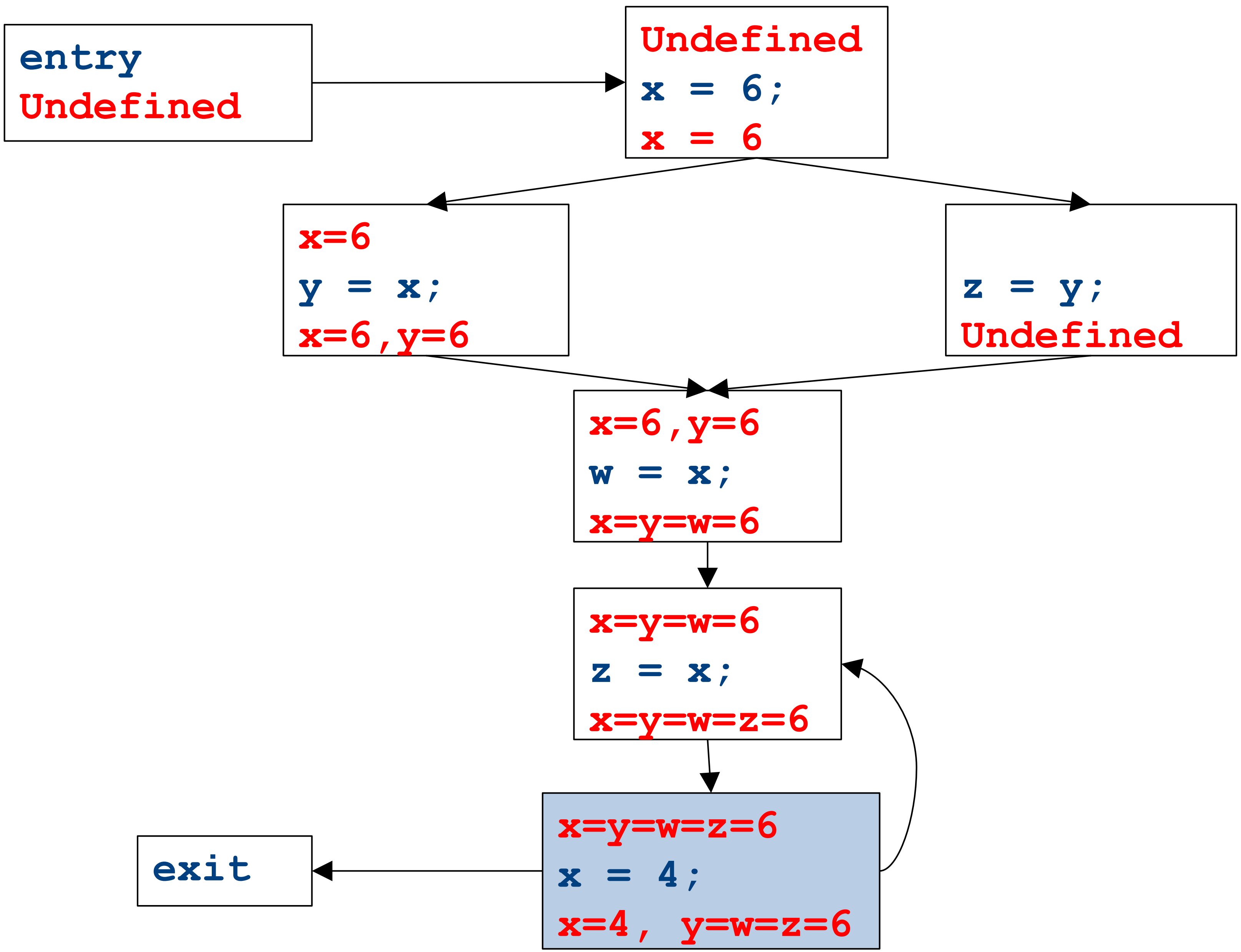

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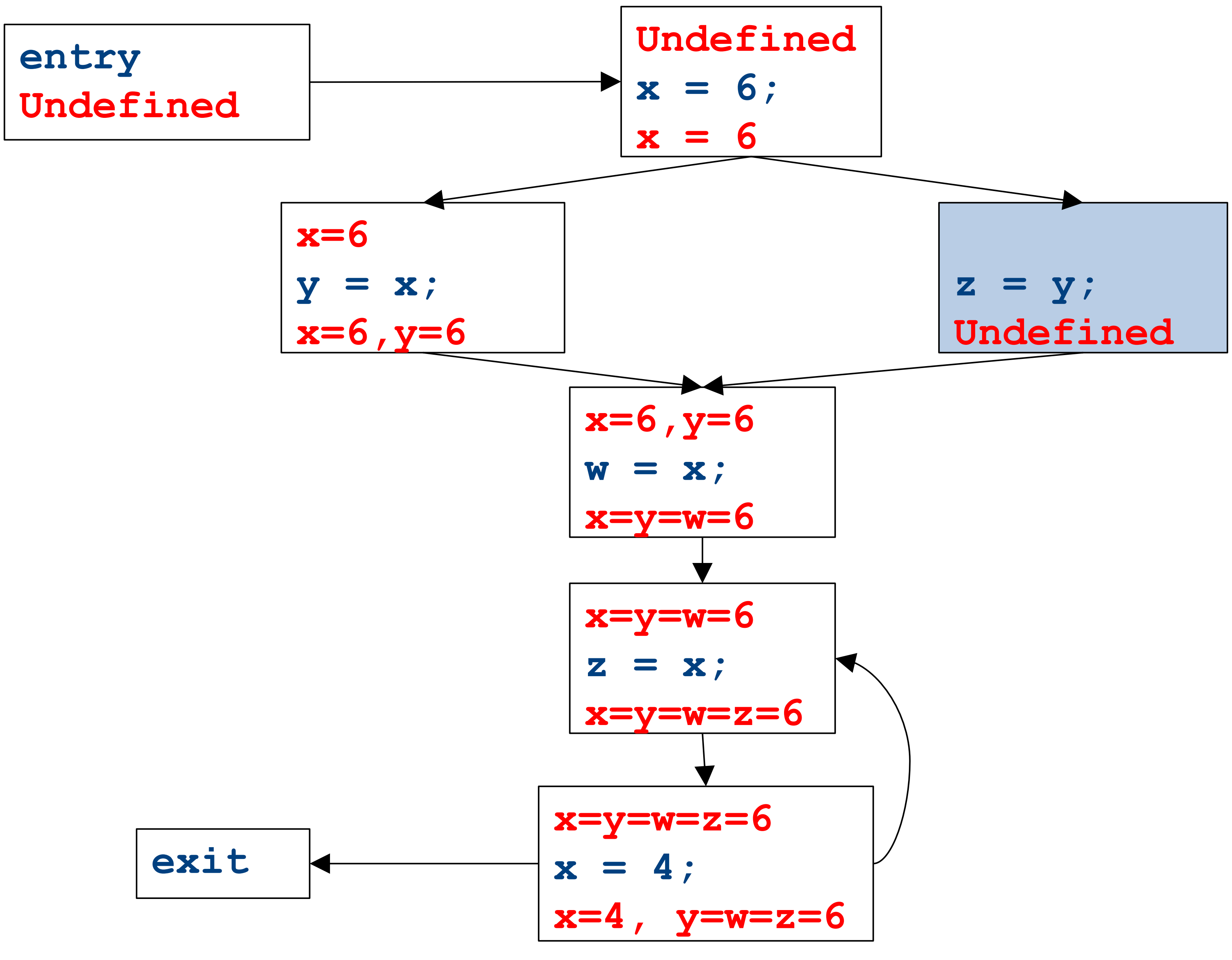

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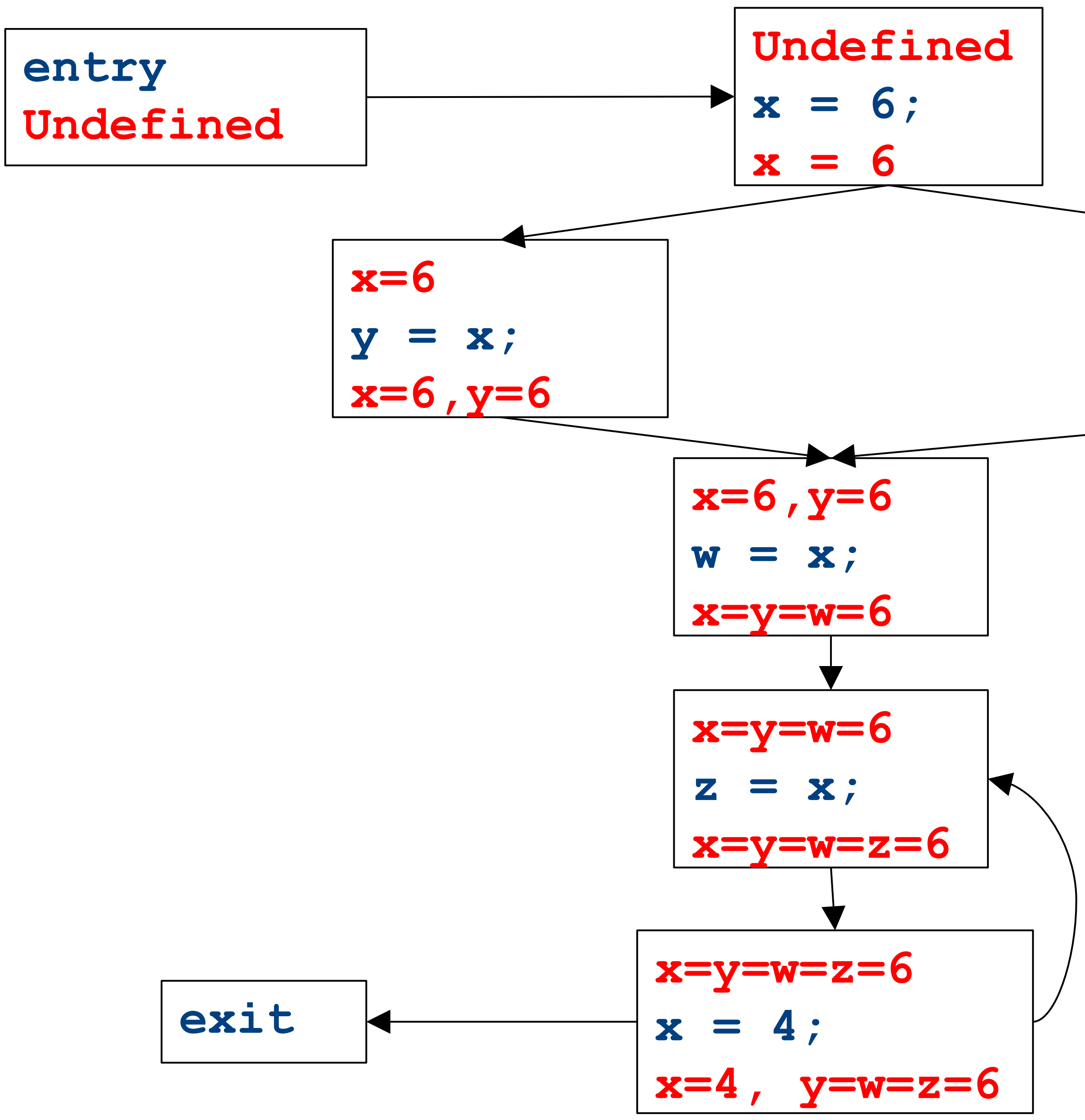

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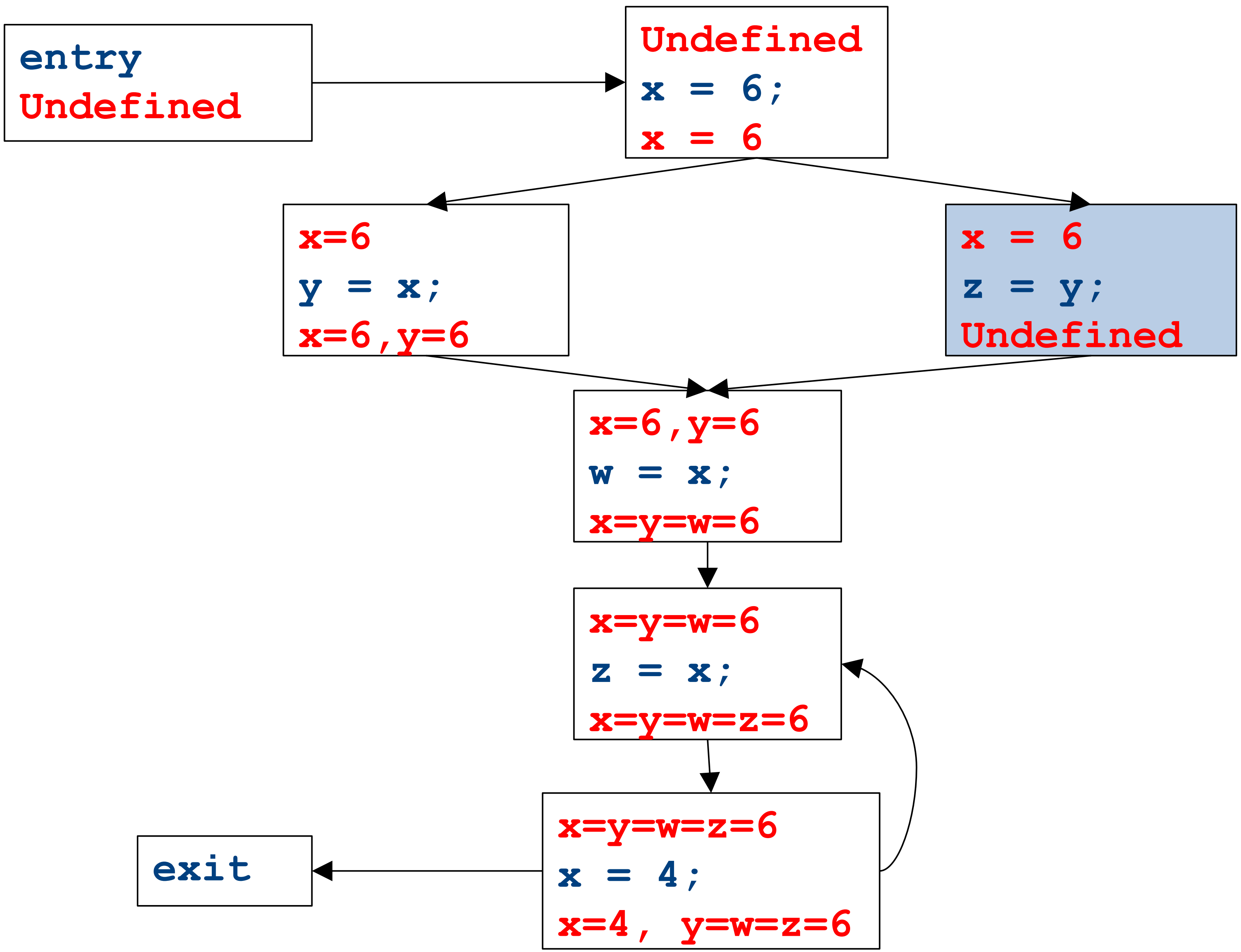

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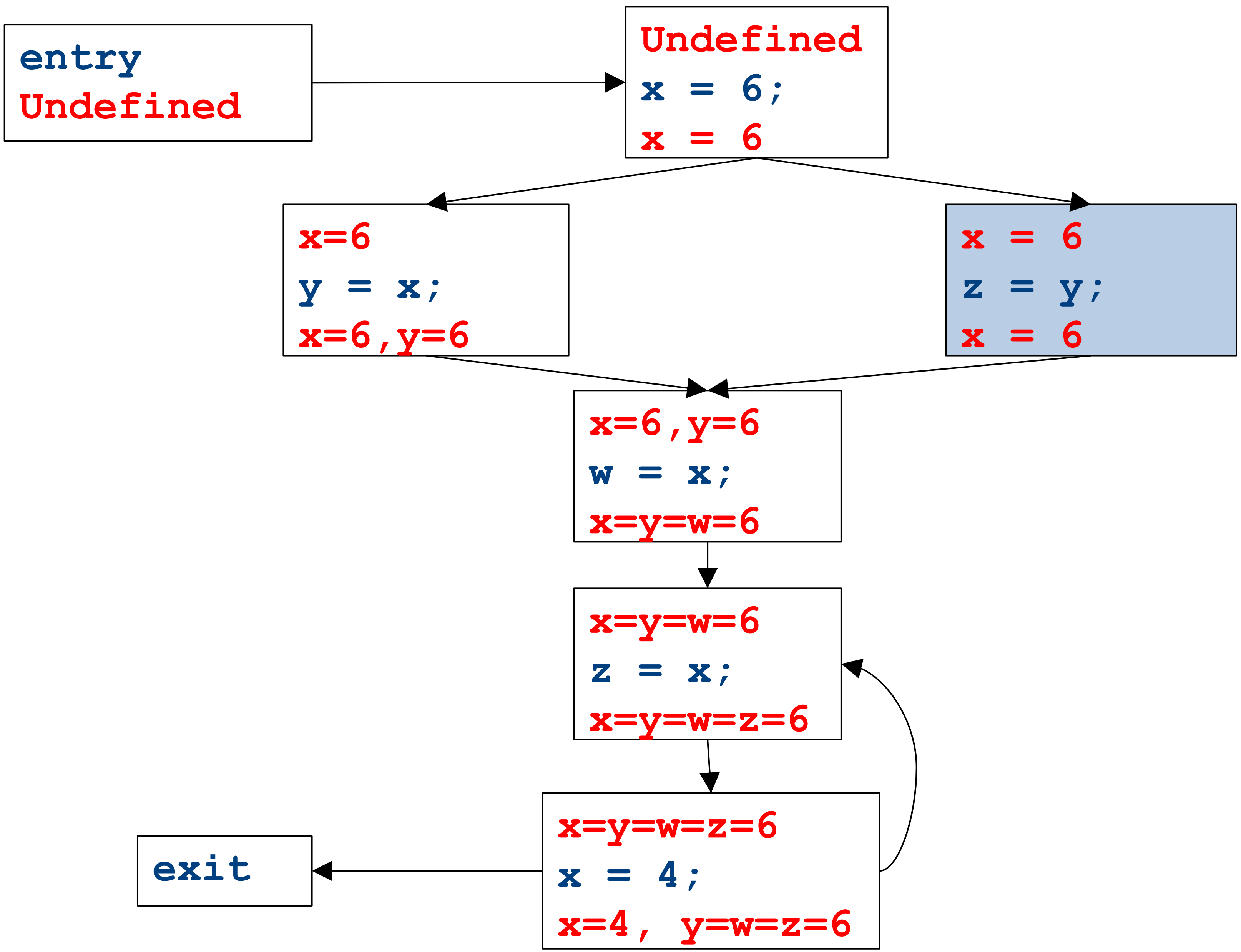

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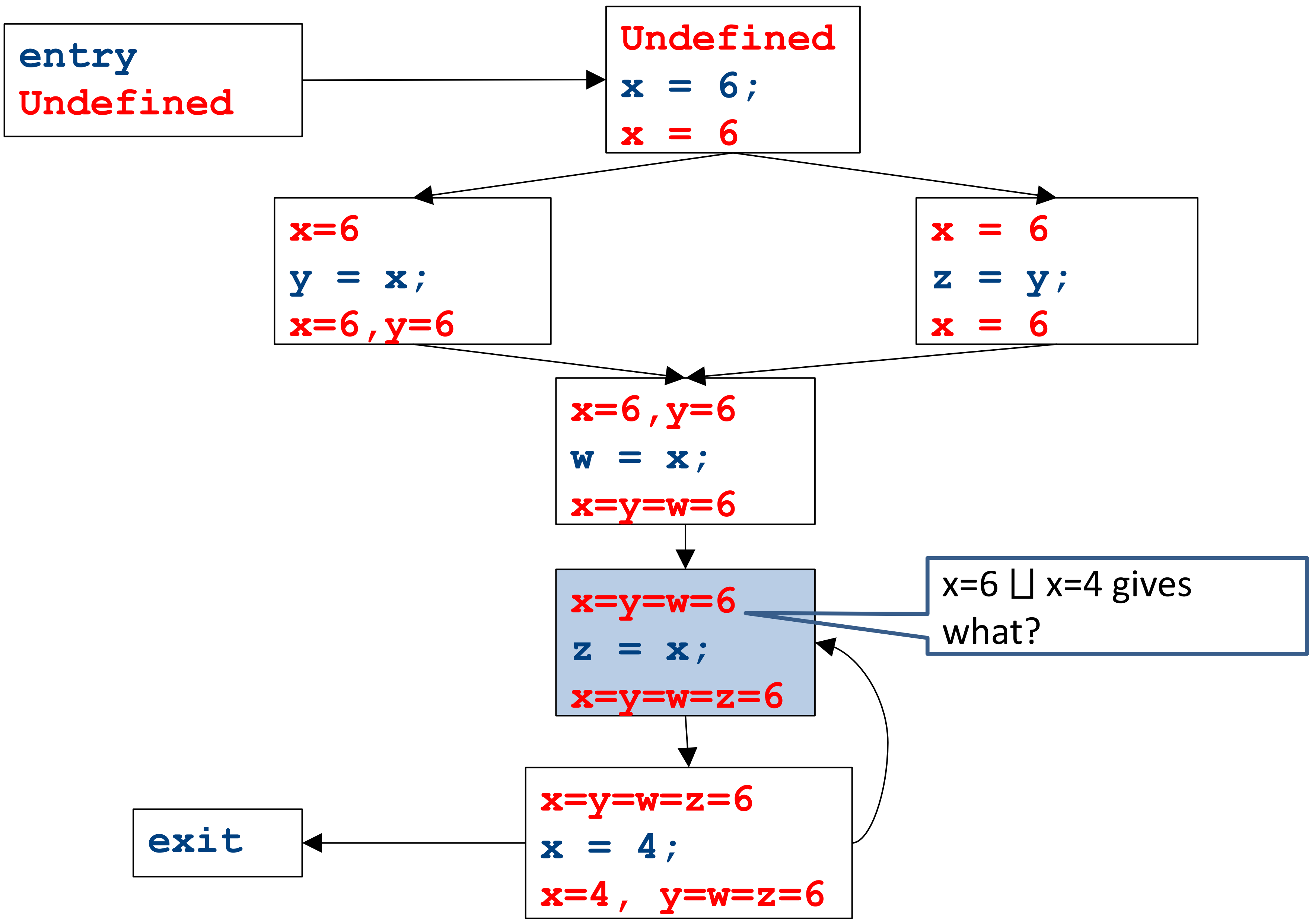

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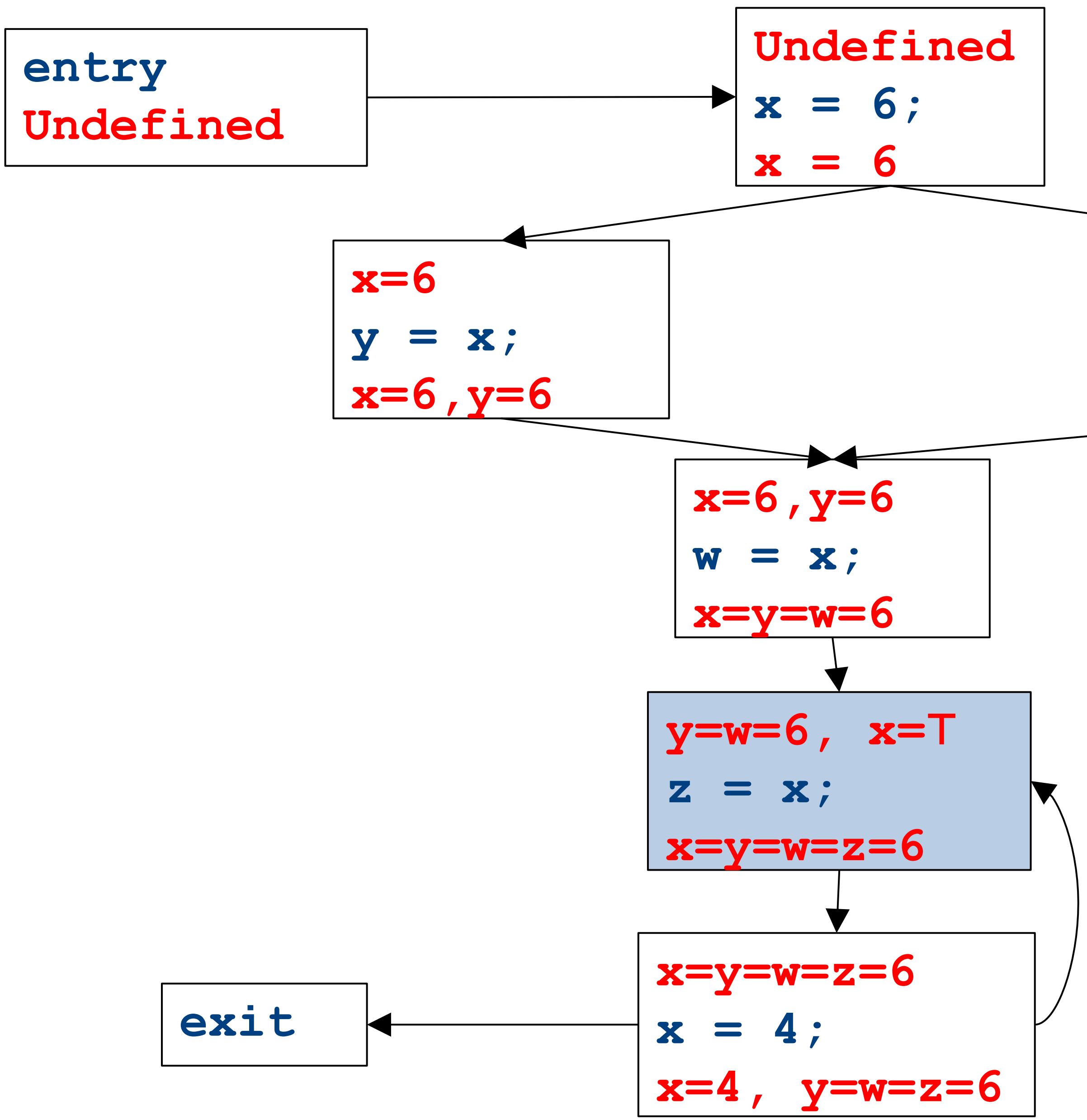

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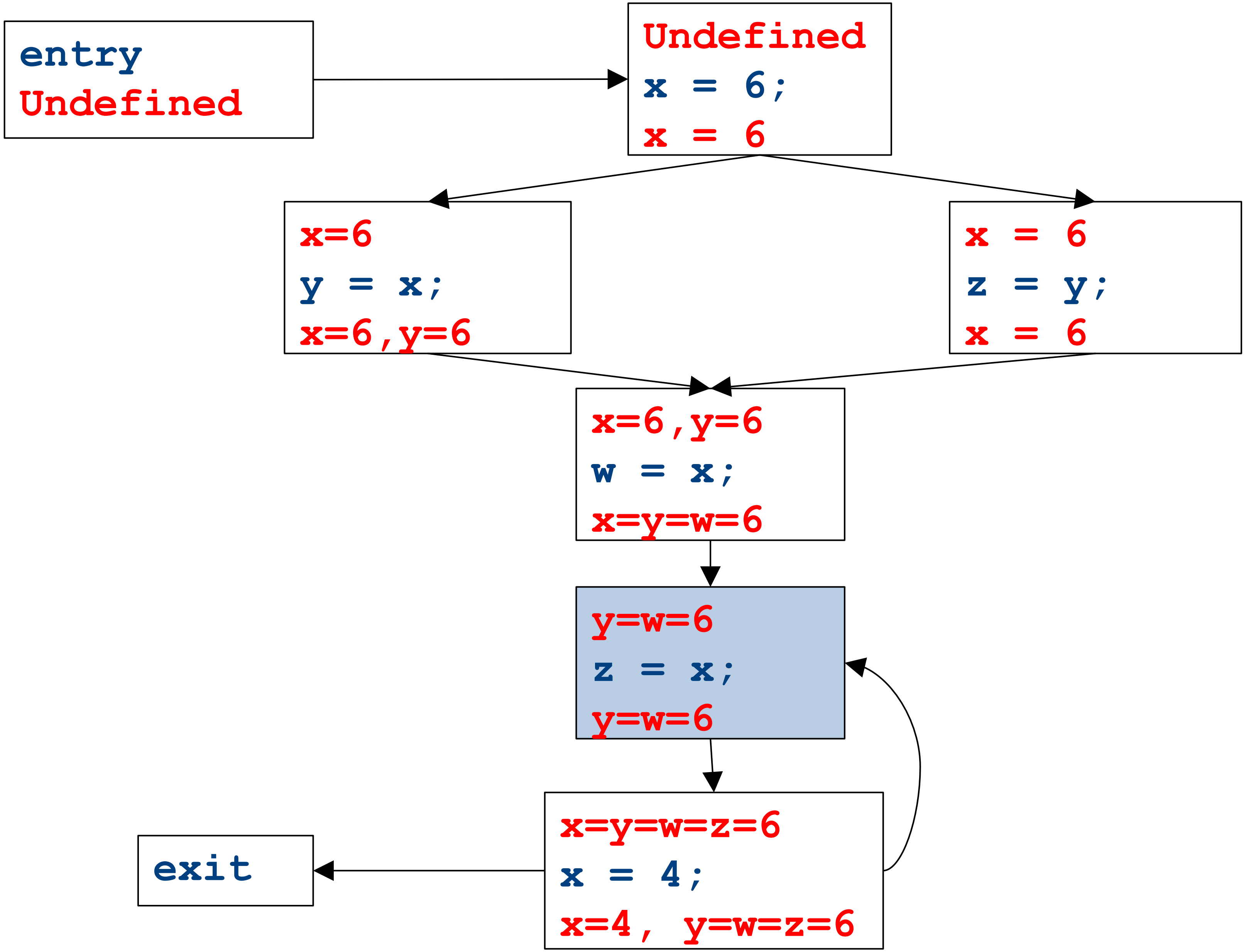

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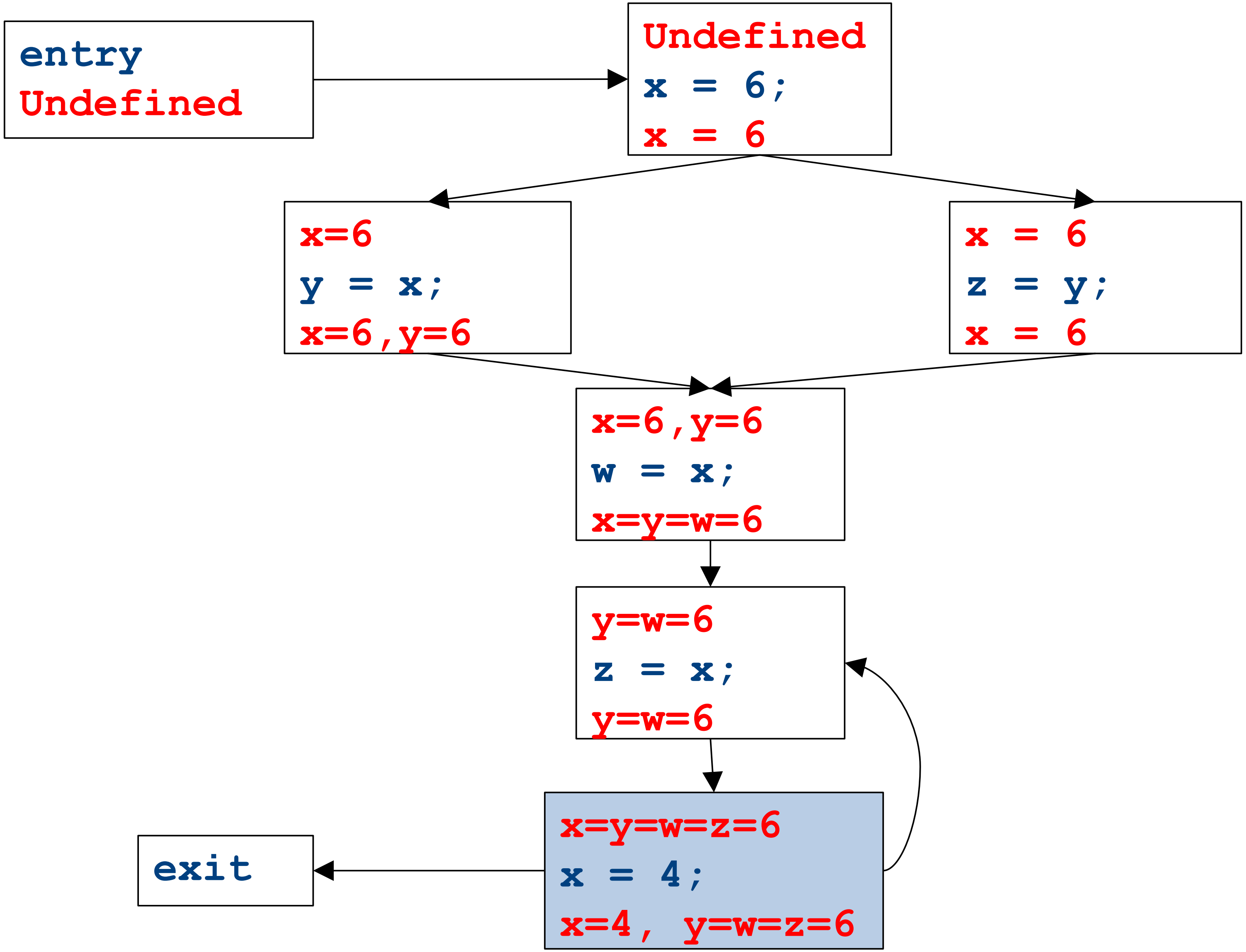

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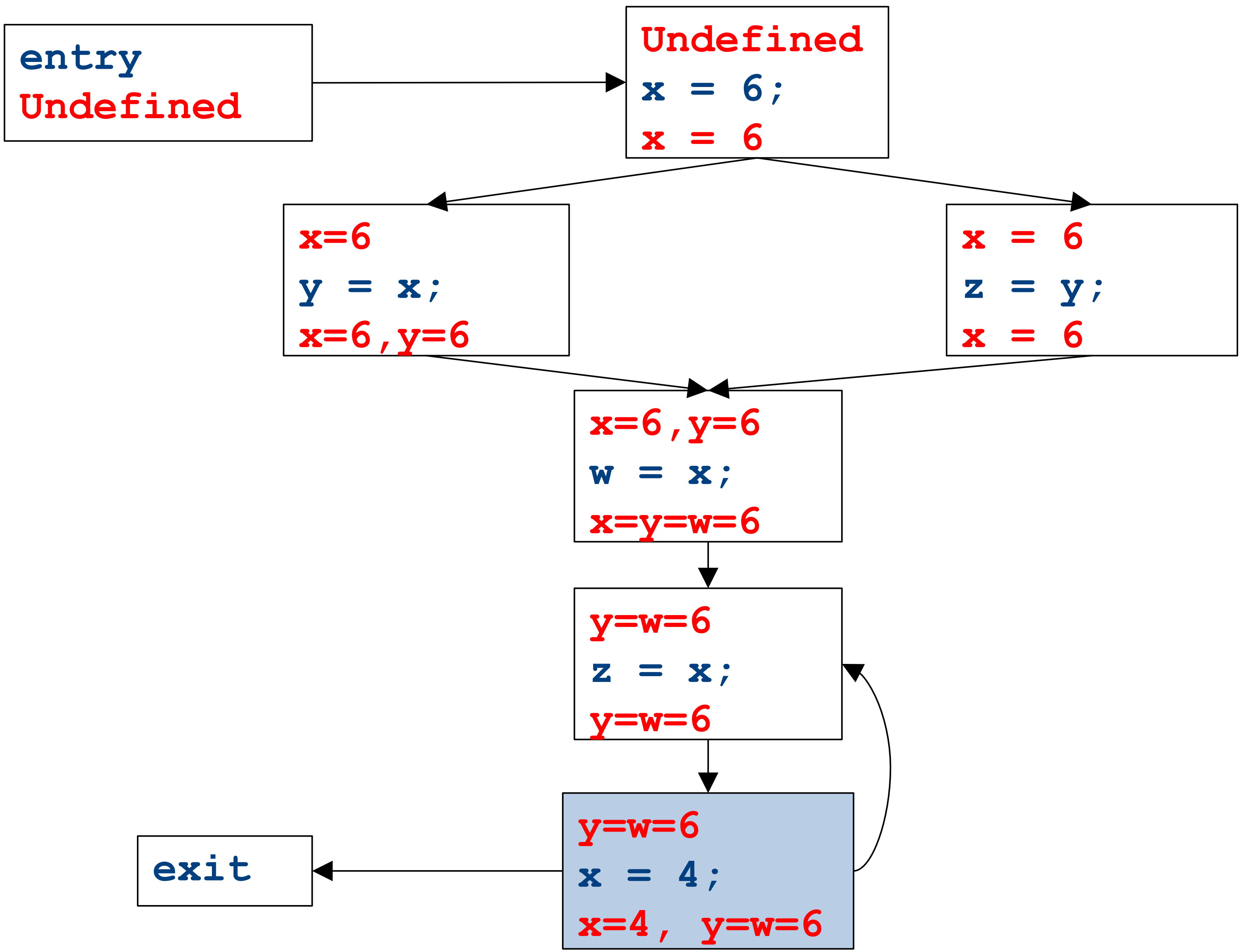

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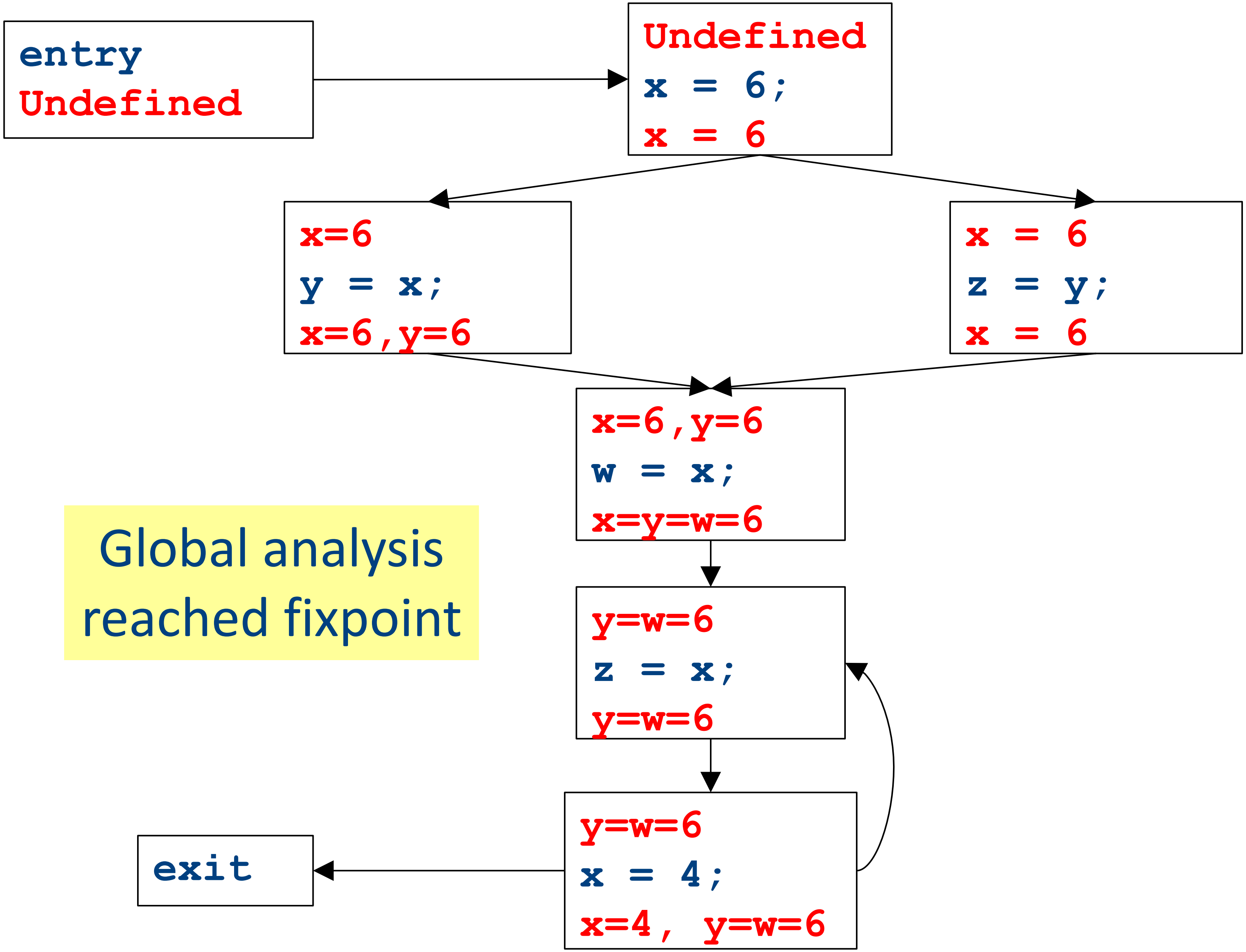
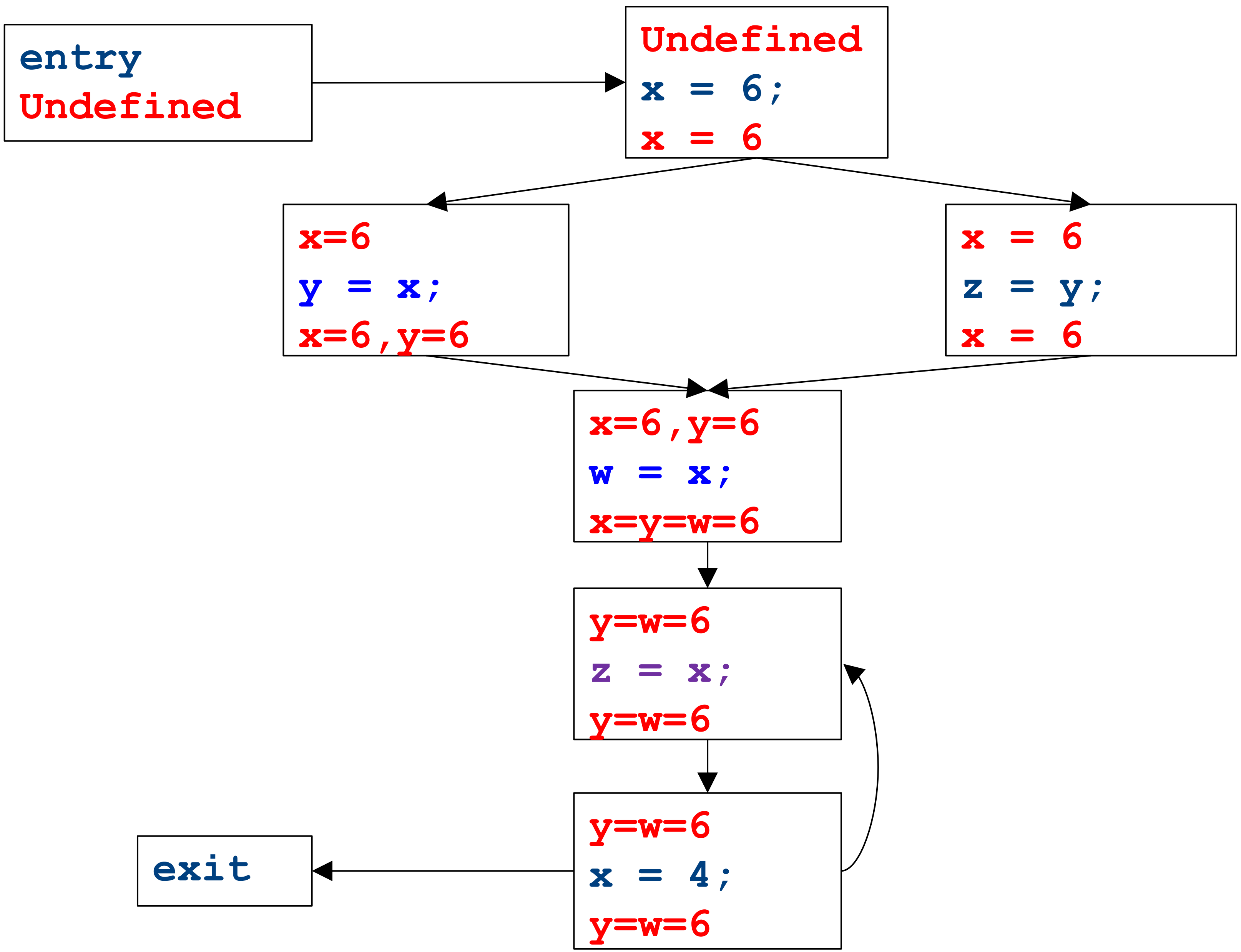


Global analysis reached fixpoint

Global constant propagation



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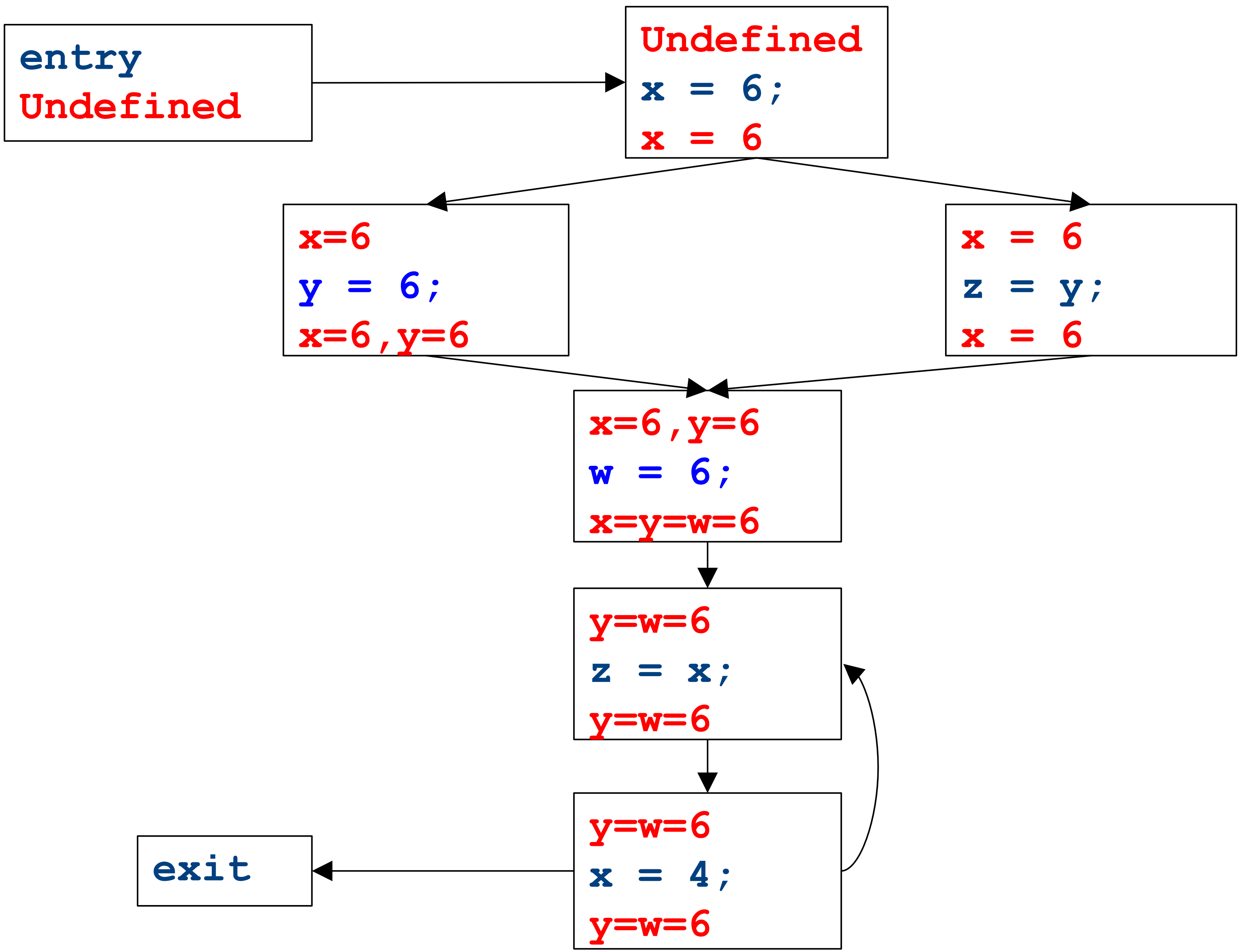

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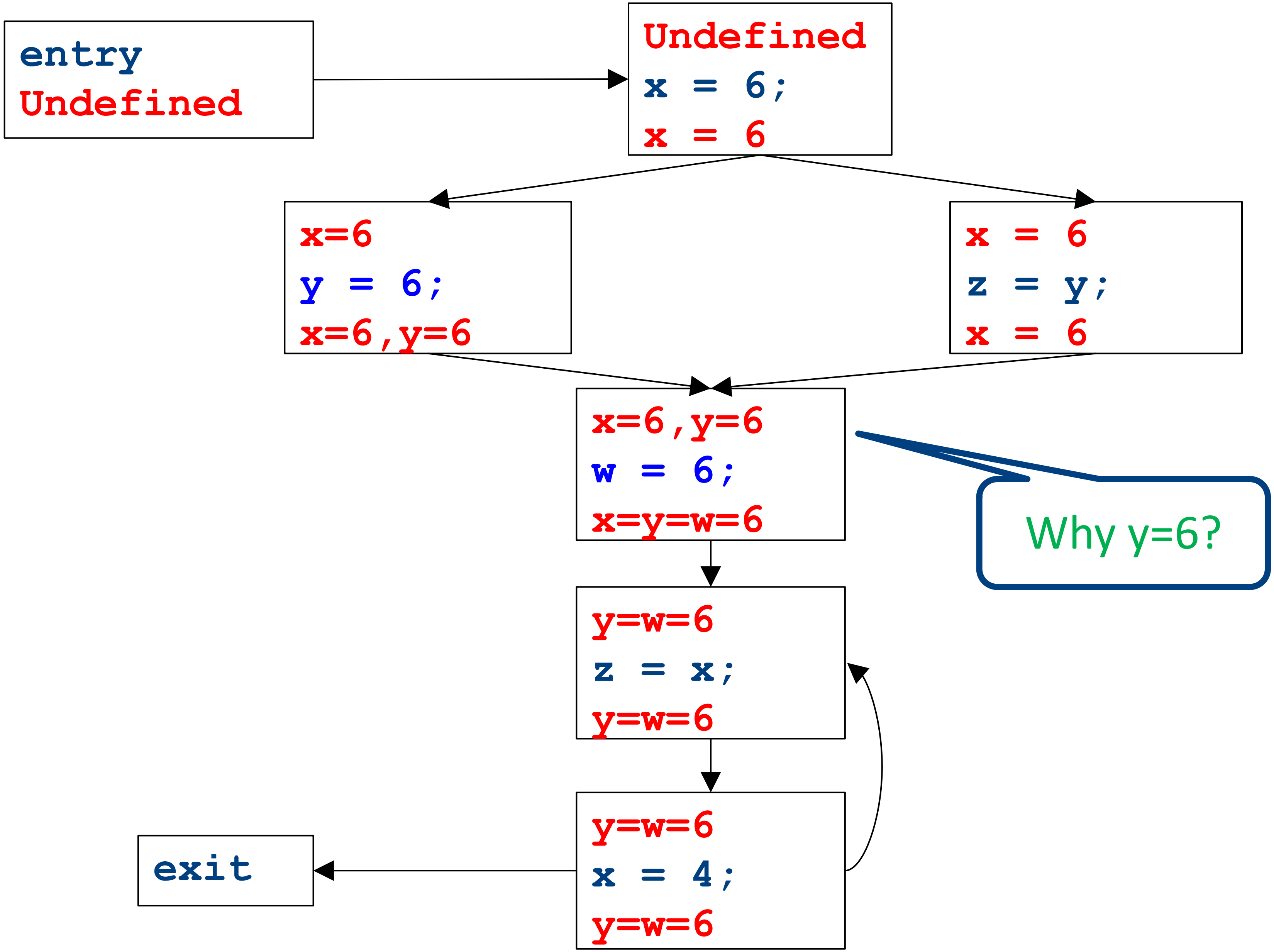

Dataflow for constant propagation

- Direction: **Forward**
- Semilattice: Vars→ {Undefined, 0, 1, -1, 2, -2, …, Not-a-Constant}
  - Join mapping for variables point-wise
    $\{x \mapsto 1, y \mapsto 1, z \mapsto 1\} \sqcup \{x \mapsto 1, y \mapsto 2, z \mapsto \text{Not-a-Constant}\} =$
    $\{x \mapsto 1, y \mapsto \text{Not-a-Constant}, z \mapsto \text{Not-a-Constant}\}$
- Transfer functions:
  - $f_{\mathtt{x=k}}(V) = V|_{x \mapsto k}$ *(update V by mapping x to k)*
  - $f_{\mathtt{x=a+b}}(V) = V|_{x \mapsto \text{Not-a-Constant}}$ *(assign Not-a-Constant)*
- Initial value: **x is Undefined**
  - (When might we use some other value?)

Proving termination

- Our algorithm for running these analyses continuously loops until no changes are detected
- Given this, how do we know the analyses will eventually terminate?
  - In general, **we don‘t**

Terminates?

Liveness Analysis

- A variable is live at a point in a program if later in the program its value will be read before it is written to again

Join semilattice definition

- A join semilattice is a pair (V, ⊔), where
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- ⊔ is a join operator that is
  - commutative: $x \sqcup y = y \sqcup x$
  - associative: $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
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- If $x \sqcup y = z$, we say that z is the join or (**L**east **U**pper **B**ound) of x and y
- Every join semilattice has a bottom element denoted $\perp$ such that $\perp \sqcup x = x$ for all x

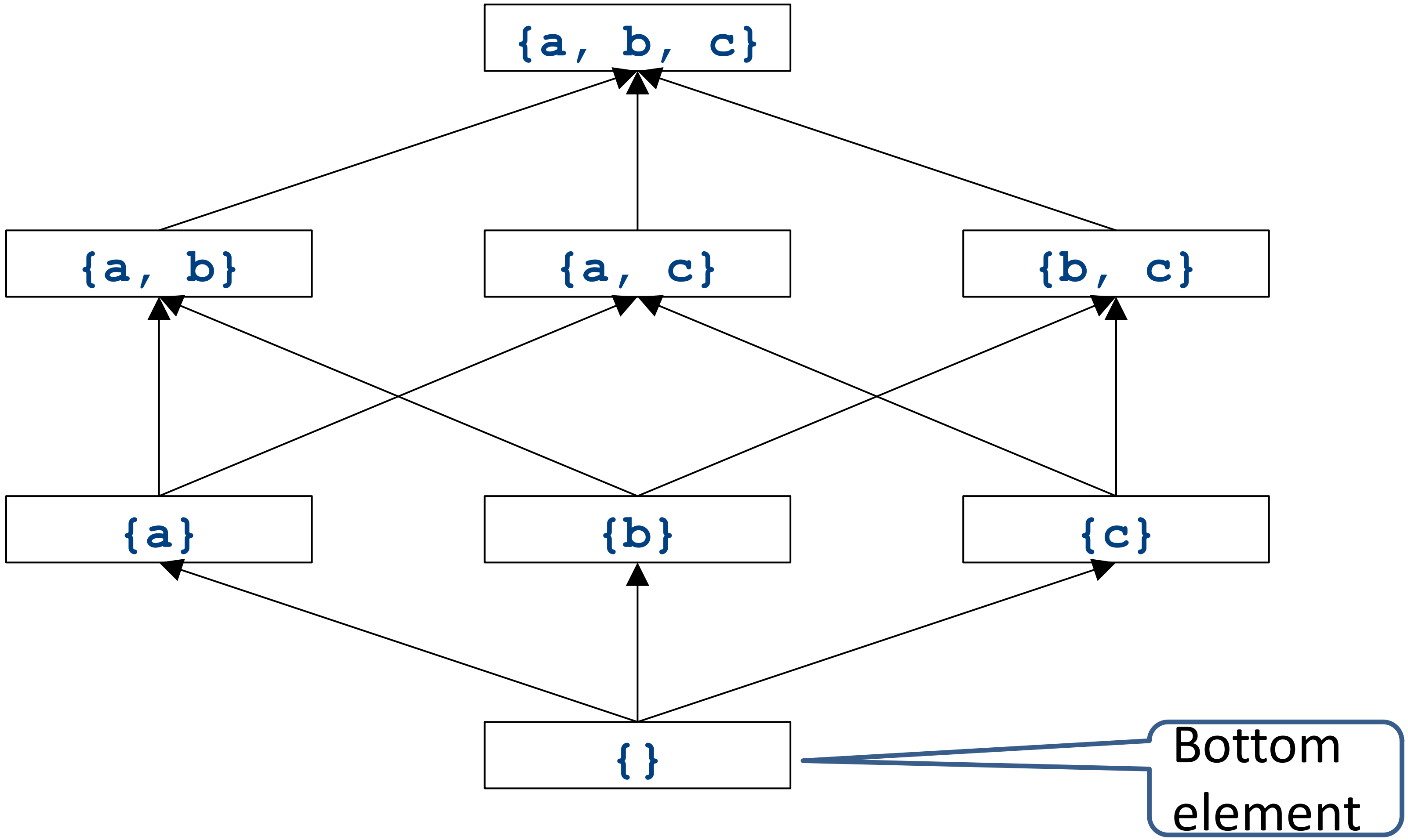
Partial ordering induced by join

- Every join semilattice $(V, \sqcup)$ induces an ordering relationship $\sqsubseteq$ over its elements
- Define $x \sqsubseteq y$ iff $x \sqcup y = y$
- Need to prove
  - Reflexivity: $x \sqsubseteq x$
  - Antisymmetry: If $x \sqsubseteq y$ and $y \sqsubseteq x$, then $x = y$
  - Transitivity: If $x \sqsubseteq y$ and $y \sqsubseteq z$, then $x \sqsubseteq z$

A join semilattice for liveness

- Sets of live variables and the set union operation
- Idempotent:
  - $x \cup x = x$
- Commutative:
  - $x \cup y = y \cup x$
- Associative:
  - $(x \cup y) \cup z = x \cup (y \cup z)$
- Bottom element:
  - The empty set: $\emptyset \cup x = x$
- Ordering over elements = subset relation

Join semilattice example for liveness



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  - $\mathbb{D}$ is a direction (forward or backward)
    - The order to visit statements within a basic block, **NOT** the order in which to visit the basic blocks
  - $\mathbb{V}$ is a set of values (sometimes called domain)
  - $\sqcup$ is a join operator over those values
  - $\mathbb{F}$ is a set of transfer functions f_s : **V → V** (for every statement s)
  - $\mathbb{I}$ is an initial value

Running global analyses

- Assume that $(\mathbb{D}, \mathbb{V}, \sqcup, \mathbb{F}, \mathbb{I})$ is a forward analysis
- For every statement s maintain values before - IN[s] - and after - OUT[s]
- Set OUT[**s**] = $\perp$ for all statements **s**
- Set OUT[**entry**] = $\mathbb{I}$
- Repeat until no values change:
  - For each statement **s** with predecessors PRED[s]={$\mathbf{p_1}$, $\mathbf{p_2}$, … , $\mathbf{p_n}$}
    - Set IN[**s**] = OUT[$\mathbf{p_1}$] $\sqcup$ OUT[$\mathbf{p_2}$] $\sqcup$ … $\sqcup$ OUT[$\mathbf{p_n}$]
    - Set OUT[**s**] = $f_{\mathbf{s}}$(IN[**s**])
- The order of this iteration does not matter
  - Chaotic iteration

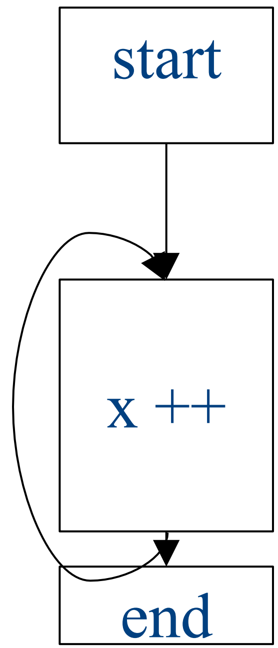
Proving termination

- Our algorithm for running these analyses continuously loops until no changes are detected
- Problem: how do we know the analyses will eventually terminate?

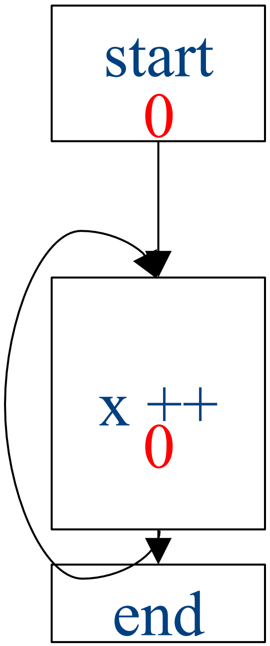
A non-terminating analysis

- The following analysis will loop infinitely on any CFG containing a loop:
- Direction: Forward
- Domain: $\mathbb{N}$
- Join operator: **max**
- Transfer function: $f(n) = n + 1$
- Initial value: 0

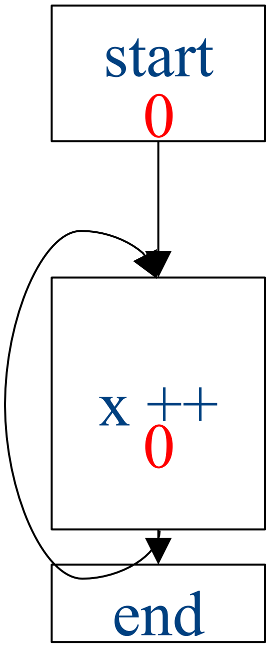
A non-terminating analysis



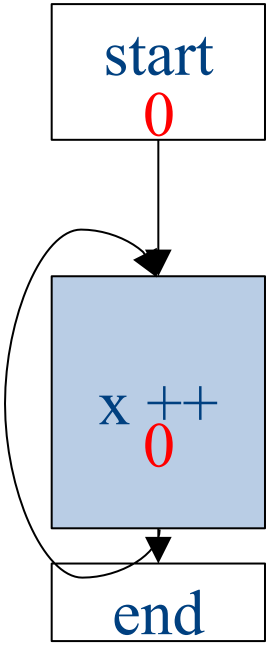
Initialization



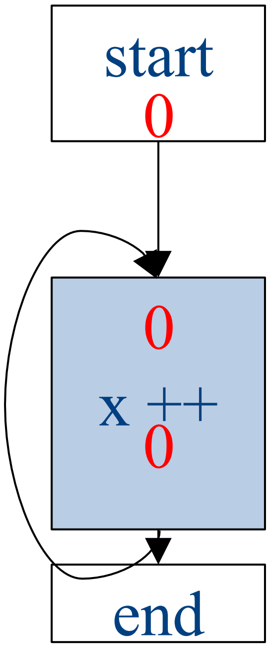
Fixed-point iteration



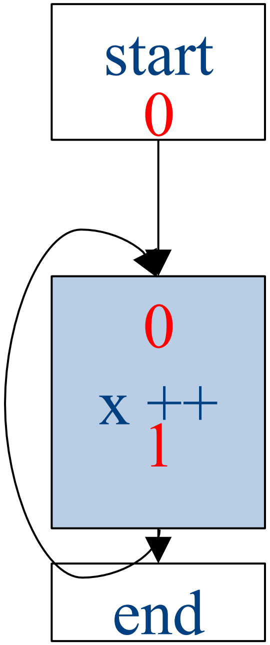
Choose a block



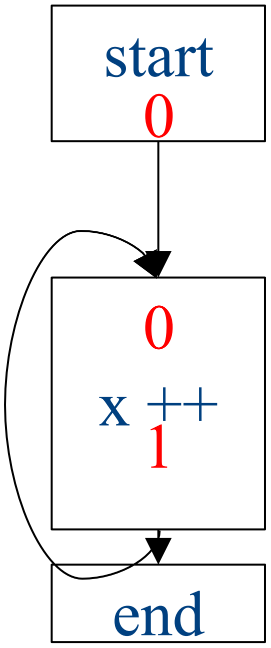
Iteration 1



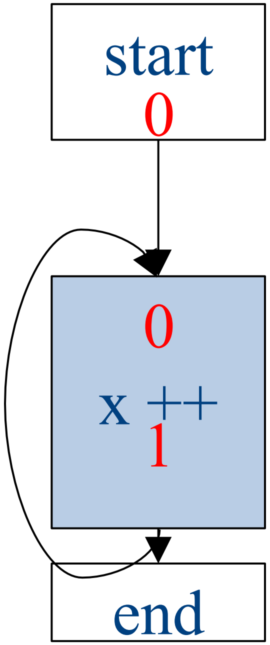
Iteration 1



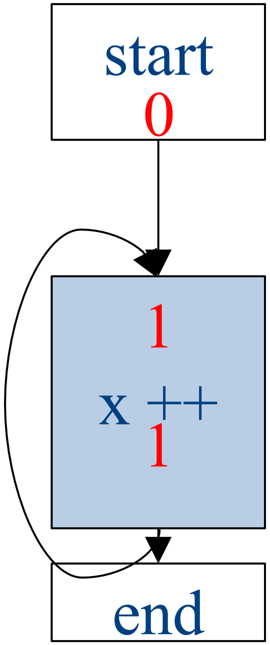
Choose a block



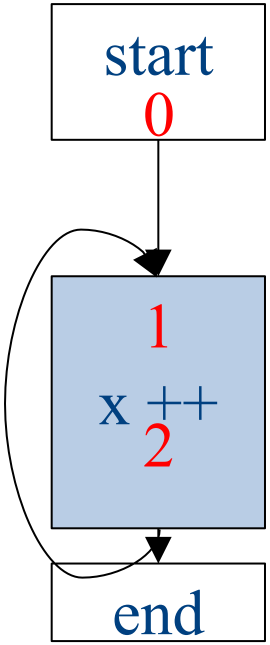
Iteration 2



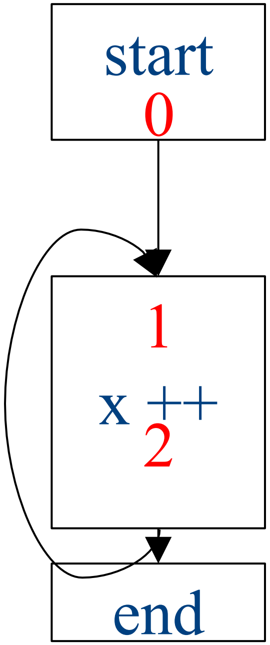
Iteration 2



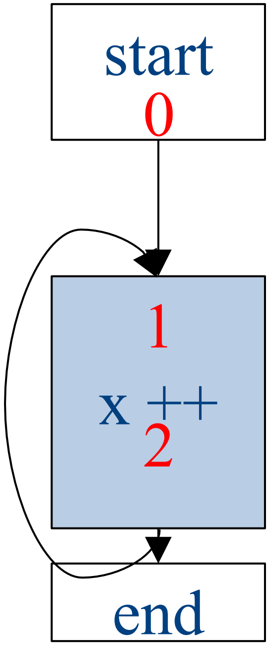
Iteration 2



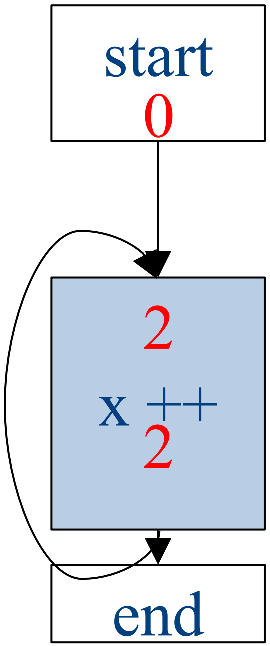
Choose a block



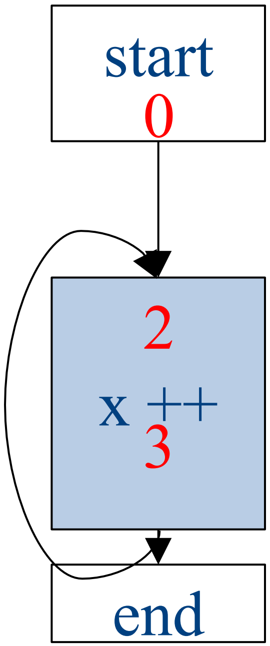
Iteration 3



Iteration 3

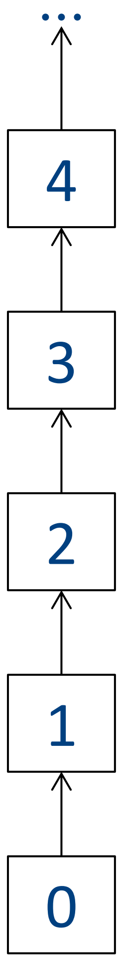


Iteration 3



Why doesn't this terminate?

- Values can increase without bound
- Note that "increase" refers to the lattice ordering, not the ordering on the natural numbers
- The height of a semilattice is the length of the longest increasing sequence in that semilattice
- The dataflow framework is not guaranteed to terminate for semilattices of infinite height
- Note that a semilattice can be infinitely large but have finite height
  - e.g. constant propagation



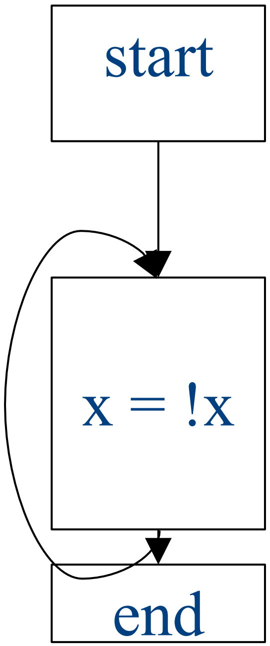
Height of a lattice

- An increasing chain is a sequence of elements $\perp \sqsubseteq a_1 \sqsubseteq a_2 \sqsubseteq \ldots \sqsubseteq a_k$
  - The length of such a chain is k
- The height of a lattice is the length of the maximal increasing chain
- For liveness with *n* program variables:
  - $\{\} \subseteq \{v_1\} \subseteq \{v_1, v_2\} \subseteq \ldots \subseteq \{v_1, \ldots, v_n\}$
- For available expressions it is the number of expressions of the form a=b op c
  - For *n* program variables and *m* operator types: mn^3

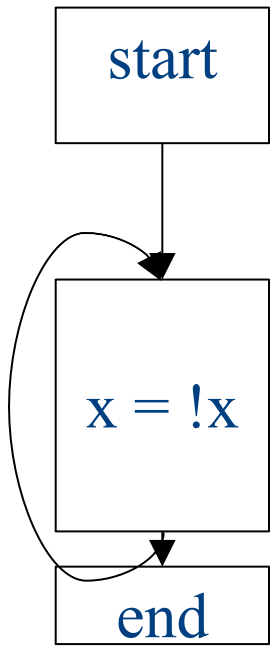
Another non-terminating analysis

- This analysis works on a finite-height semilattice, but will not terminate on certain CFGs:
- Direction: Forward
- Domain: Boolean values **`true`** and **`false`**
- Join operator: Logical OR
- Transfer function: Logical NOT
- Initial value: **`false`**

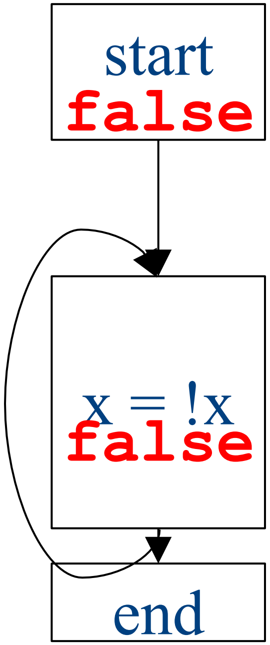
A non-terminating analysis



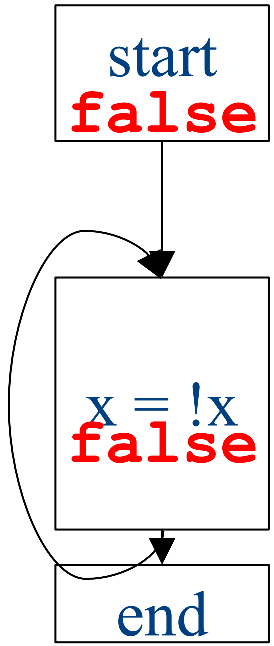
A non-terminating analysis



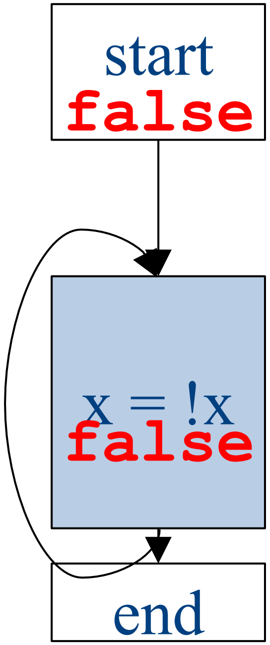
Initialization



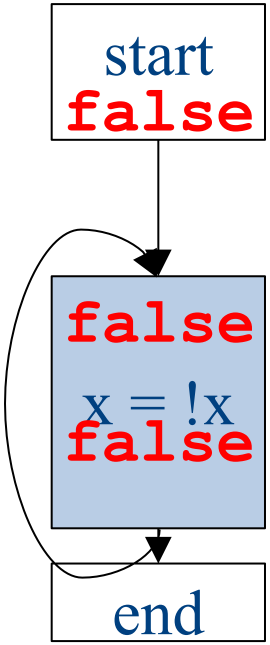
Fixed-point iteration



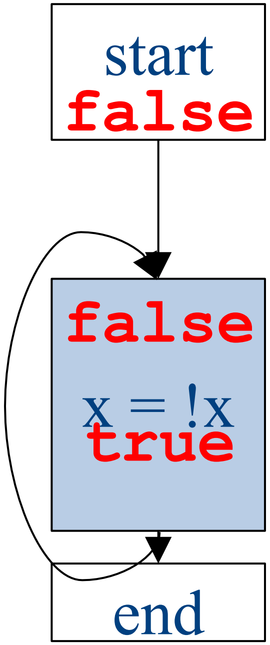
Choose a block



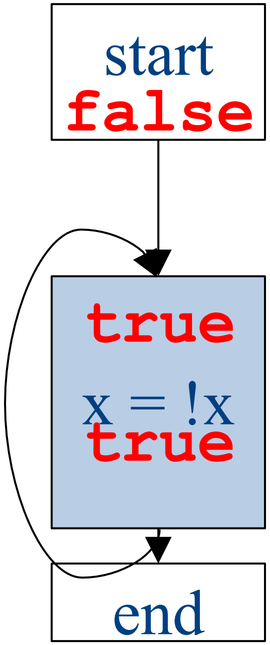
Iteration 1



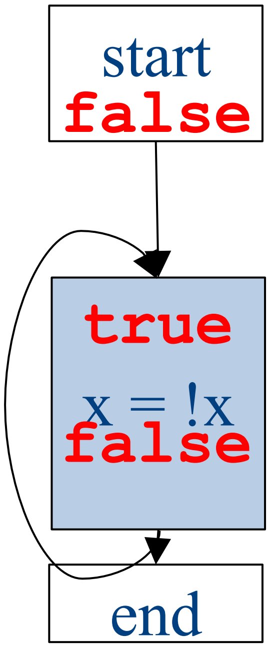
Iteration 1



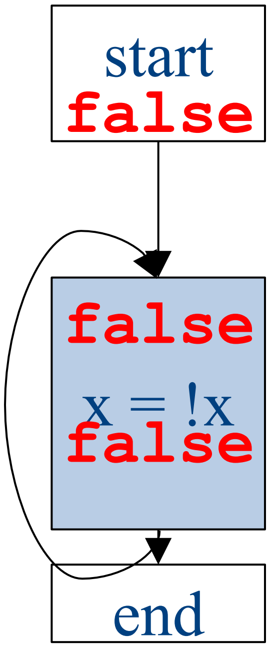
Iteration 2



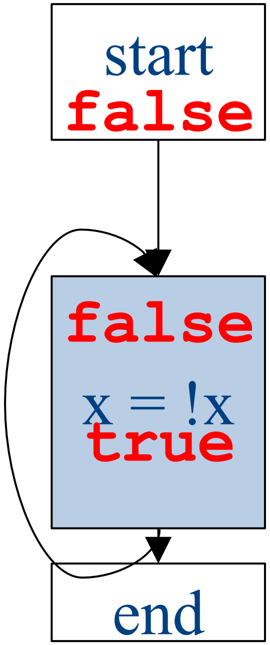
Iteration 2



Iteration 3

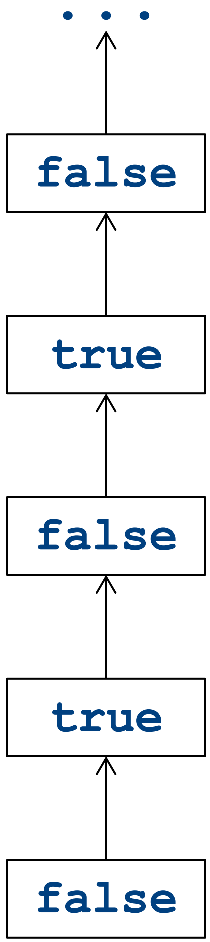


Iteration 3



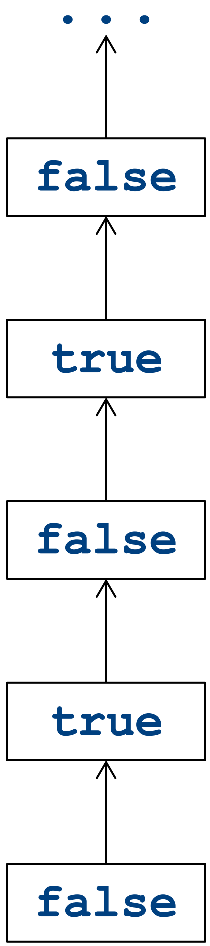
Why doesn't it terminate?

- Values can loop indefinitely
- Intuitively, the join operator keeps pulling values up
- If the transfer function can keep pushing values back down again, then the values might cycle forever



Why doesn't it terminate?

- Values can loop indefinitely
- Intuitively, the join operator keeps pulling values up
- If the transfer function can keep pushing values back down again, then the values might cycle forever
- How can we fix this?



Monotone transfer functions

- A transfer function f is monotone iff

  if $x \sqsubseteq y$, then $f(x) \sqsubseteq f(y)$

- Intuitively, if you know less information about a program point, you can't "gain back" more information about that program point
- Many transfer functions are monotone, including those for liveness and constant propagation
- Note: Monotonicity does **not** mean that $x \sqsubseteq f(x)$
  - (This is a different property called extensivity)

Liveness and monotonicity

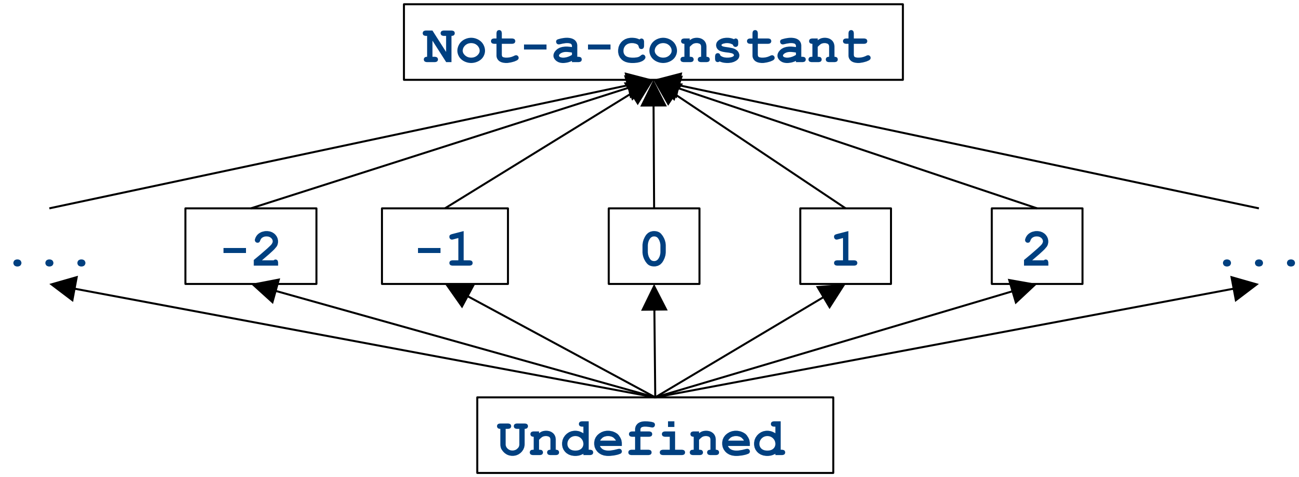
- A transfer function f is monotone iff

  if $x \sqsubseteq y$, then $f(x) \sqsubseteq f(y)$
- Recall our transfer function for **a = b + c** is
  - $f_{a = b + c}(V) = (V - \{a\}) \cup \{b, c\}$
- Recall that our join operator is set union and induces an ordering relationship

  $X \sqsubseteq Y$ iff $X \subseteq Y$
- Is this monotone?

Is constant propagation monotone?

- A transfer function *f* is monotone iff
  if $x \sqsubseteq y$, then $f(x) \sqsubseteq f(y)$
- Recall our transfer functions
  - $f_{\mathbf{x=k}}(V) = V[x \mapsto k]$ *(update V by mapping x to k)*
  - $f_{\mathbf{x=a+b}}(V) = V[x \mapsto \text{Not-a-Constant}]$ *(assign Not-a-Constant)*
- Is this monotone?



The grand result

- Theorem: A dataflow analysis with a **finite-height semilattice** and family of **monotone transfer functions** always terminates
- Proof sketch:
  - The join operator can only bring values up
  - Transfer functions can never lower values back down below where they were in the past (monotonicity)
  - Values cannot increase indefinitely (finite height)

An "optimality" result

- A transfer function f is distributive if
  $$f(a \sqcup b) = f(a) \sqcup f(\mathrm{b})$$
  for every domain elements a and b
- If all transfer functions are distributive then the fixed-point solution is the solution that would be computed by joining results from all (potentially infinite) control-flow paths
  - Join over all paths
- Optimal if we ignore program conditions

An “optimality” result

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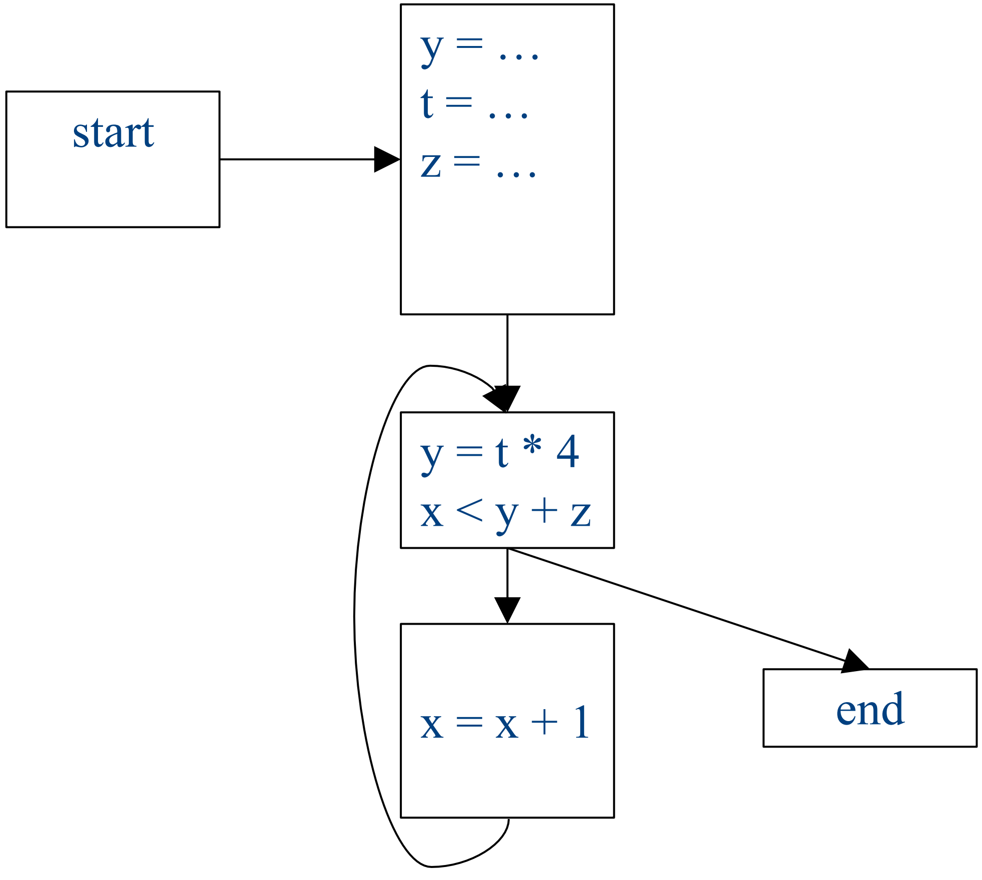
  $$f(a \sqcup b) = f(a) \sqcup f(\mathrm{b})$$
  for every domain elements a and b
- If all transfer functions are distributive then the fixed-point solution is equal to the solution computed by joining results from all (potentially infinite) control-flow paths
  - Join over all paths
- Optimal if we pretend all control-flow paths can be executed by the program
- Which analyses use distributive functions?

Loop Optimizations

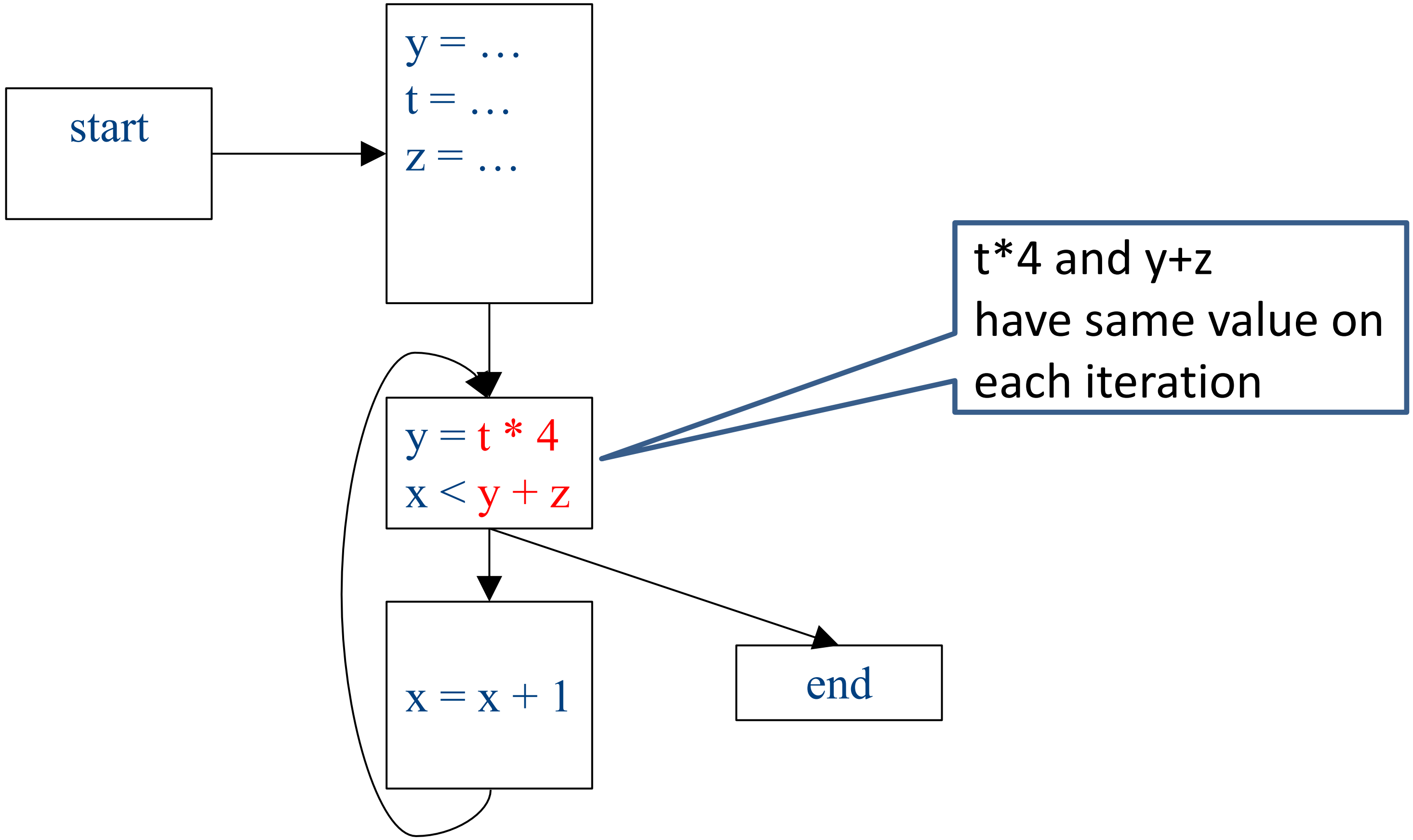
Loop optimizations

- Most of a program's computations are done inside loops
  - Focus optimizations effort on loops
- The optimizations we've seen so far are independent of the control structure
- Some optimizations are specialized to loops
  - Loop-invariant code motion
  - (Strength reduction via induction variables)
- Require another type of analysis to find out where expressions get their values from
  - Reaching definitions
    - (Also useful for improving register allocation)

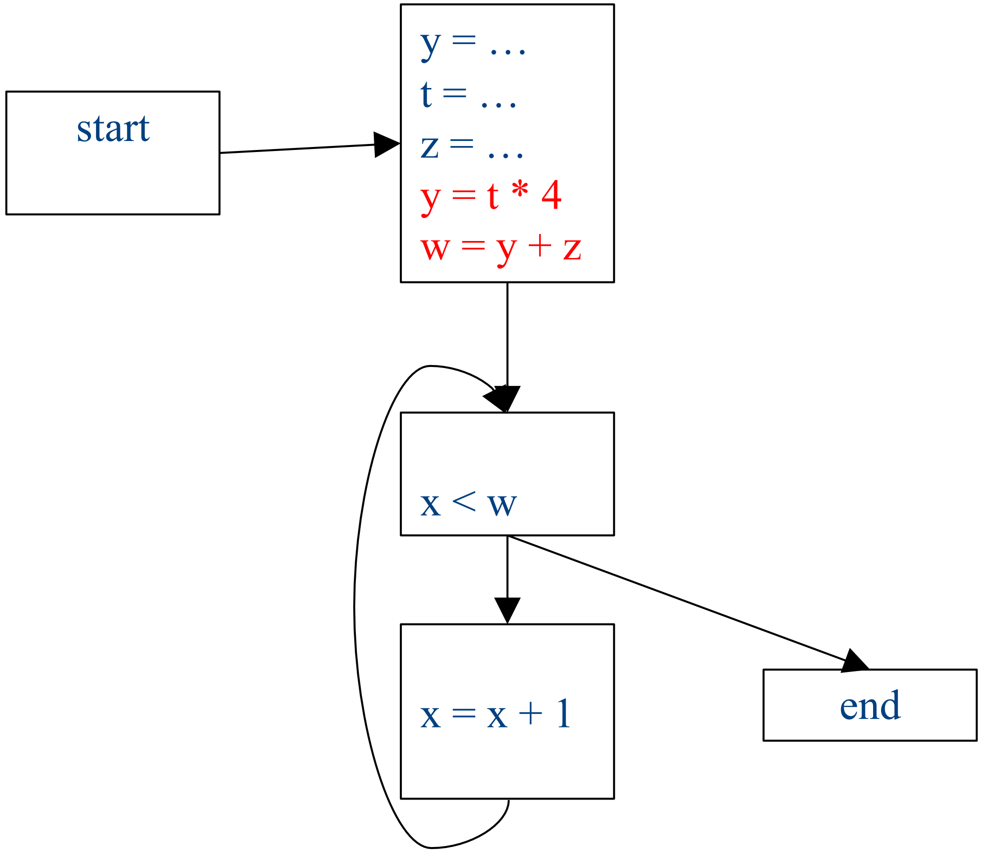
Loop invariant computation



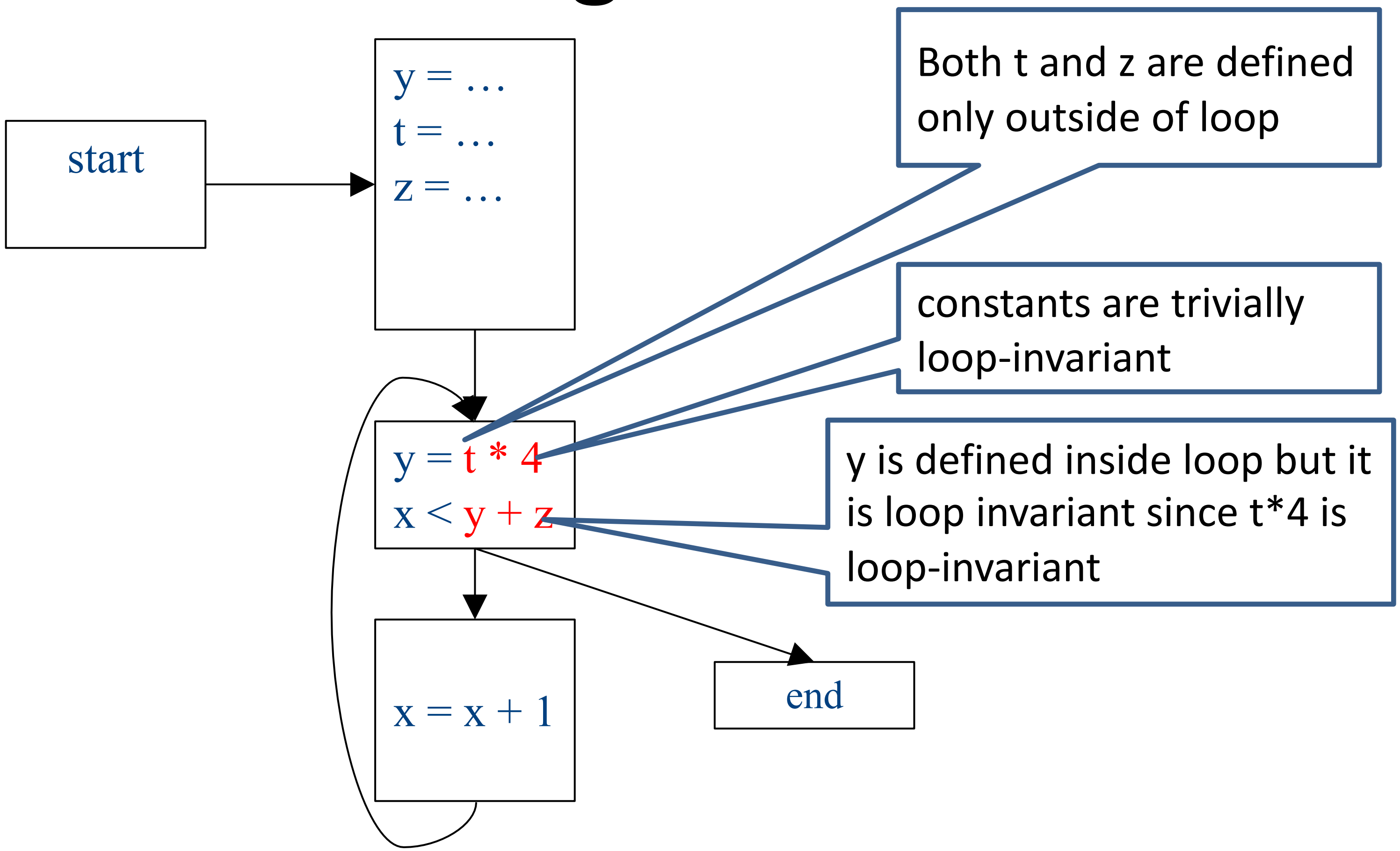
Loop invariant computation



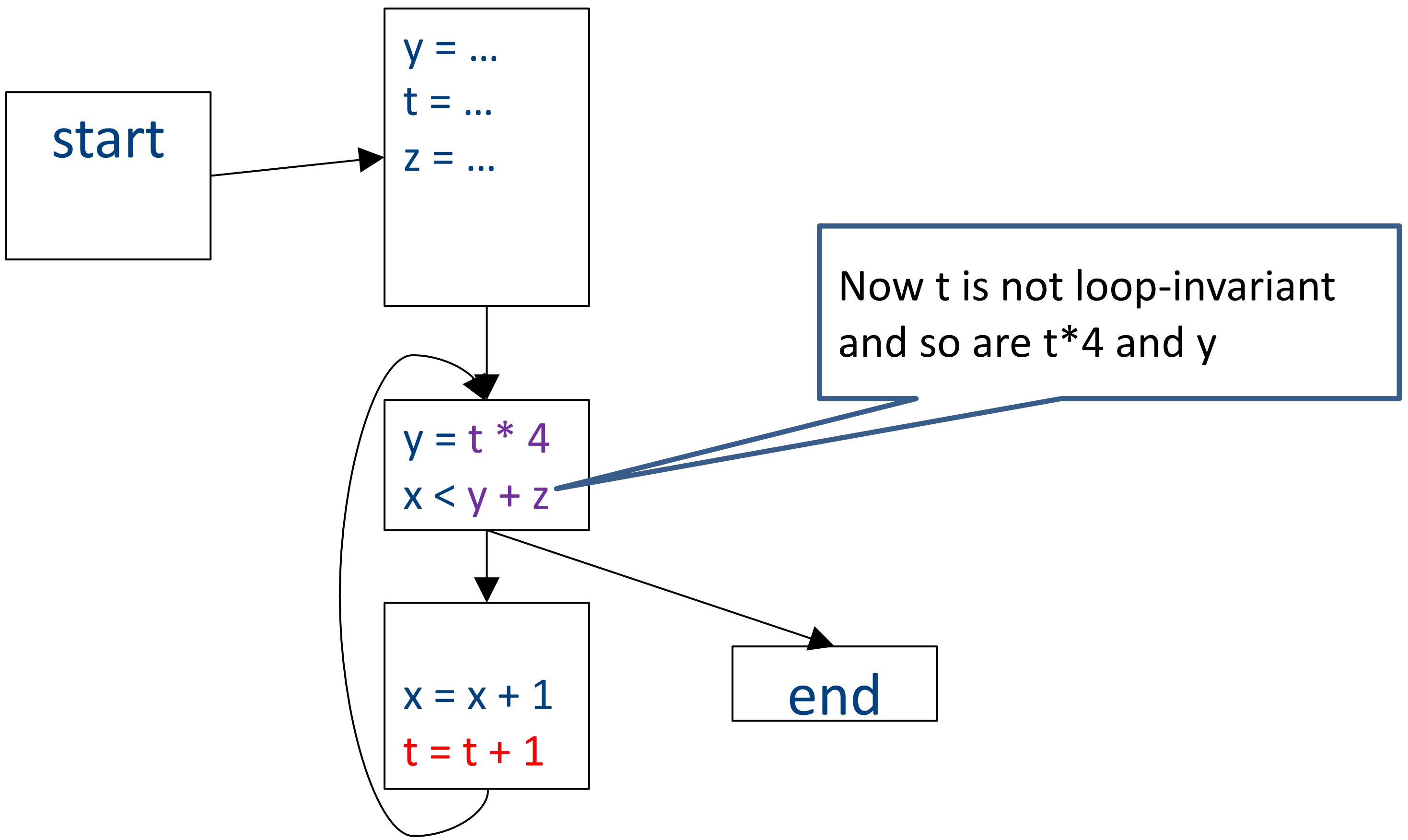
Code hoisting



What reasoning did we use?



What about now?



Loop-invariant code motion

- d: t = a_1 op a_2
  - d is a program location
- a_1 op a_2 loop-invariant (for a loop L) if computes the same value in each iteration
  - Hard to know in general
- Conservative approximation
  - Each a_i is a constant, or
  - All definitions of a_i that reach d are outside L, or
  - Only one definition of of a_i reaches d, and is loop-invariant itself
- Transformation: hoist the loop-invariant code outside of the loop

Reaching definitions analysis

- A definition *d*: t = … reaches a program location if there is a path from the definition to the program location, along which the defined variable is never redefined

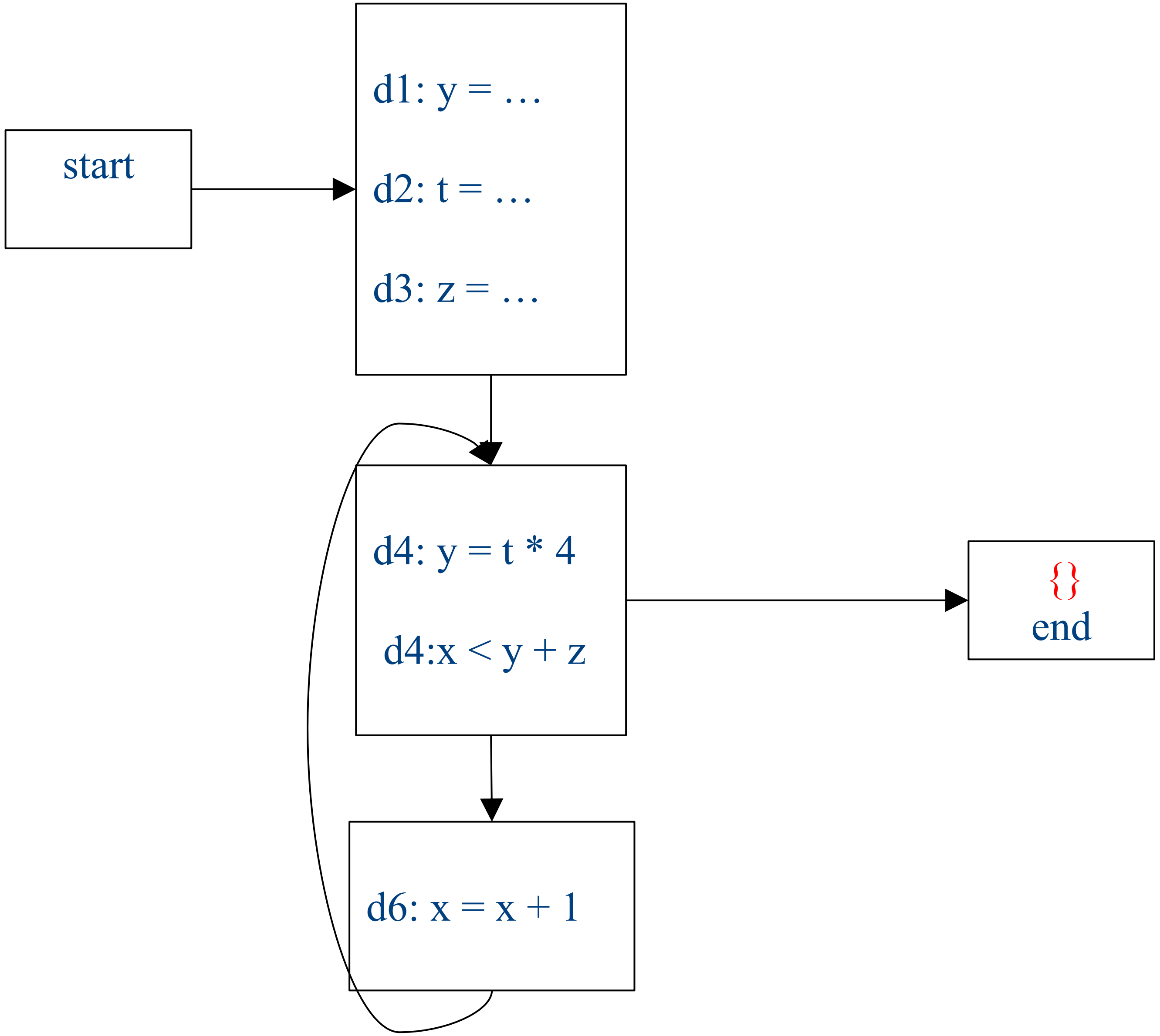
Reaching definitions analysis

- A definition *d*: t = … reaches a program location if there is a path from the definition to the program location, along which the defined variable is never redefined
- Direction: Forward
- Domain: sets of program locations that are definitions `
- Join operator: union
- Transfer function:

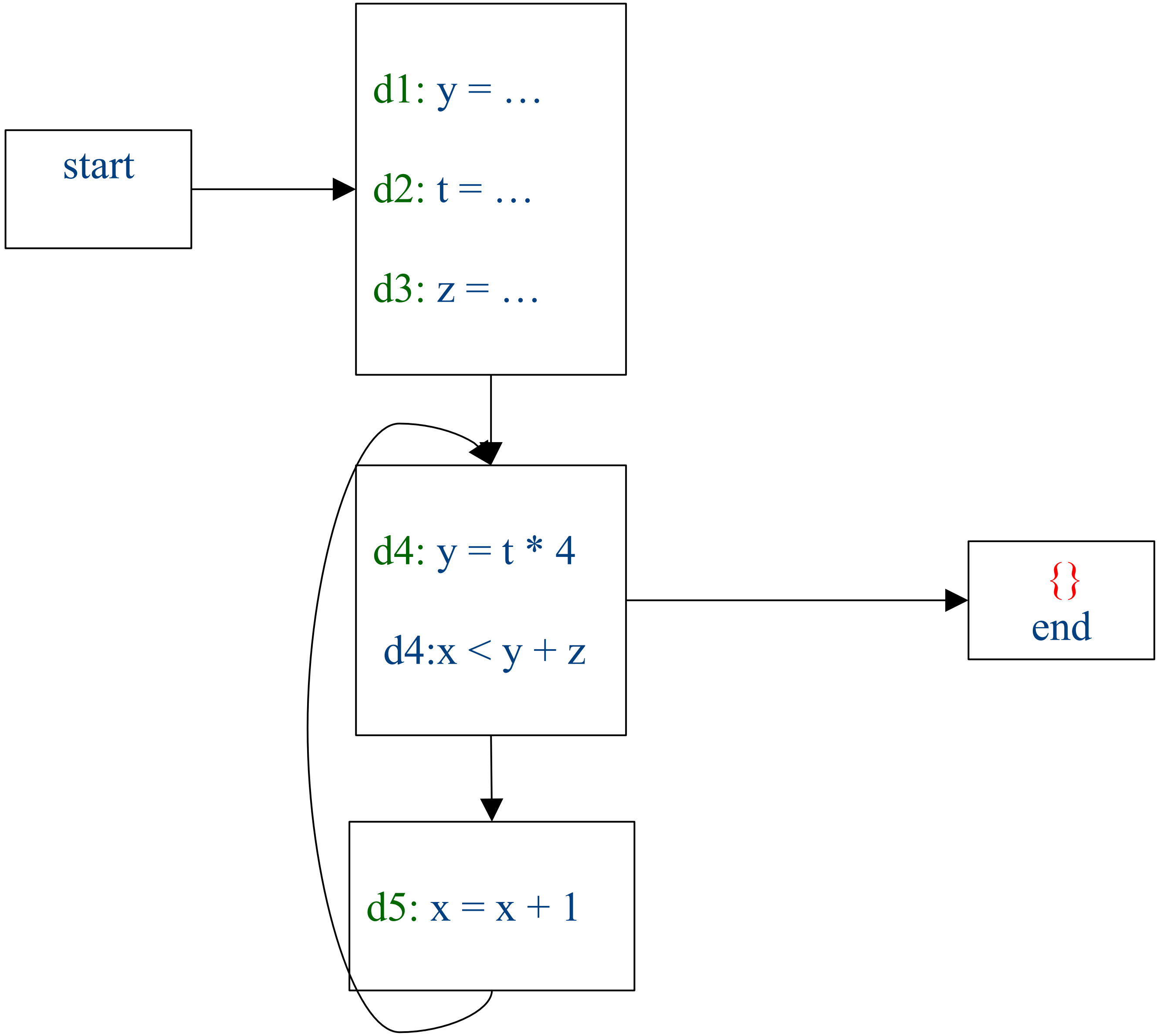
  $$f_{d:\ a=b\ op\ c}(RD) = (RD - \mathit{defs}(a)) \cup \{d\}$$
  $$f_{d:\ not\text{-}a\text{-}def}(RD) = RD$$

  – Where *defs*(*a*) is the set of locations defining *a* (statements of the form *a*=...)
- Initial value: {}

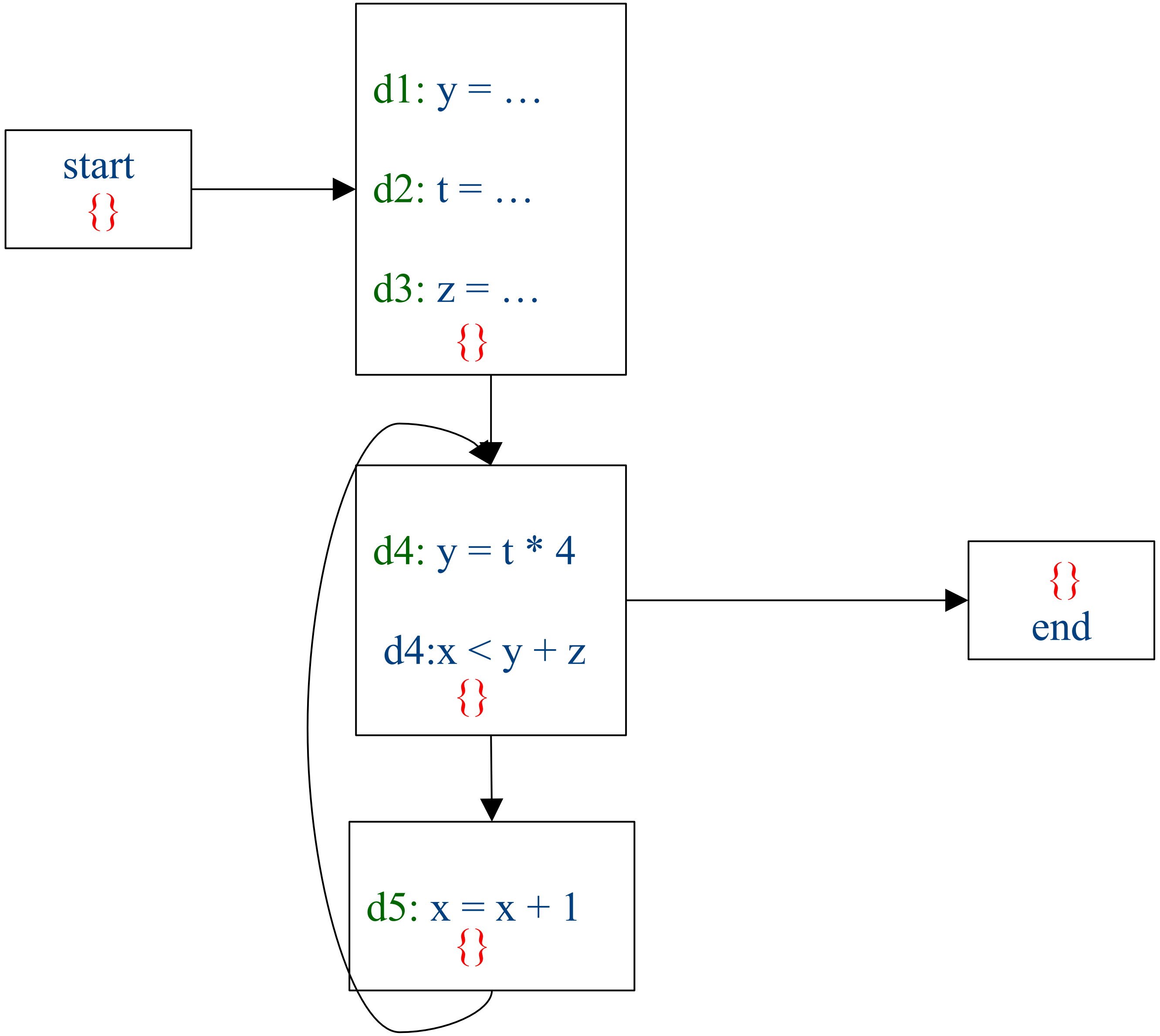
Reaching definitions analysis



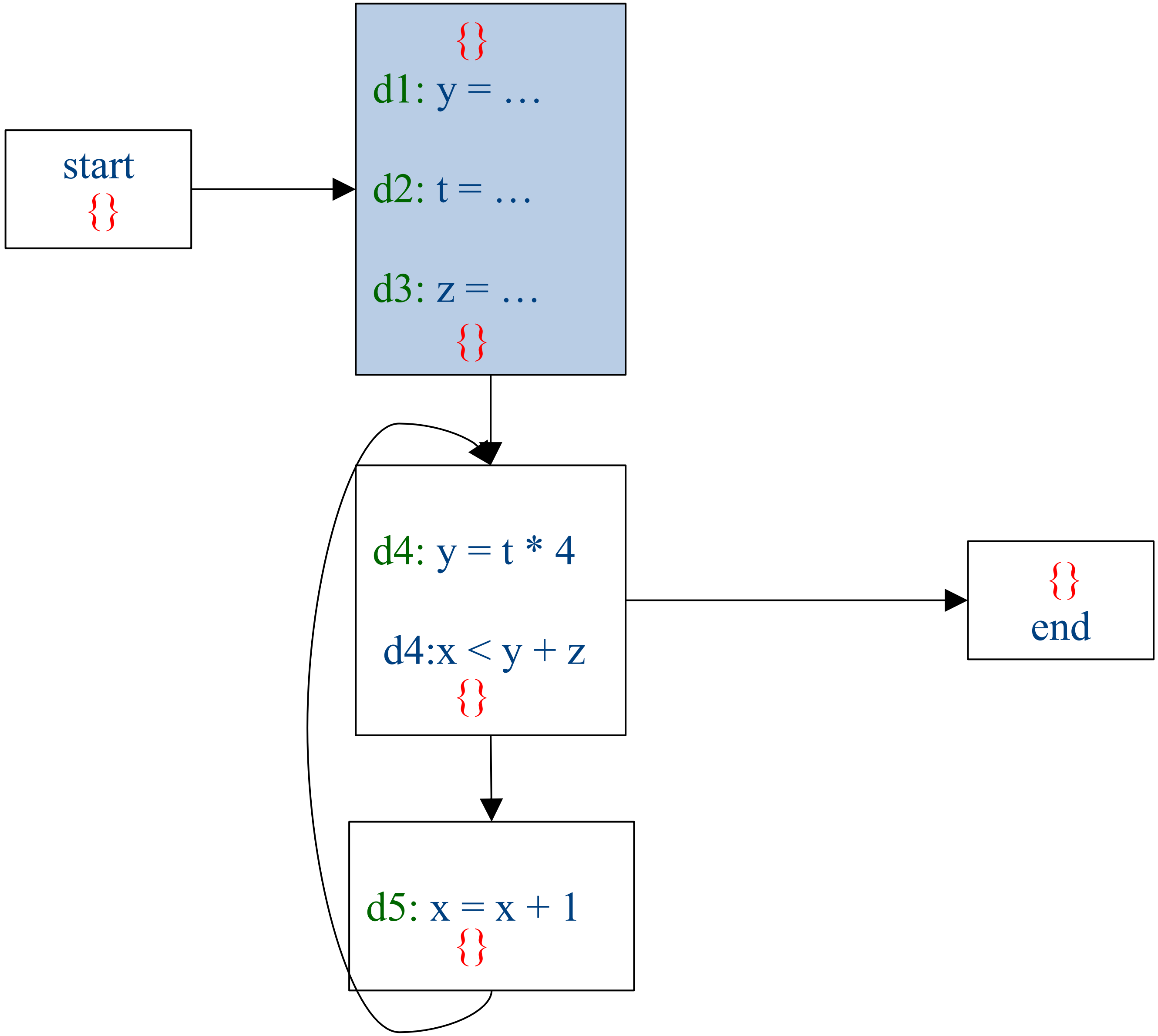
Reaching definitions analysis



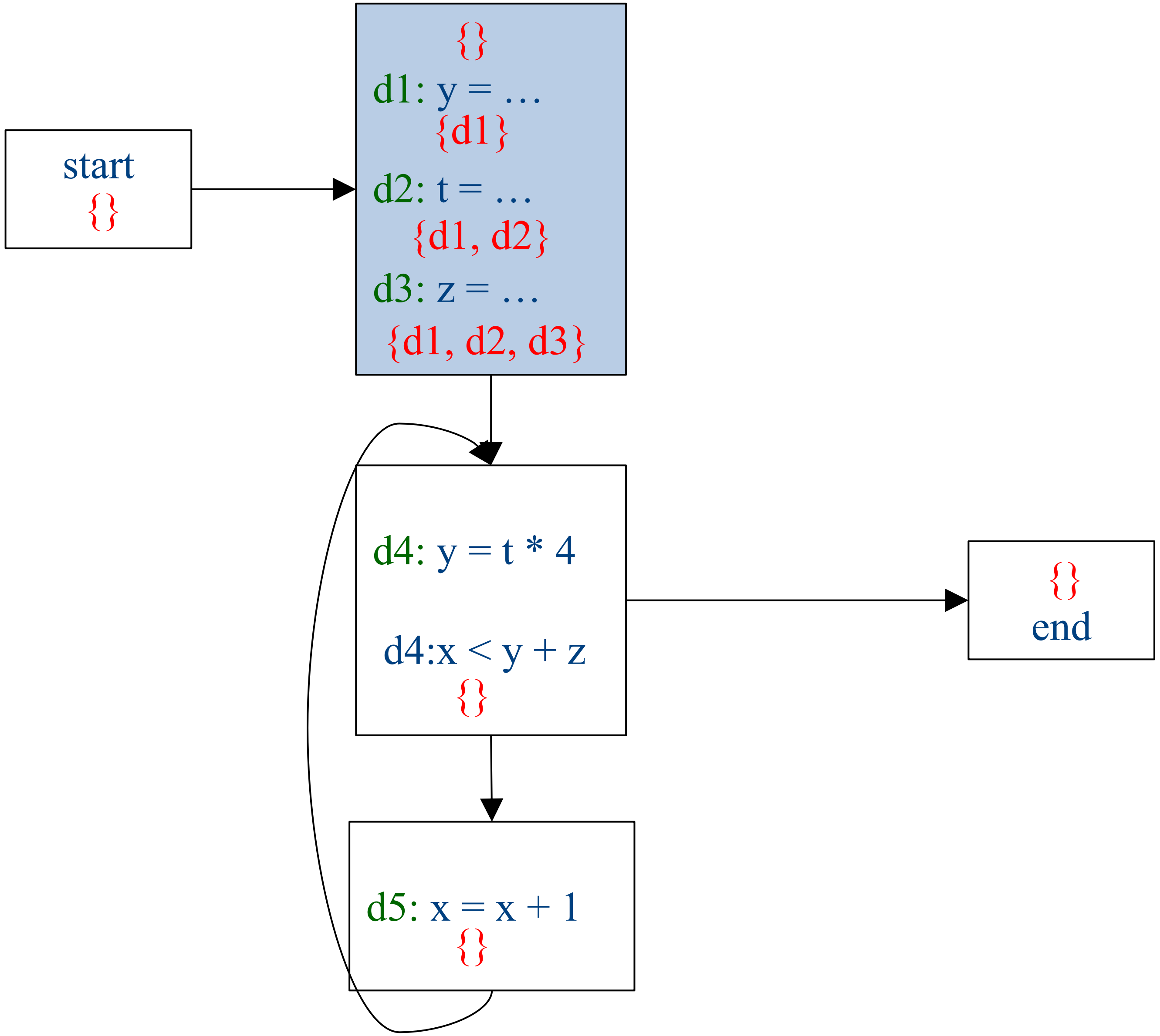
Initialization



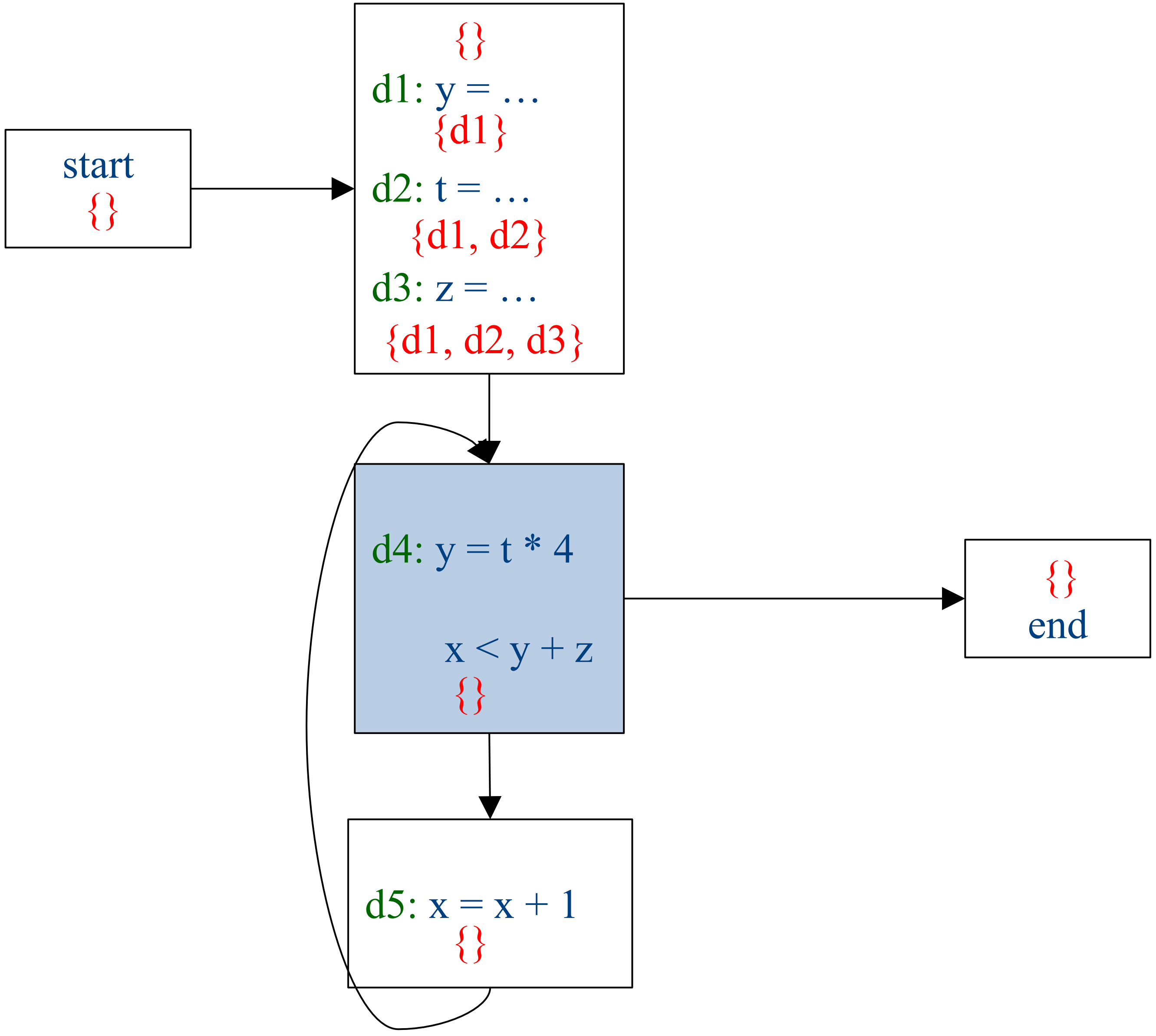
Iteration 1



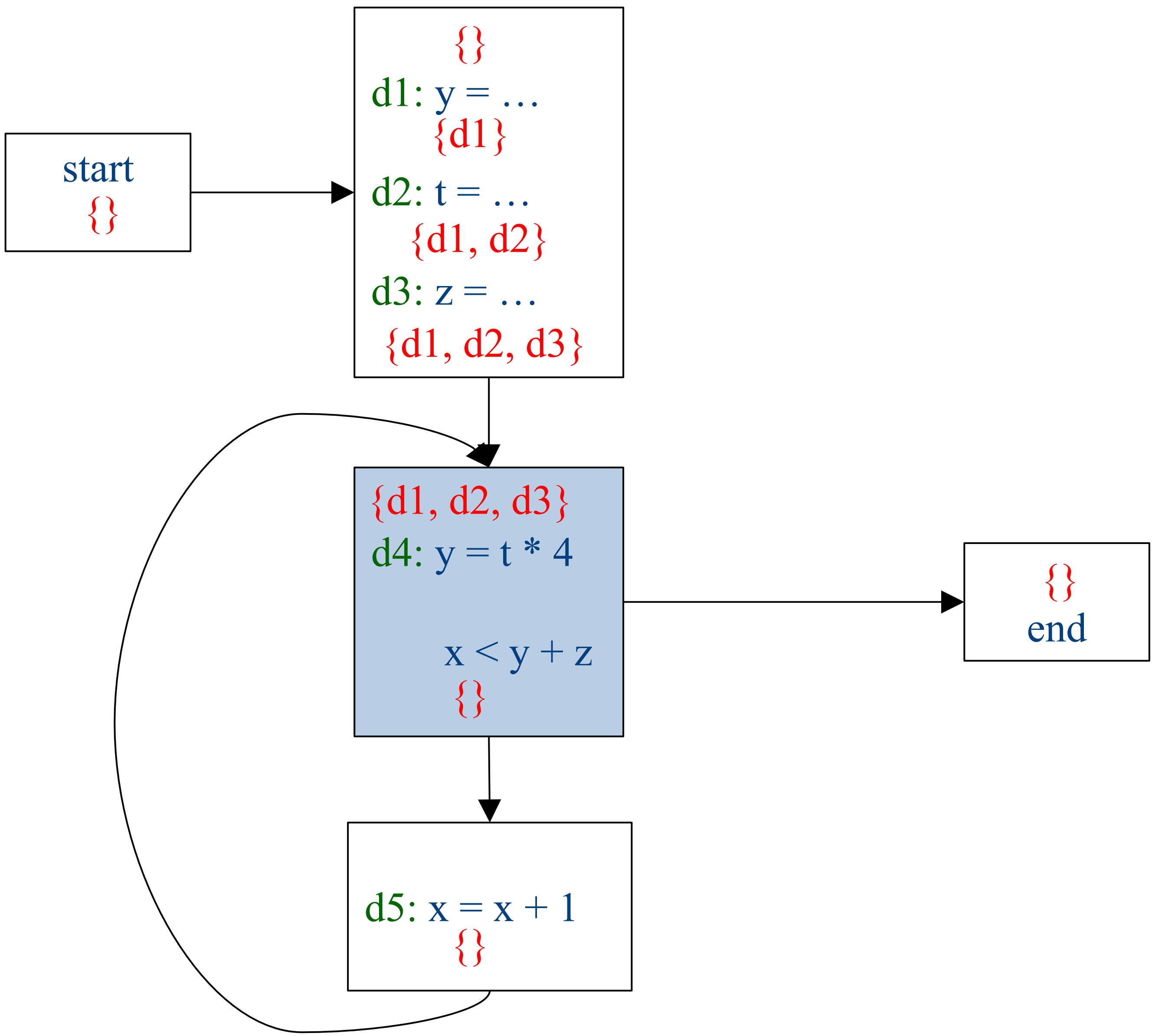
Iteration 1



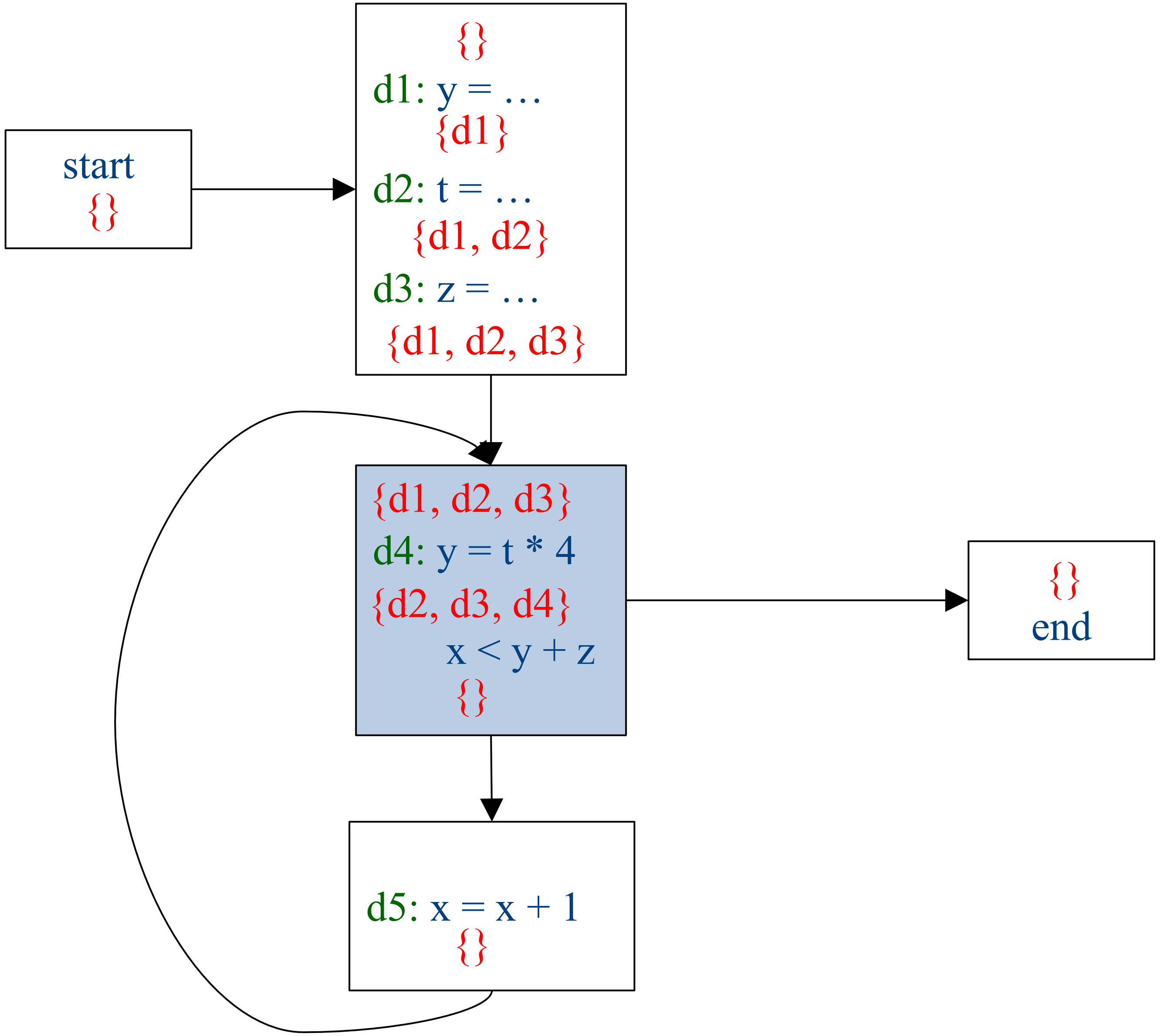
Iteration 2



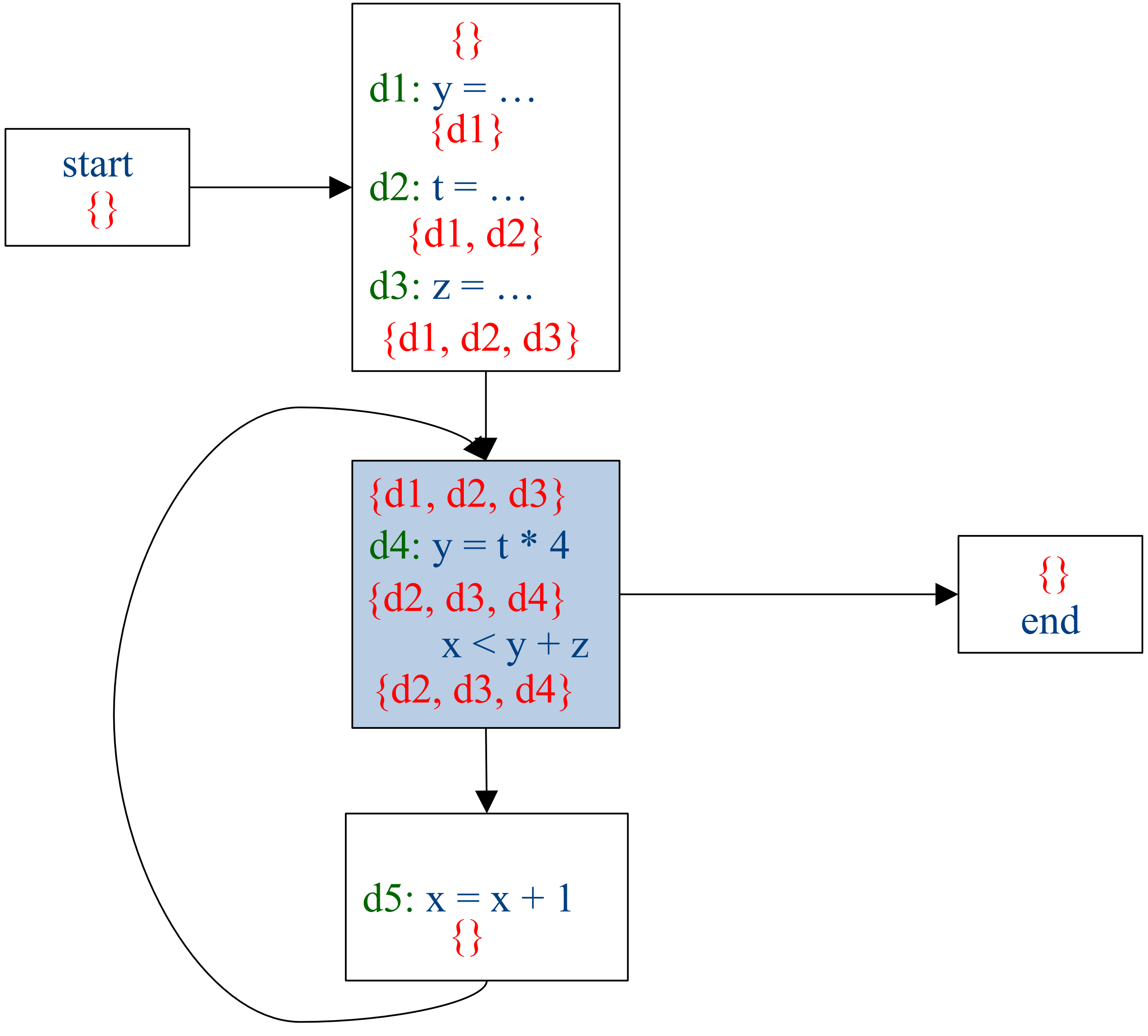
Iteration 2



Iteration 2


start
{}
{}
d1: y = …
{d1}
d2: t = …
{d1, d2}
d3: z = …
{d1, d2, d3}
{d1, d2, d3}
d4: y = t * 4
{d2, d3, d4}
x < y + z
{}
{}
end
d5: x = x + 1
{}

Iteration 2



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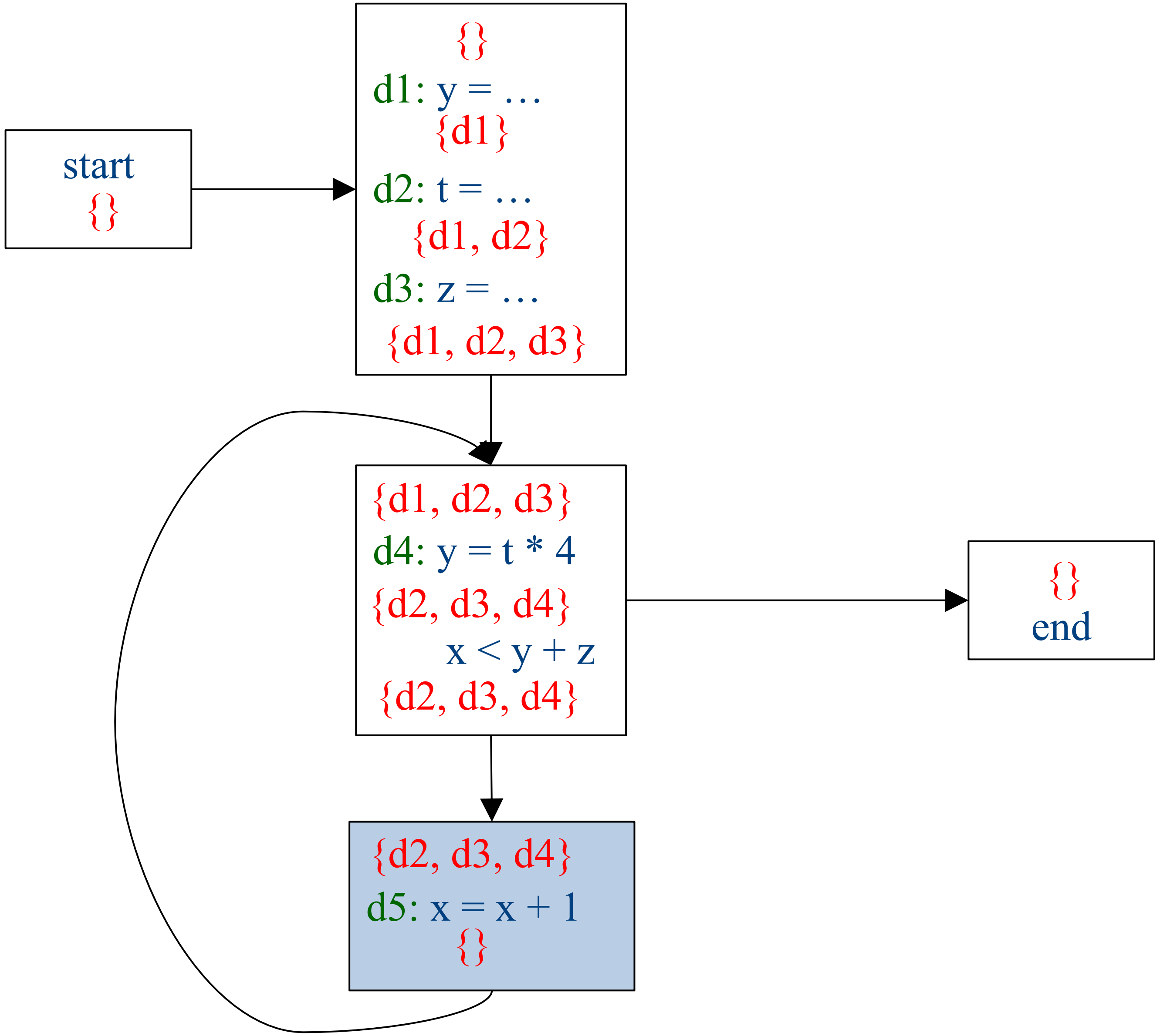
{d2, d3, d4}
x < y + z
{d2, d3, d4}
{}
end
d5: x = x + 1
{}

Iteration 3



start
{}
{}
d1: y = …
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{d1, d2}
d3: z = …
{d1, d2, d3}
{d1, d2, d3}
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end
{d2, d3, d4}
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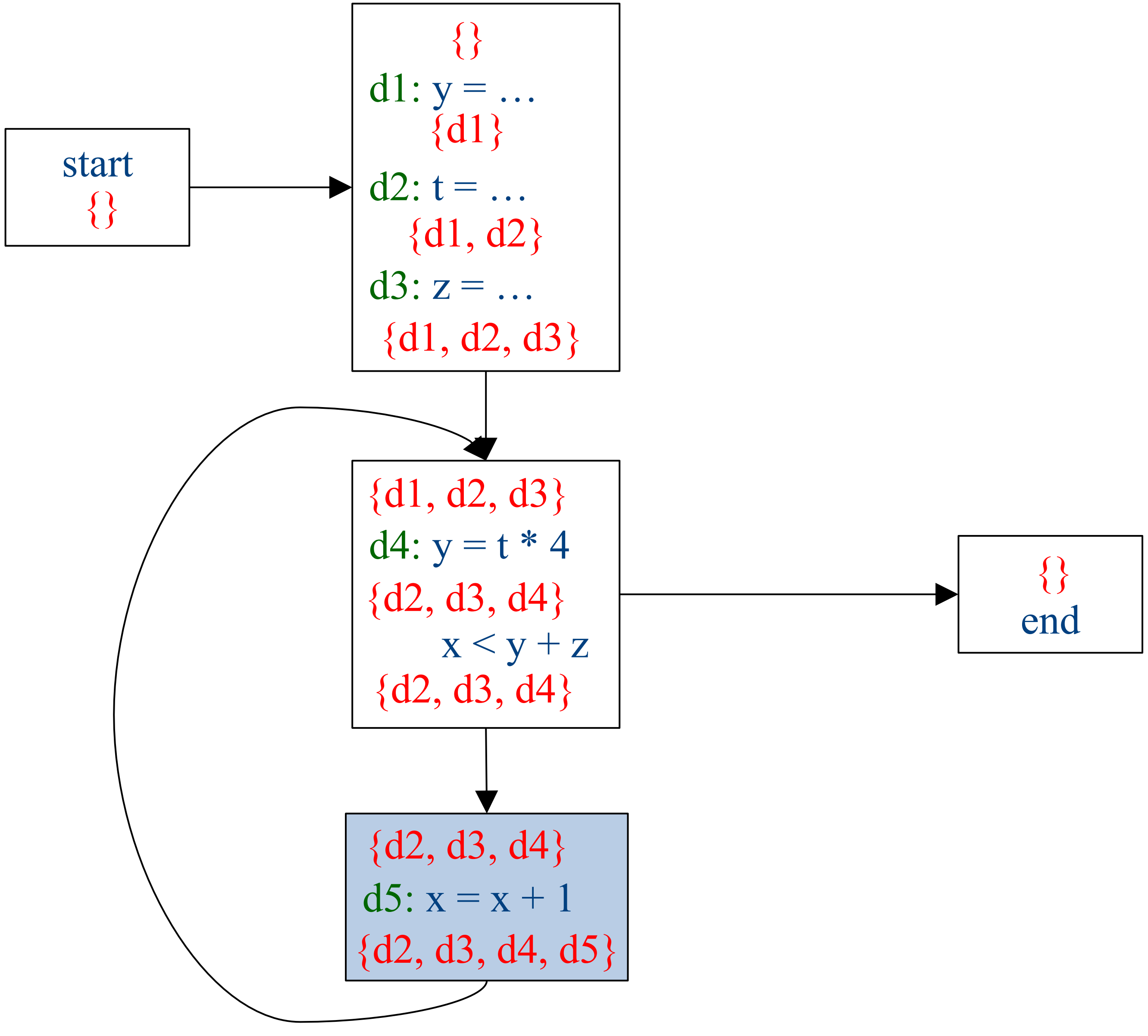
Iteration 3



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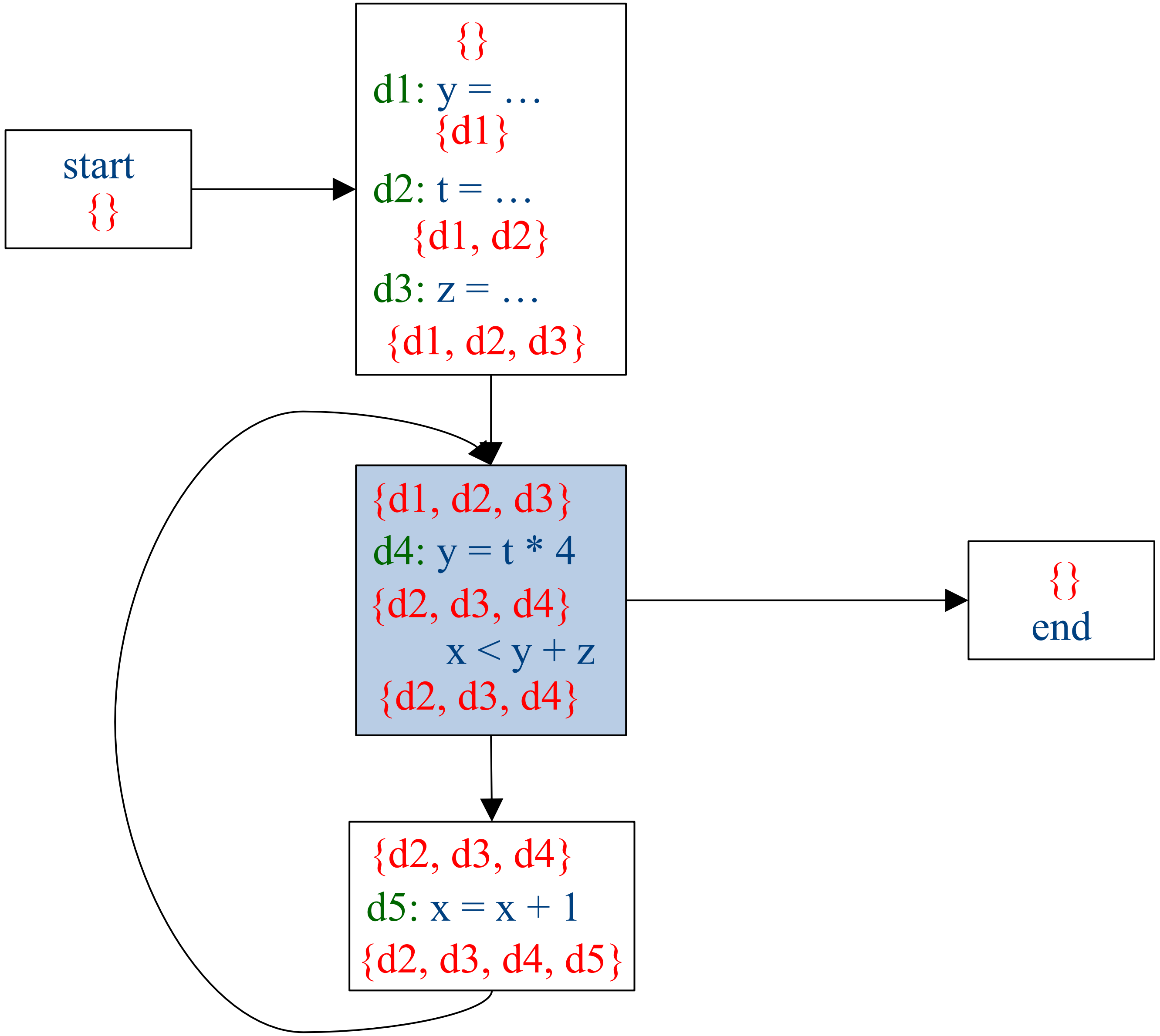
{d2, d3, d4}
x < y + z
{d2, d3, d4}
{}
end
{d2, d3, d4}
d5: x = x + 1
{d2, d3, d4, d5}

Iteration 4



start
{}
{}
d1: y = …
{d1}
d2: t = …
{d1, d2}
d3: z = …
{d1, d2, d3}
{d1, d2, d3}
d4: y = t * 4
{d2, d3, d4}
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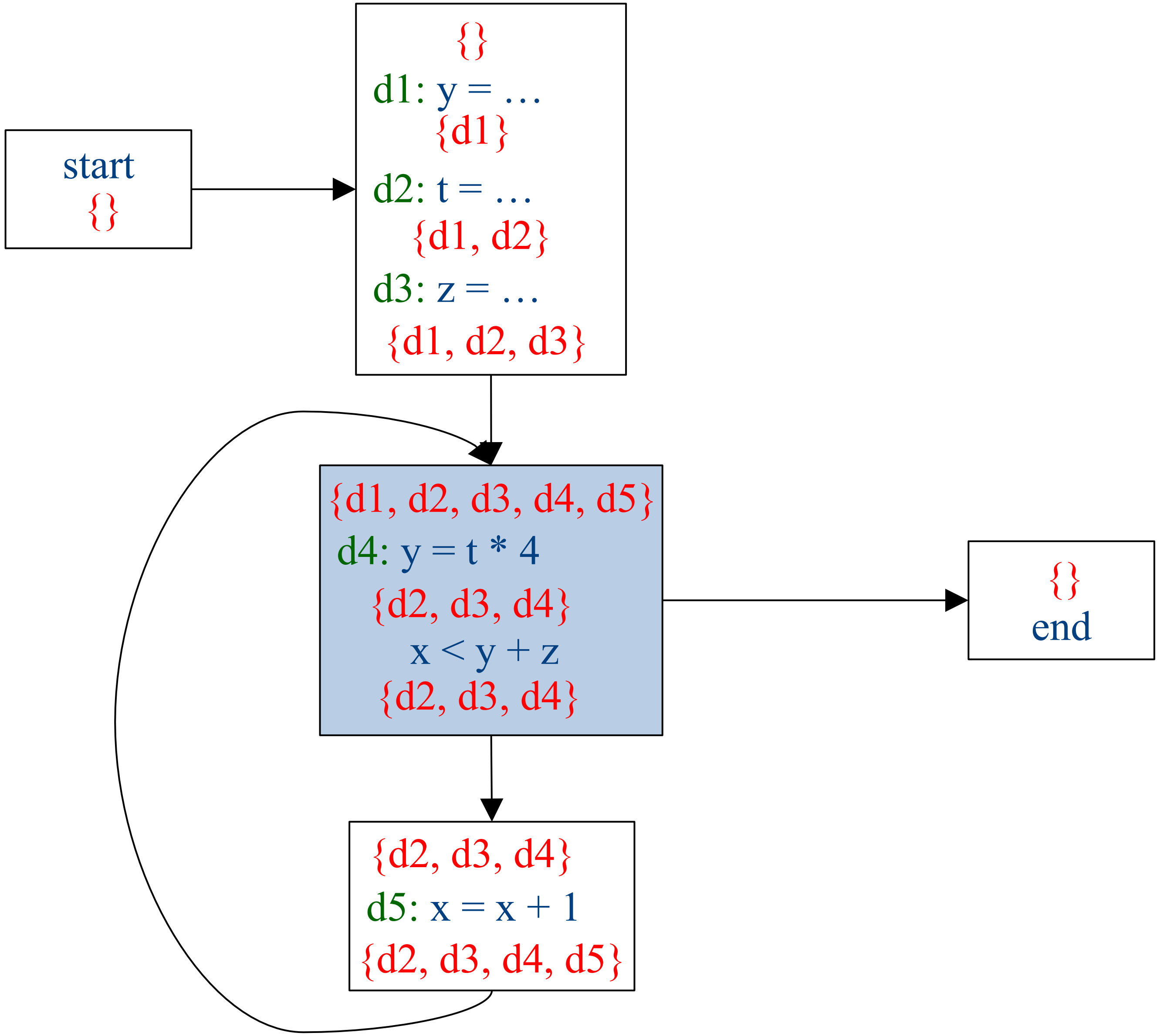
Iteration 4



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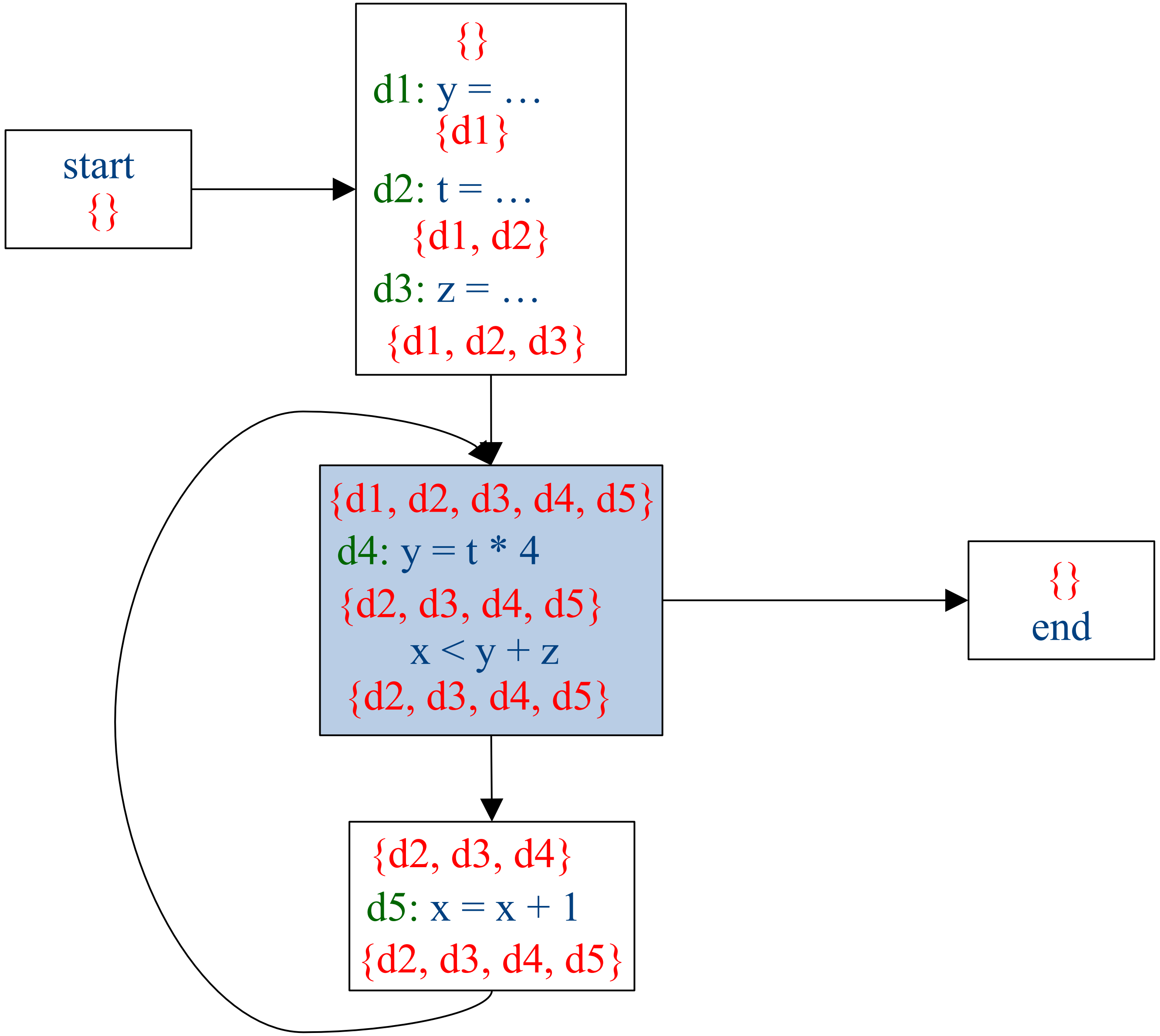
Iteration 4


start
{}
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{d1, d2}
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{d1, d2, d3}
{d1, d2, d3, d4, d5}
d4: y = t * 4
{d2, d3, d4, d5}
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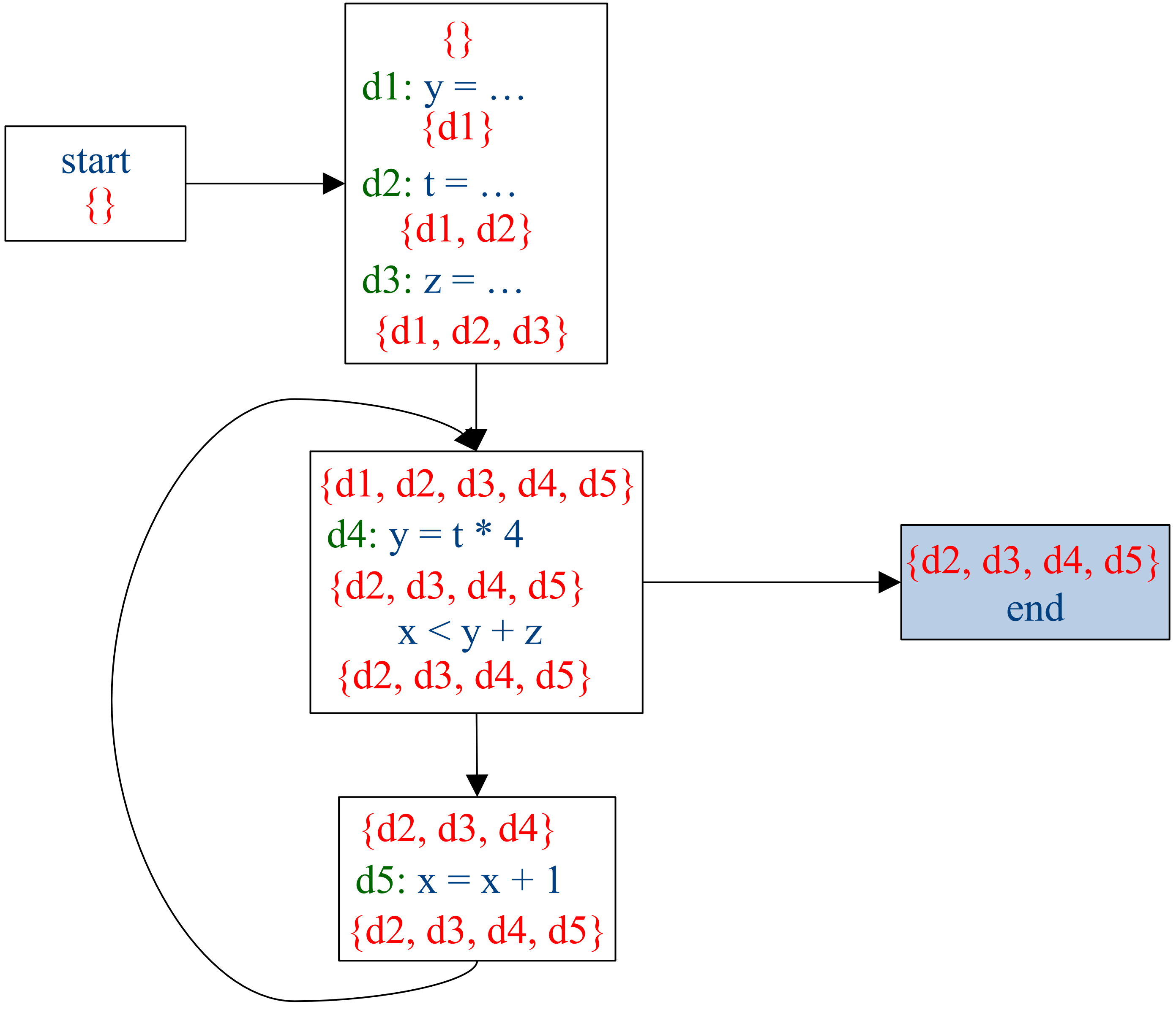
{d2, d3, d4, d5}
{}
end
{d2, d3, d4}
d5: x = x + 1
{d2, d3, d4, d5}

Iteration 5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{}
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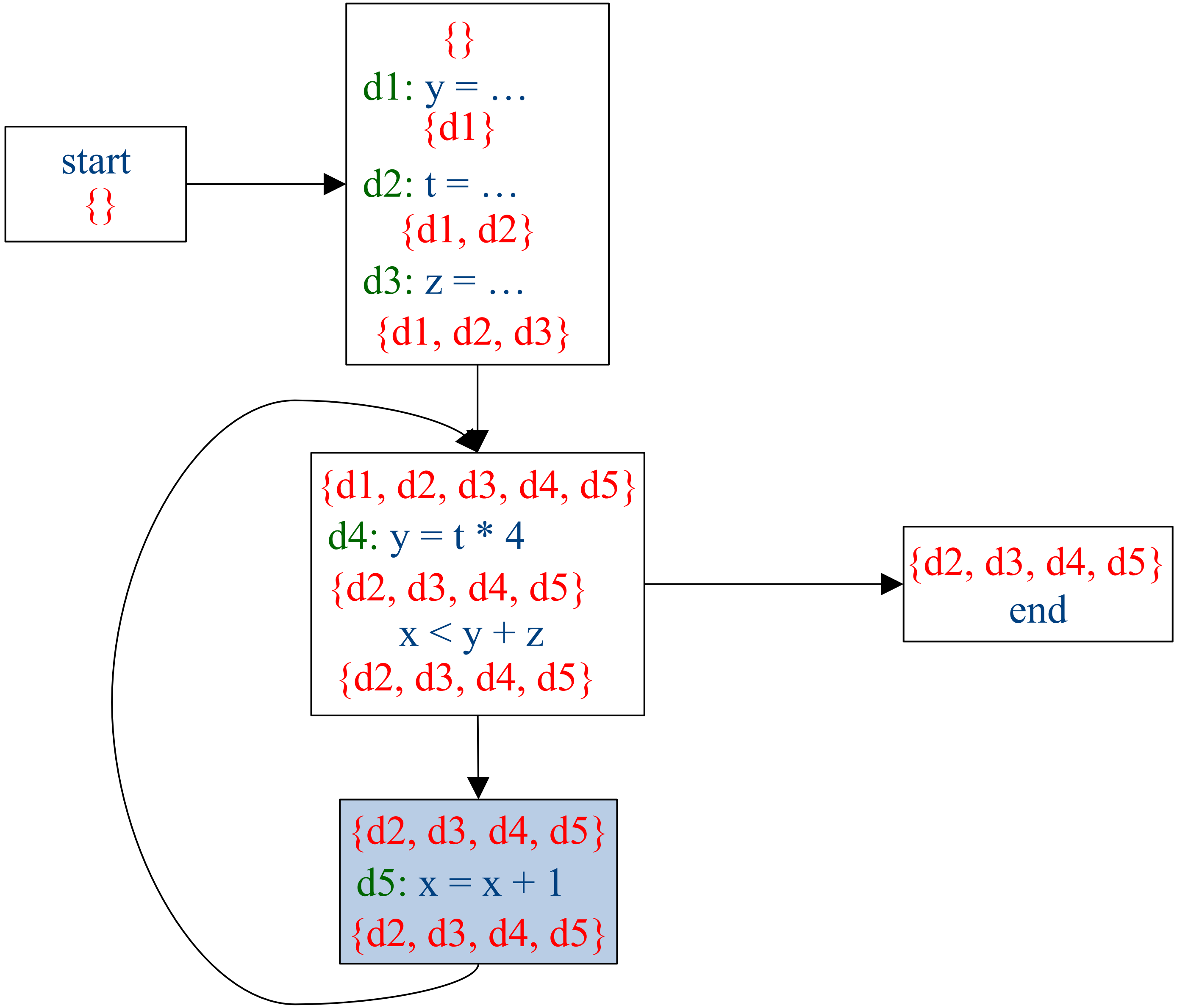
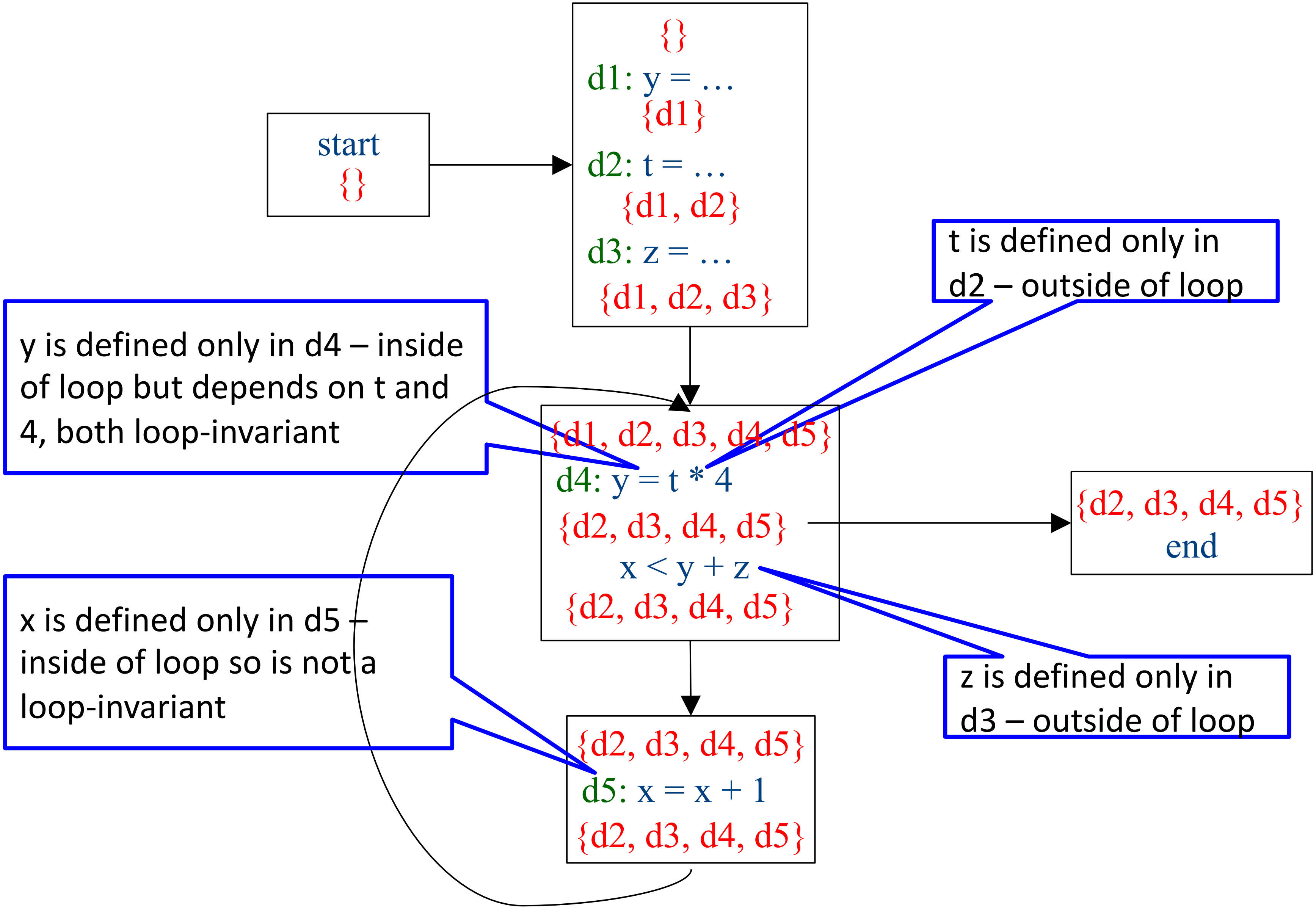
d1: y = …
{d1}
d2: t = …
{d1, d2}
d3: z = …
{d1, d2, d3}
{d1, d2, d3, d4, d5}
d4: y = t * 4
{d2, d3, d4, d5}
x < y + z
{d2, d3, d4, d5}
{d2, d3, d4, d5}
end
{d2, d3, d4}
d5: x = x + 1
{d2, d3, d4, d5}

Iteration 6



start
{}
{}
d1: y = …
{d1}
d2: t = …
{d1, d2}
d3: z = …
{d1, d2, d3}
{d1, d2, d3, d4, d5}
d4: y = t * 4
{d2, d3, d4, d5}
x < y + z
{d2, d3, d4, d5}
{d2, d3, d4, d5}
end
{d2, d3, d4, d5}
d5: x = x + 1
{d2, d3, d4, d5}

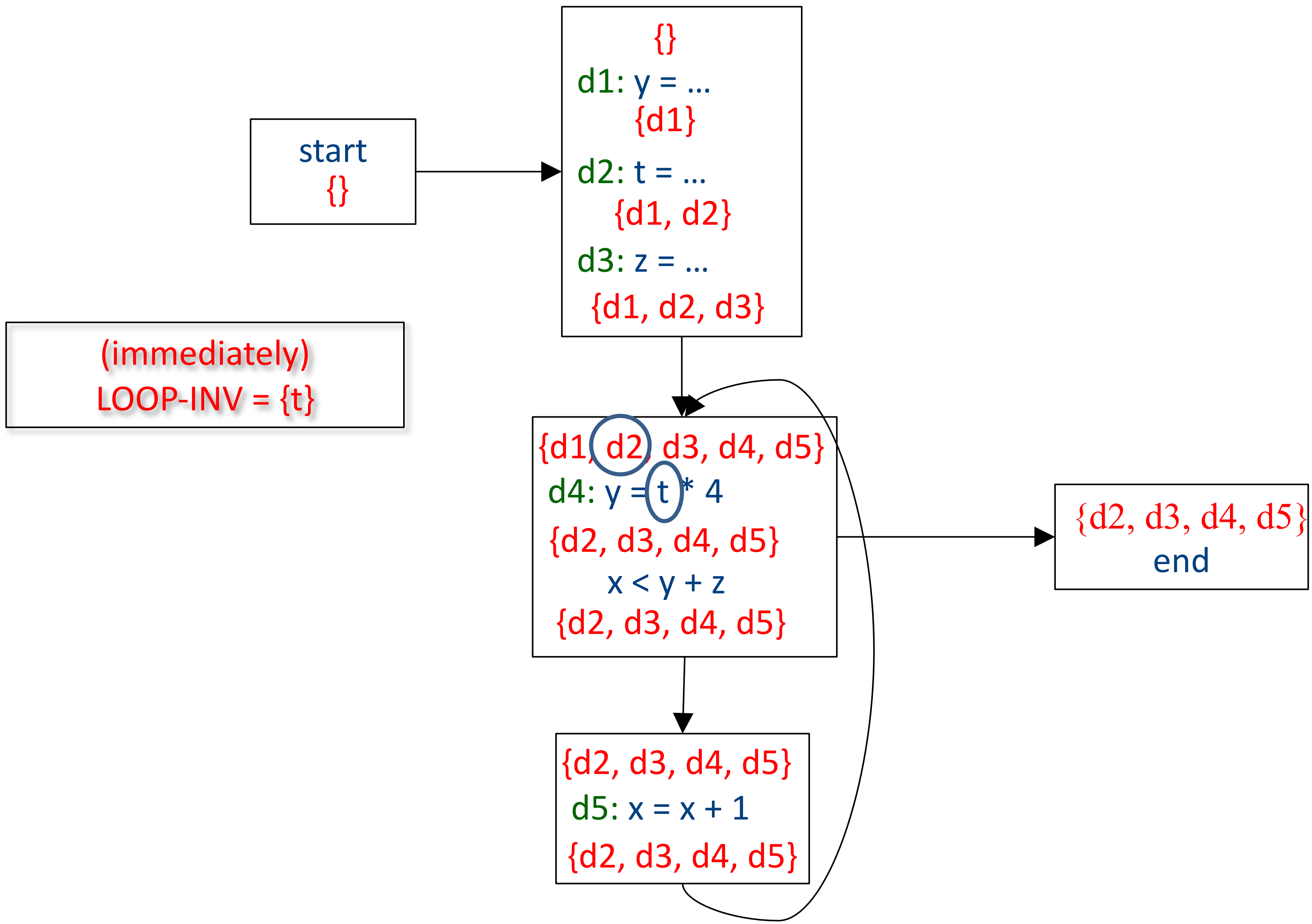
Which expressions are loop invariant?



Inferring loop-invariant expressions

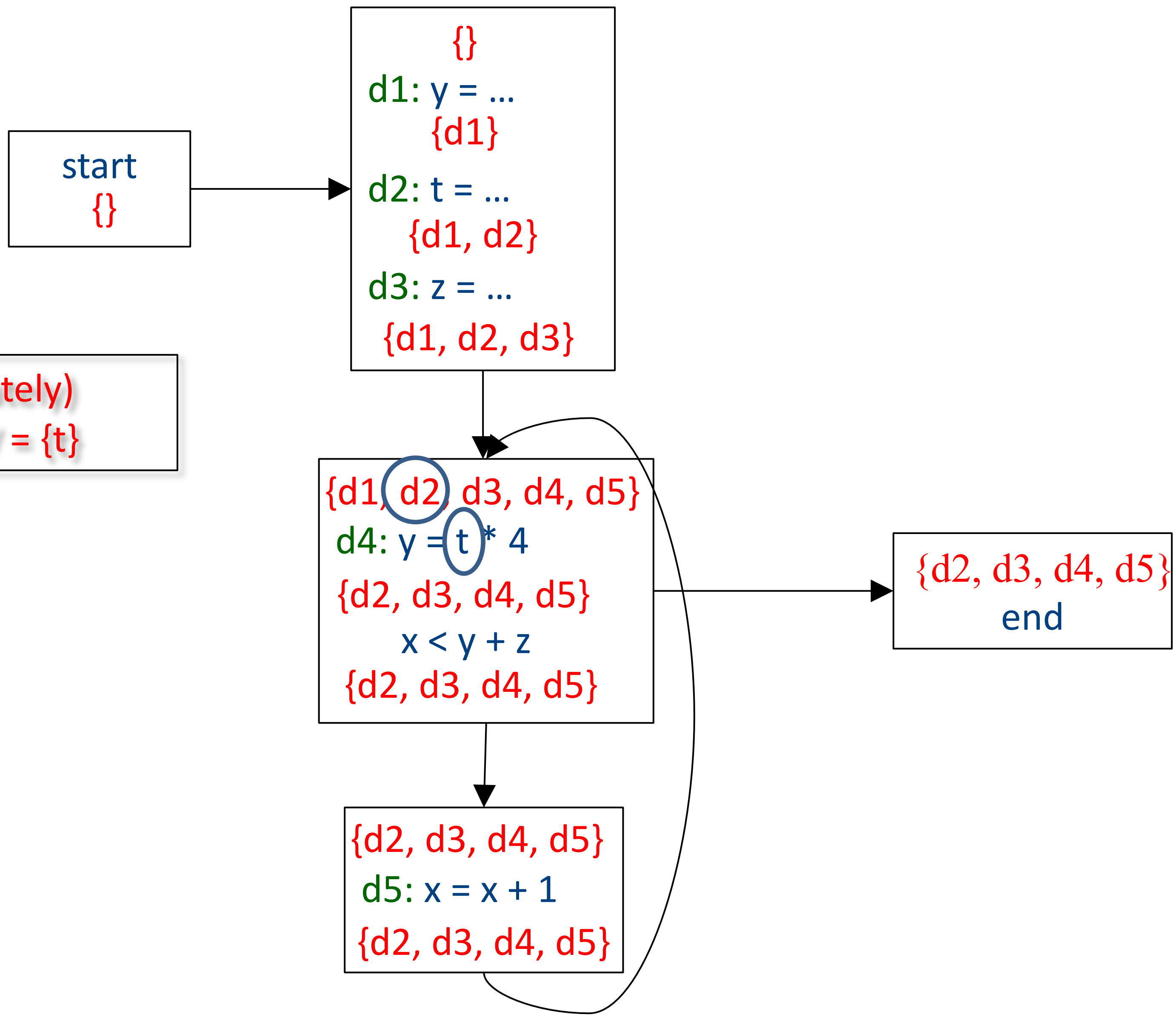
- For a statement *s* of the form t = a_1 op a_2
- A variable a_i is immediately loop-invariant if all reaching definitions IN[*s*]={d_1,…,d_k} for a_i are outside of the loop
- LOOP-INV = immediately loop-invariant variables and constants
  LOOP-INV = LOOP-INV $\cup$ {x | d: x = a_1 op a_2, d is in the loop, and both a_1 and a_2 are in LOOP-INV}
  - Iterate until fixed-point
- An expression is loop-invariant if all operands are loop-invariants

Computing LOOP-INV

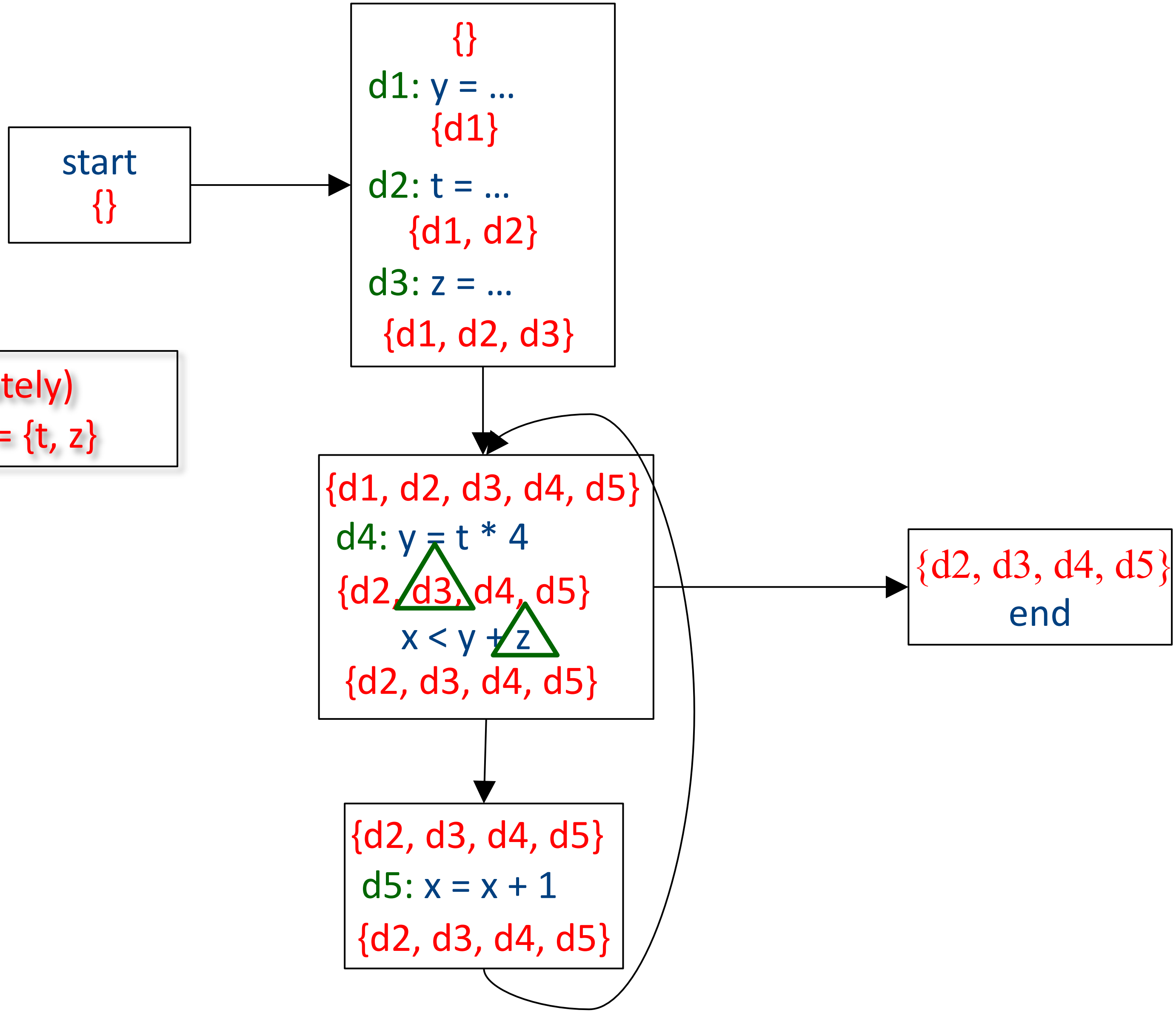


Computing LOOP-INV

(immediately)
LOOP-INV = {t}



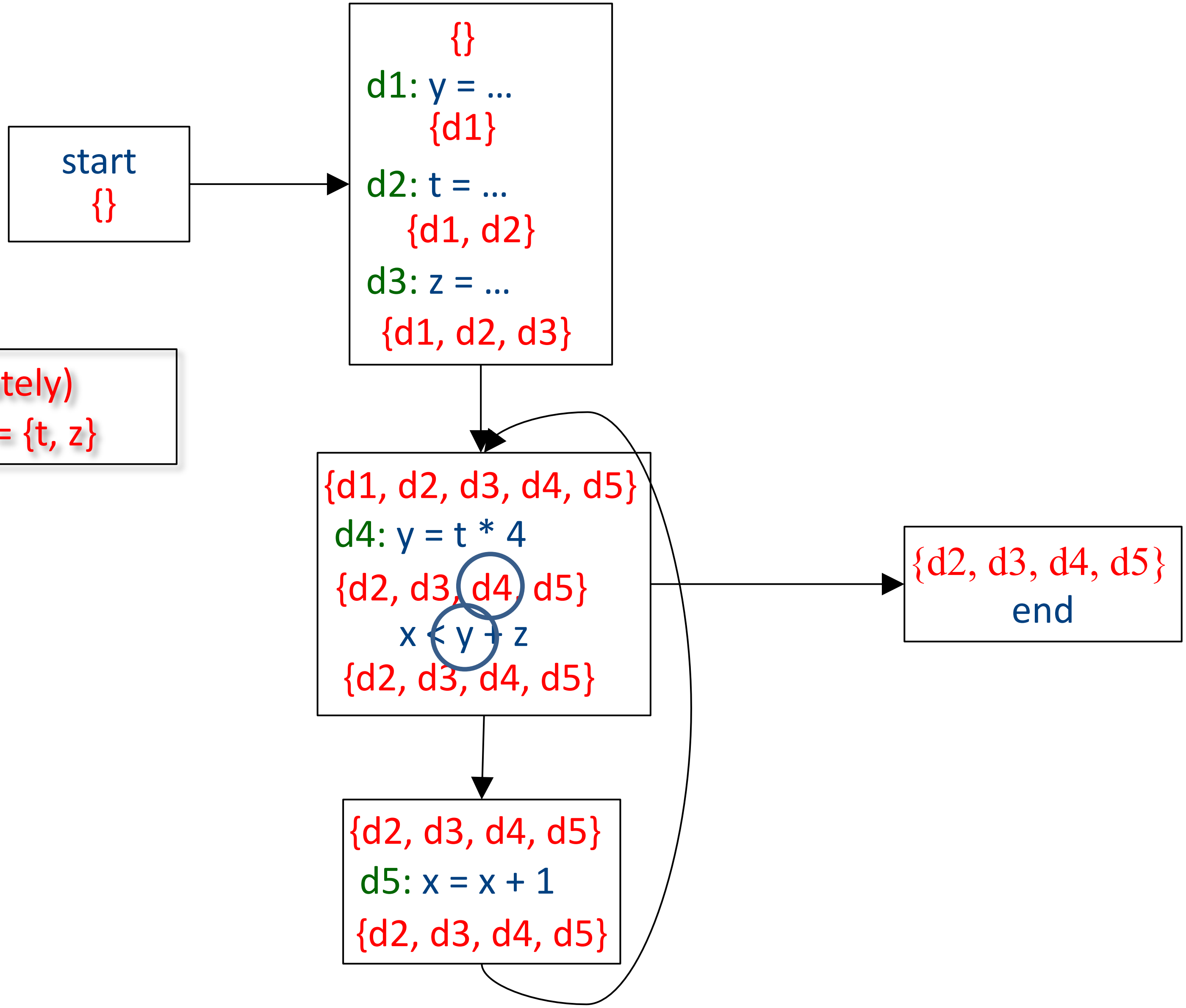
Computing LOOP-INV



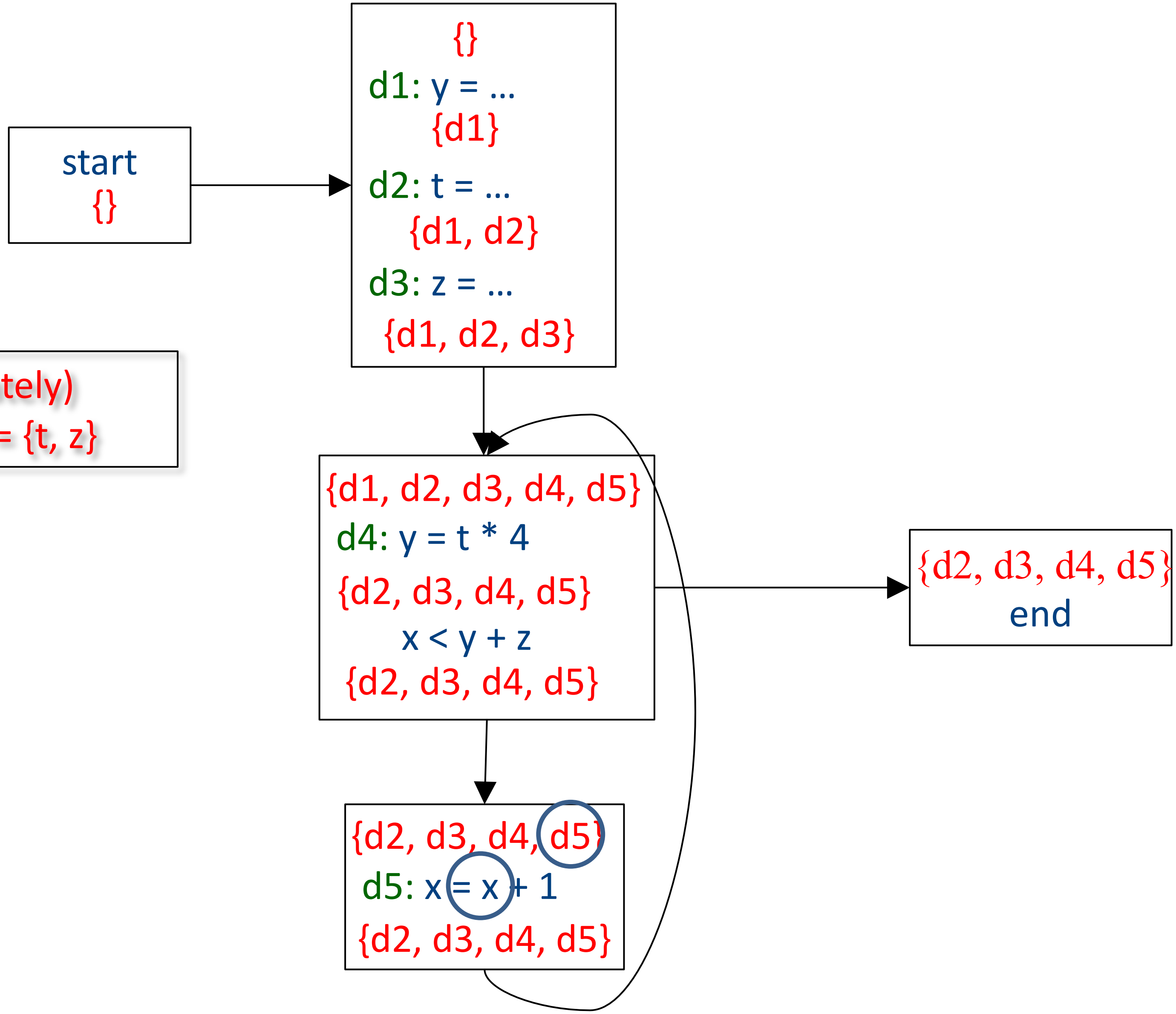
(immediately)
LOOP-INV = {t, z}

Computing LOOP-INV

(immediately)
LOOP-INV = {t, z}

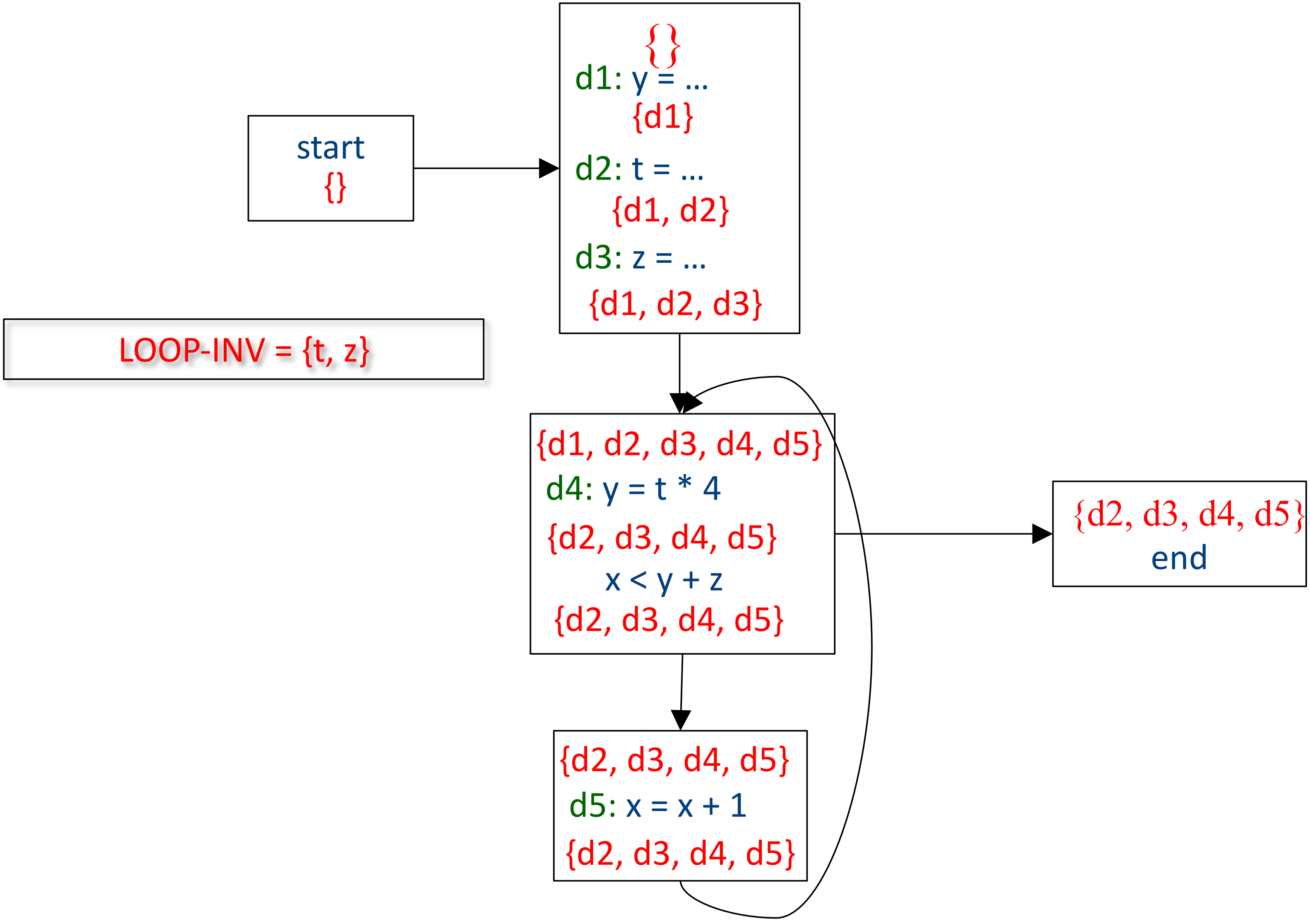


Computing LOOP-INV

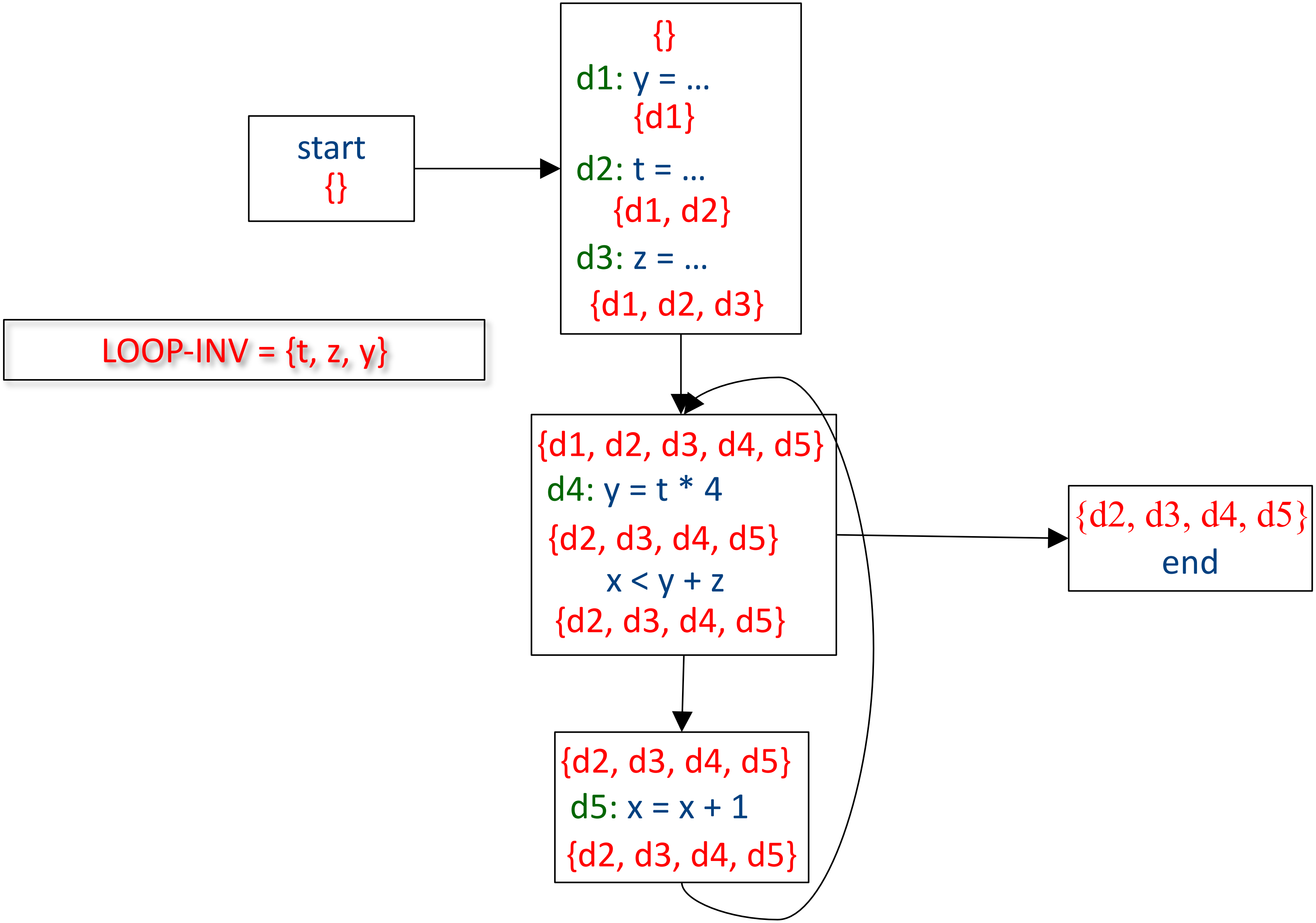


(immediately)
LOOP-INV = {t, z}

Computing LOOP-INV



Computing LOOP-INV



Induction variables

j is a linear function of the induction variable with multiplier 4

```
while (i < x) {
    j = a + 4 * i
    a[j] = j
    i = i + 1
}
```

i is incremented by a loop-invariant expression on each iteration – this is called an induction variable

Strength-reduction

Prepare initial value

```
j = a + 4 * i
while (i < x) {
    j = j + 4
    a[j] = j
    i = i + 1
}
```

Increment by multiplier