Compilation

Lecture 10a

Abstract Interpretation

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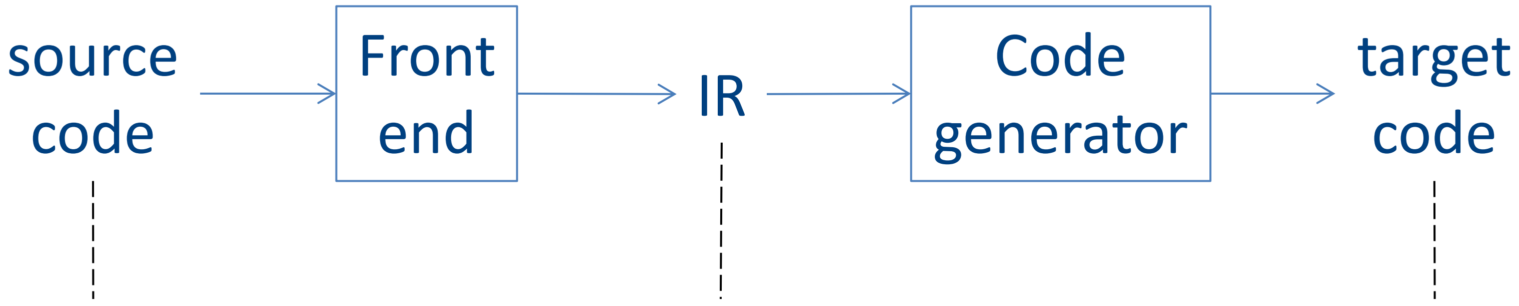
Optimization points



User
profile program
change algorithm

Compiler
intraprocedural IR
Interprocedural IR
IR optimizations

Compiler
register allocation
instruction selection
peephole transformations

now

IR Optimization

- Making code “better”

Overview of IR optimization

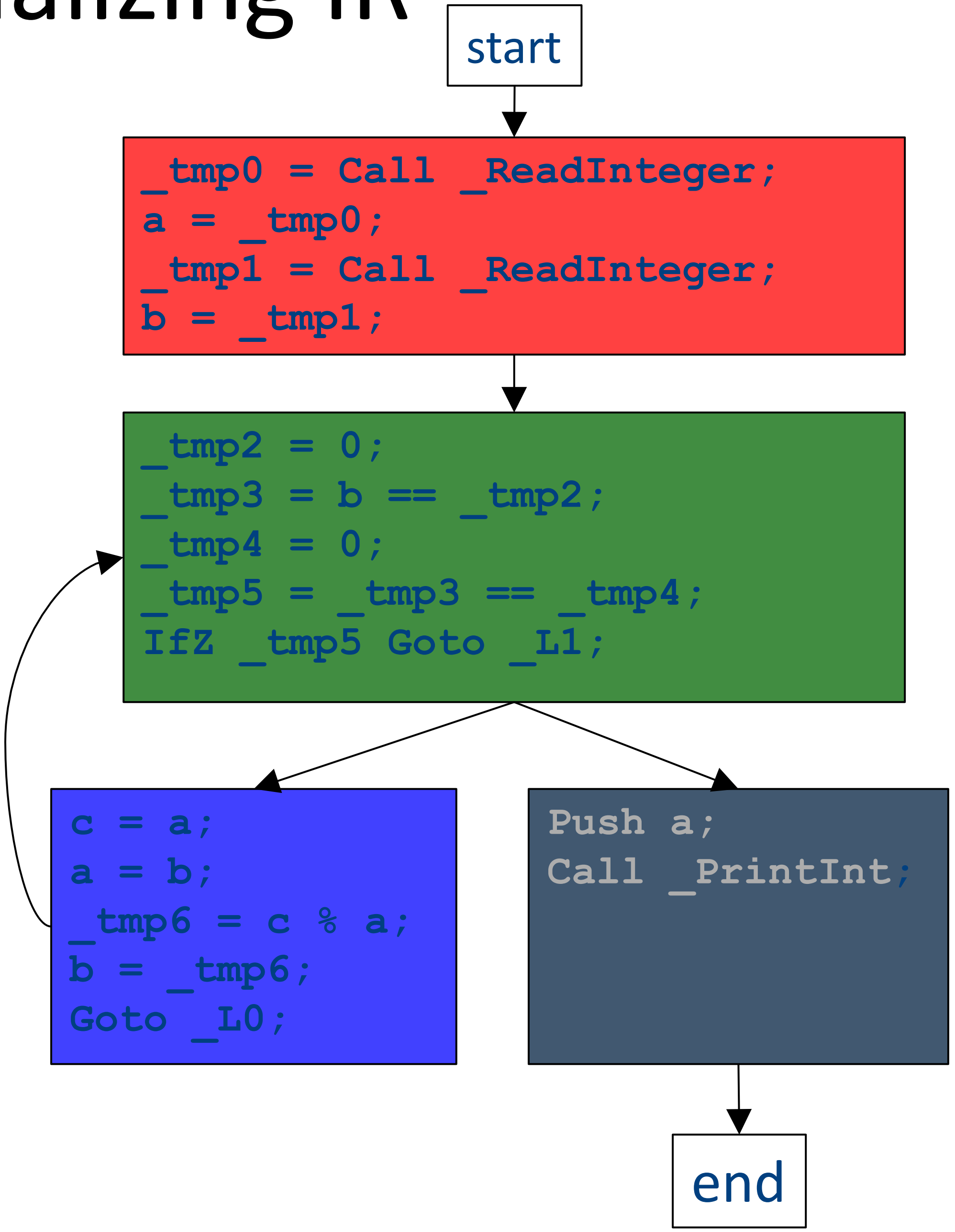
- Formalisms and Terminology
  - Control-flow graphs
  - Basic blocks
- Local optimizations
  - Speeding up small pieces of a procedure
- Global optimizations
  - Speeding up procedure as a whole
- The dataflow framework
  - Defining and implementing a wide class of optimizations

Program Analysis

- In order to optimize a program, the compiler has to be able to reason about the properties of that program
- An analysis is called **sound** if it never asserts an incorrect fact about a program
- All the analyses we will discuss in this class are sound
  - *(Why?)*

Visualizing IR

```
main:
    _tmp0 = Call _ReadInteger;
    a = _tmp0;
    _tmp1 = Call _ReadInteger;
    b = _tmp1;
_L0:
    _tmp2 = 0;
    _tmp3 = b == _tmp2;
    _tmp4 = 0;
    _tmp5 = _tmp3 == _tmp4;
    IfZ _tmp5 Goto _L1;
    c = a;
    a = b;
    _tmp6 = c % a;
    b = _tmp6;
    Goto _L0;
_L1:
    Push a;
    Call _PrintInt;
```



Common Subexpression Elimination

- If we have two variable assignments
  ```
  v1 = a op b
  …
  v2 = a op b
  ```
- and the values of v1, a, and b have not changed between the assignments, rewrite the code as
  ```
  v1 = a op b
  …
  v2 = v1
  ```
- Eliminates useless recalculation
- Paves the way for later optimizations

Common Subexpression Elimination

- If we have two variable assignments
  v1 = a op b   [or: v1 = a]
  …
  v2 = a op b   [or: v2 = a]
- and the values of v1, a, and b have not changed between the assignments, rewrite the code as
  v1 = a op b   [or: v1 = a]
  …
  v2 = v1
- Eliminates useless recalculation
- Paves the way for later optimizations

Copy Propagation

- If we have a variable assignment
  v1 = v2
  then as long as v1 and v2 are not reassigned, we can rewrite expressions of the form
  a = … v1 …
  as
  a = … v2 …
  provided that such a rewrite is legal

Dead Code Elimination

- An assignment to a variable v is called dead if the value of that assignment is never read anywhere
- Dead code elimination removes dead assignments from IR
- Determining whether an assignment is dead depends on what variable is being assigned to and when it's being assigned

Abstract Interpretation

- Theoretical foundations of program analysis
- Cousot and Cousot 1977
- Abstract meaning of programs
  - Executed at compile time

Another view of local optimization

- In local optimization, we want to reason about some property of the runtime behavior of the program
- Could we run the program and just watch what happens?
- **Idea:** Redefine the semantics of our programming language to give us information about our analysis

Assigning new semantics

- Example: Available Expressions
- Redefine the statement **a = b + c to** mean "**a now holds the value of b + c,** and any variable holding the value **a is** now invalid"
- Run the program assuming these new semantics
- Treat the optimizer as an interpreter for these new semantics

Join semilattices

- A join semilattice is a ordering defined on a set of elements
- Any two elements have some join that is the smallest element larger than both elements
- There is a unique bottom element, which is smaller than all other elements
- Intuitively:
  - The join of two elements represents combining information from two elements by an overapproximation
- The bottom element represents “no information yet” or “the least conservative possible answer”

Join semilattices and ordering

$$\begin{tikzcd}
 & \{a, b, c\} & \\
\{a, b\} \arrow[ur] & \{a, c\} \arrow[u] & \{b, c\} \arrow[ul] \\
\{a\} \arrow[u] \arrow[ur] & \{b\} \arrow[ul] \arrow[ur] & \{c\} \arrow[ul] \arrow[u] \\
 & \{\} \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Least precise

$$\updownarrow$$

Most precise

Formal definitions

- A join semilattice is a pair (V, $\sqcup$), where
- V is a domain of elements
- $\sqcup$ is a join operator that is
  - commutative: $x \sqcup y = y \sqcup x$
  - associative: $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
  - idempotent: $x \sqcup x = x$
- If $x \sqcup y = z$, we say that z is the join or (**l**east **u**pper **b**ound) of x and y
- Every join semilattice has a bottom element denoted $\perp$ such that $\perp \sqcup x = x$ for all x

Join semilattices and orderings

- Every join semilattice $(V, \sqcup)$ induces an ordering relationship $\sqsubseteq$ over its elements
- Define $x \sqsubseteq y$ iff $x \sqcup y = y$
- Need to prove
  - Reflexivity: $x \sqsubseteq x$
  - Antisymmetry: If $x \sqsubseteq y$ and $y \sqsubseteq x$, then $x = y$
  - Transitivity: If $x \sqsubseteq y$ and $y \sqsubseteq z$, then $x \sqsubseteq z$

A general framework

- A global analysis is a tuple $(\mathtt{D}, \mathtt{V}, \sqsubseteq, \mathtt{F}, \mathtt{I})$, where
  - $\mathtt{D}$ is a direction (forward or backward)
    - The order to visit statements within a basic block, not the order in which to visit the basic blocks
  - $\mathtt{V}$ is a set of values
  - $\sqcup$ is a join operator over those values
  - $\mathtt{F}$ is a set of transfer functions $f : \mathbf{V} \rightarrow \mathbf{V}$
  - $\mathtt{I}$ is an initial value
- The only difference from local analysis is the introduction of the join operator

Running global analyses

- Assume that $(\mathbb{D}, \mathbb{V}, \sqcup, \mathbb{F}, \mathbb{I})$ is a forward analysis
- Set OUT[**s**] = $\bot$ for all statements **s**
- Set OUT[**entry**] = $\mathbb{I}$
- Repeat until no values change:
  - For each statement **s** with predecessors $\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n}$:
    - Set IN[**s**] = OUT[$\mathbf{p_1}$] $\sqcup$ OUT[$\mathbf{p_2}$] $\sqcup$ … $\sqcup$ OUT[$\mathbf{p_n}$]
    - Set OUT[**s**] = f_s (IN[**s**])
- The order of this iteration does not matter
  - This is sometimes called chaotic iteration

Global constant propagation

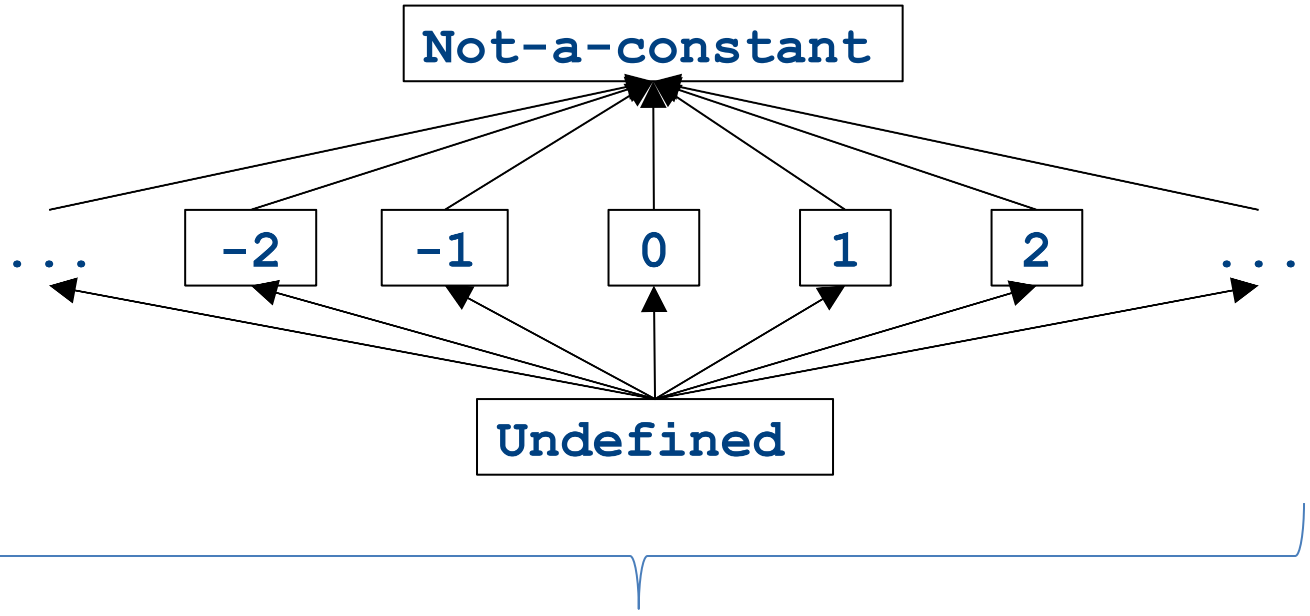
- Constant propagation is an optimization that replaces each variable that is known to be a constant value with that constant
- An elegant example of the dataflow framework

Defining a join operator

- The join of any two different constants is **Not-a-Constant**
  - (If the variable might have two different values on entry to a statement, it cannot be a constant)
- The join of **Not a Constant** and any other value is **Not-a-Constant**
  - (If on some path the value is known not to be a constant, then on entry to a statement its value can't possibly be a constant)
- The join of **Undefined** and any other value is that other value
  - (If **x** has no value on some path and does have a value on some other path, we can just pretend it always had the assigned value)

A semilattice for constant propagation

- One possible semilattice for this analysis is shown here (for each variable):



The lattice is infinitely wide

A semilattice for constant propagation

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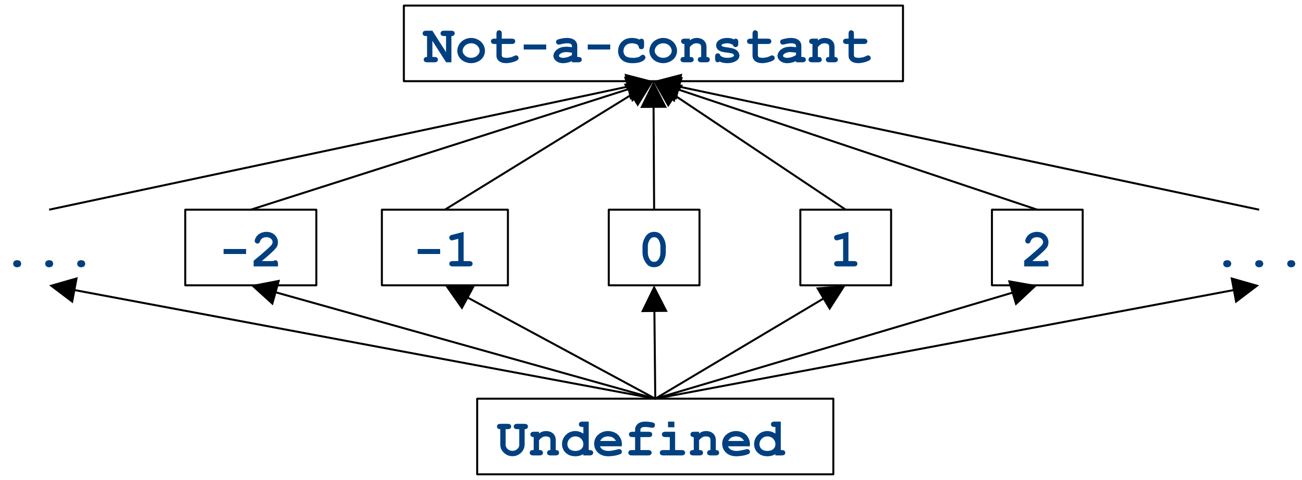
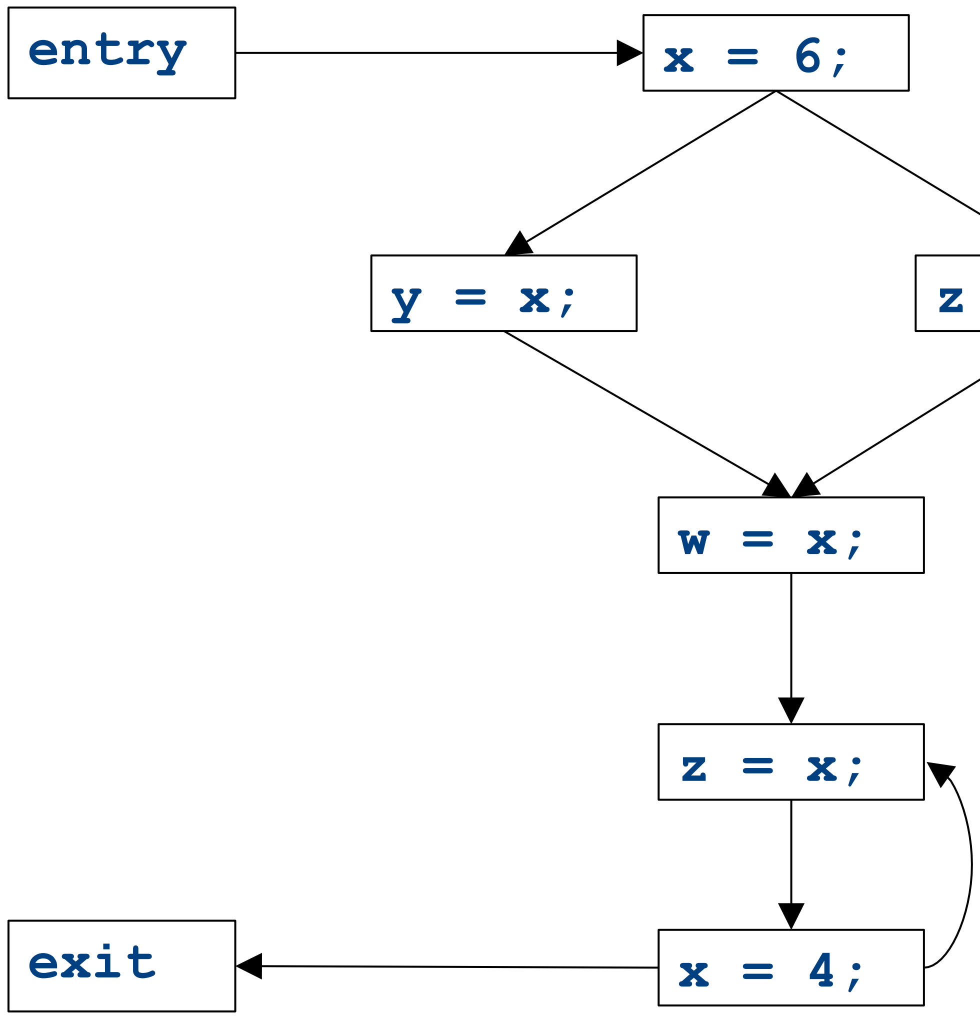
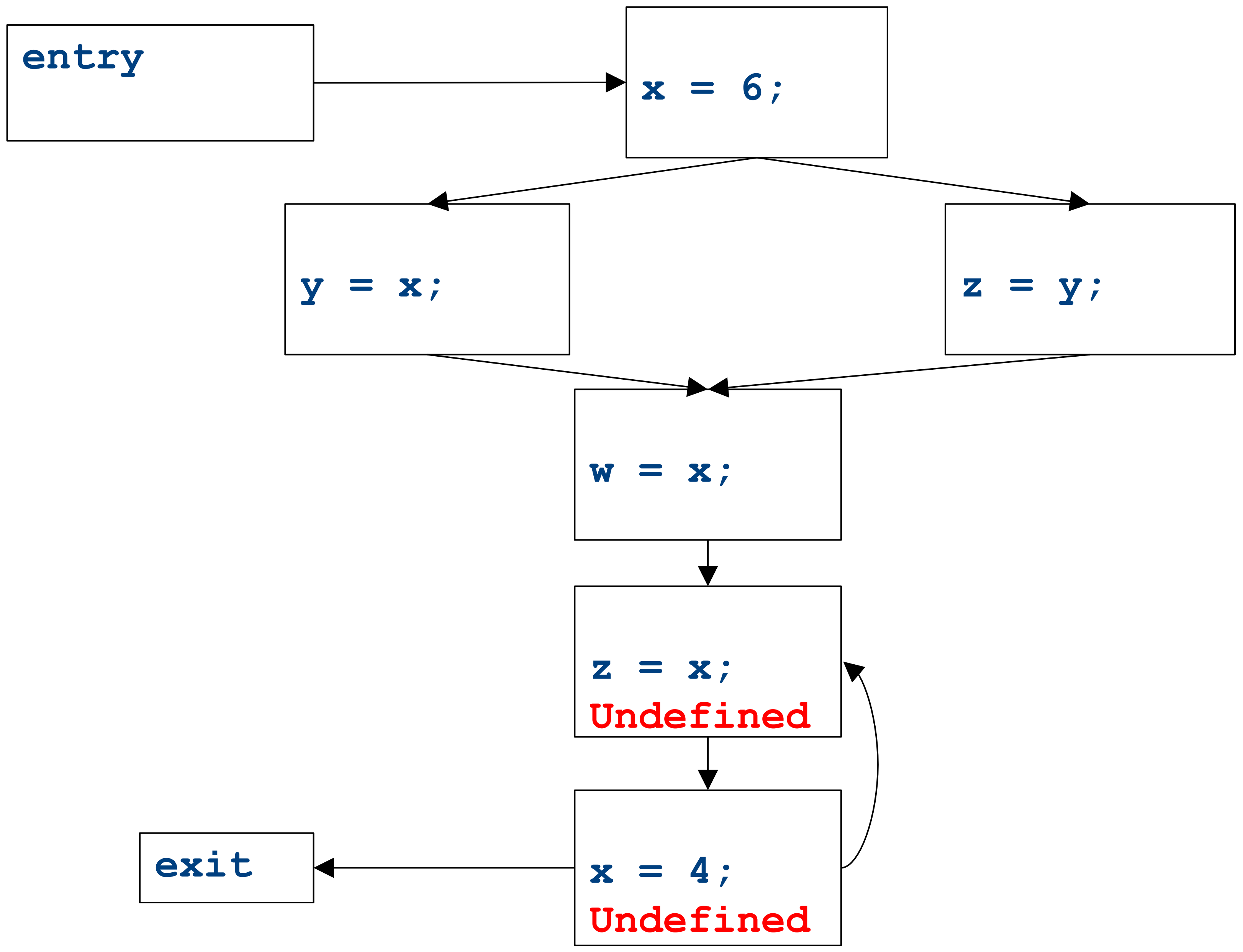


- Note:
  - The join of any two different constants is **Not-a-Constant**
  - The join of **Not a Constant** and any other value is **Not-a-Constant**
  - The join of **Undefined** and any other value is that other value

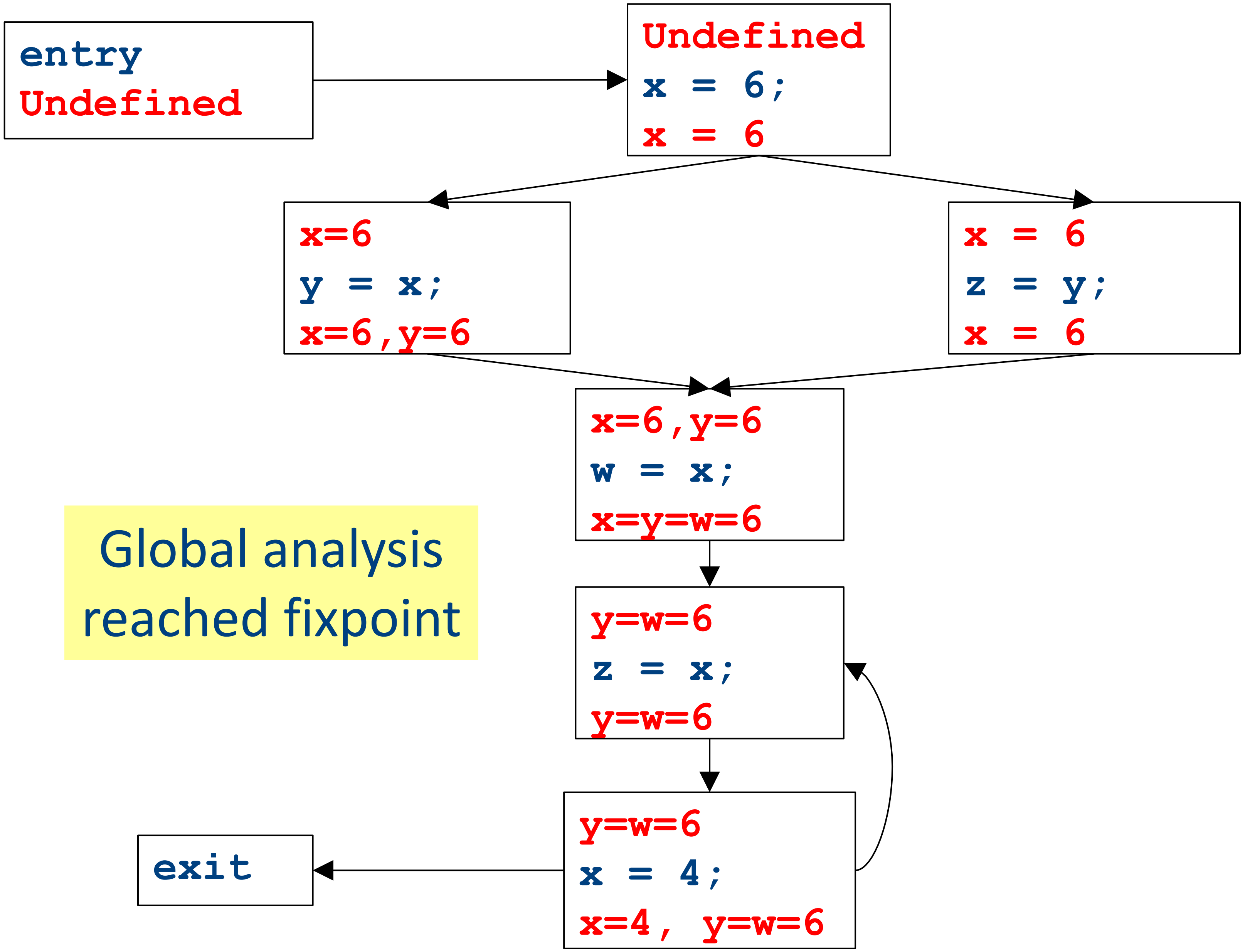
Global constant propagation



Global constant propagation

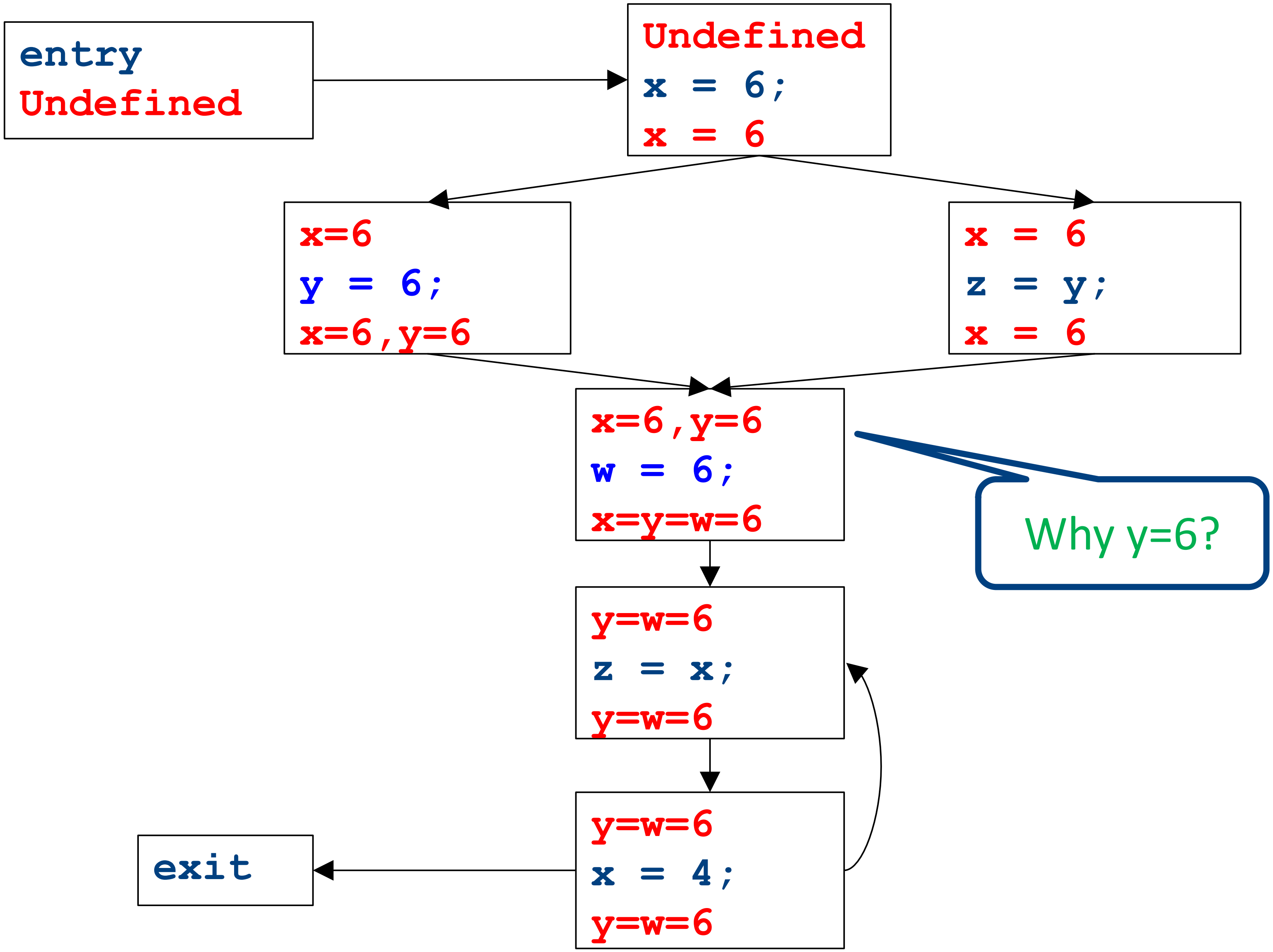


Global constant propagation



Global analysis reached fixpoint

Global constant propagation



Dataflow for constant propagation

- Direction: **Forward**
- Semilattice: Vars→ {Undefined, 0, 1, -1, 2, -2, …, Not-a-Constant}
  - Join mapping for variables point-wise
    $\{x \mapsto 1, y \mapsto 1, z \mapsto 1\} \sqcup \{x \mapsto 1, y \mapsto 2, z \mapsto \text{Not-a-Constant}\} =$
    $\{x \mapsto 1, y \mapsto \text{Not-a-Constant}, z \mapsto \text{Not-a-Constant}\}$
- Transfer functions:
  - $f_{\mathtt{x=k}}(V) = V|_{x \mapsto k}$ *(update V by mapping x to k)*
  - $f_{\mathtt{x=a+b}}(V) = V|_{x \mapsto \text{Not-a-Constant}}$ *(assign Not-a-Constant)*
- Initial value: **x is Undefined**
  - (When might we use some other value?)

Proving termination

- Our algorithm for running these analyses continuously loops until no changes are detected
- Given this, how do we know the analyses will eventually terminate?
  - In general, **we don‘t**

Terminates?

Liveness Analysis

- A variable is live at a point in a program if later in the program its value will be read before it is written to again

Join semilattice definition

- A join semilattice is a pair $(V, \sqcup)$, where
- V is a domain of elements
- $\sqcup$ is a join operator that is
  - commutative: $x \sqcup y = y \sqcup x$
  - associative: $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
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- If $x \sqcup y = z$, we say that z is the join
  or (**L**east **U**pper **B**ound) of x and y
- Every join semilattice has a bottom element
  denoted $\perp$ such that $\perp \sqcup x = x$ for all x

Partial ordering induced by join

- Every join semilattice $(V, \sqcup)$ induces an ordering relationship $\sqsubseteq$ over its elements
- Define $x \sqsubseteq y$ iff x ◀◀ y = y
- Need to prove
  - Reflexivity: $x \sqsubseteq x$
  - Antisymmetry: If $x \sqsubseteq y$ and $y \sqsubseteq x$, then $x = y$
  - Transitivity: If $x \sqsubseteq y$ and $y \sqsubseteq z$, then $x \sqsubseteq z$

A join semilattice for liveness

- Sets of live variables and the set union operation
- Idempotent:
  - $x \cup x = x$
- Commutative:
  - $x \cup y = y \cup x$
- Associative:
  - $(x \cup y) \cup z = x \cup (y \cup z)$
- Bottom element:
  - The empty set: $\emptyset \cup x = x$
- Ordering over elements = subset relation

Join semilattice example for liveness

$$\begin{tikzcd}
 & \boxed{\{a,\ b,\ c\}} & \\
\boxed{\{a,\ b\}} \arrow[ur] & \boxed{\{a,\ c\}} \arrow[u] & \boxed{\{b,\ c\}} \arrow[ul] \\
\boxed{\{a\}} \arrow[u] \arrow[ur] & \boxed{\{b\}} \arrow[ul] \arrow[ur] & \boxed{\{c\}} \arrow[ul] \arrow[u] \\
 & \boxed{\{\}} \arrow[ul] \arrow[u] \arrow[ur] & \text{Bottom element} \arrow[l, dash]
\end{tikzcd}$$

Dataflow framework

- A global analysis is a tuple $(\mathtt{D}, \mathtt{V}, \sqcup, \mathtt{F}, \mathtt{I})$, where
  - $\mathtt{D}$ is a direction (forward or backward)
    - The order to visit statements within a basic block, **NOT** the order in which to visit the basic blocks
  - $\mathtt{V}$ is a set of values (sometimes called domain)
  - $\sqcup$ is a join operator over those values
  - $\mathtt{F}$ is a set of transfer functions f_s : **V → V** (for every statement s)
  - $\mathtt{I}$ is an initial value

Running global analyses

- Assume that $(\mathbb{D}, \mathbb{V}, \sqcup, \mathbb{F}, \mathbb{I})$ is a forward analysis
- For every statement s maintain values before - IN[s] - and after - OUT[s]
- Set OUT[**s**] = $\perp$ for all statements **s**
- Set OUT[**entry**] = $\mathbb{I}$
- Repeat until no values change:
  - For each statement **s** with predecessors PRED[s]={$\mathbf{p}_1$, $\mathbf{p}_2$, … , $\mathbf{p}_n$}
    - Set IN[**s**] = OUT[$\mathbf{p}_1$] $\sqcup$ OUT[$\mathbf{p}_2$] $\sqcup$ … $\sqcup$ OUT[$\mathbf{p}_n$]
    - Set OUT[**s**] = f_s(IN[**s**])
- The order of this iteration does not matter
  - Chaotic iteration

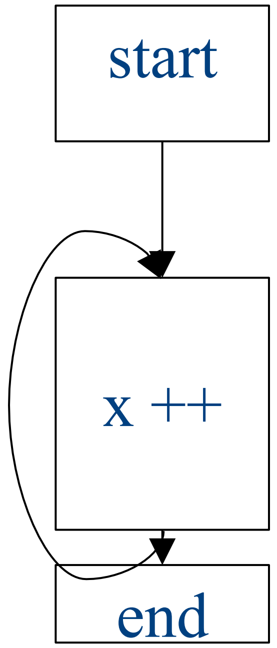
Proving termination

- Our algorithm for running these analyses continuously loops until no changes are detected
- Problem: how do we know the analyses will eventually terminate?

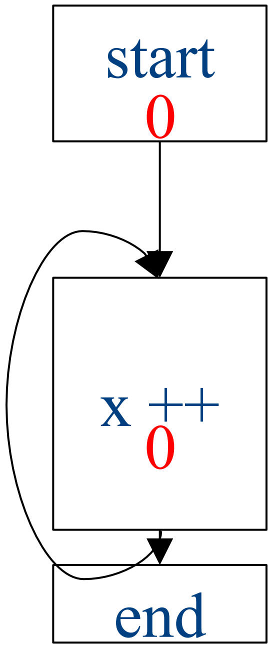
A non-terminating analysis

- The following analysis will loop infinitely on any CFG containing a loop:
- Direction: Forward
- Domain: $\mathbb{N}$
- Join operator: **max**
- Transfer function: $f(n) = n + 1$
- Initial value: 0

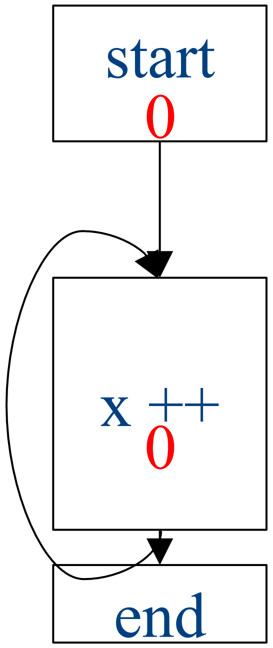
A non-terminating analysis



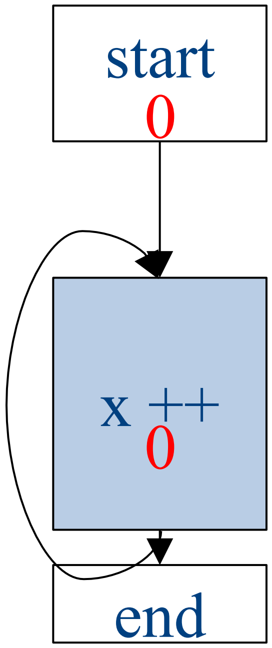
Initialization



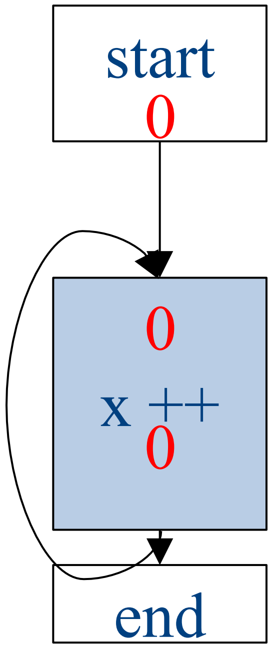
Fixed-point iteration



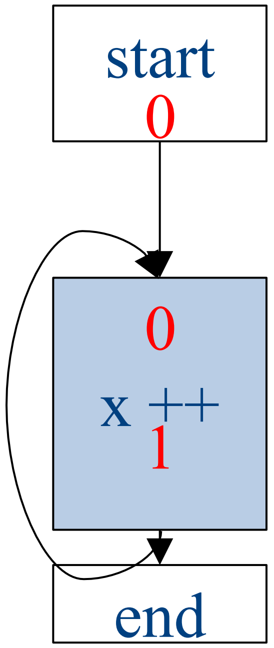
Choose a block



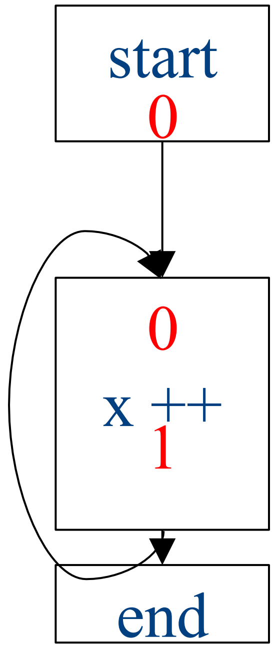
Iteration 1



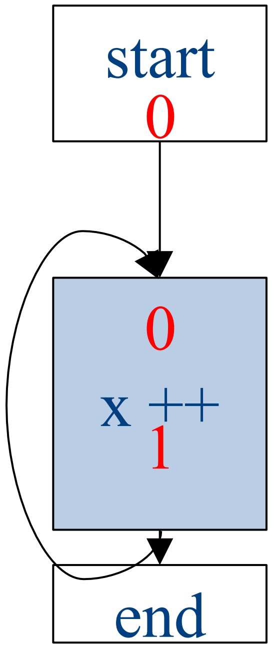
Iteration 1



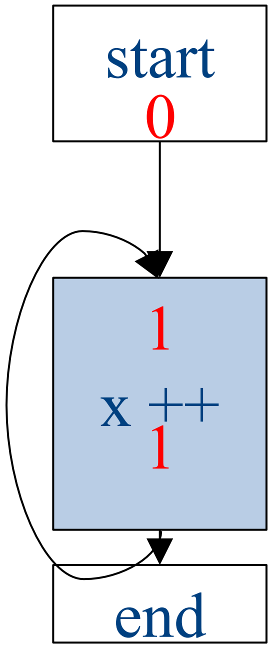
Choose a block



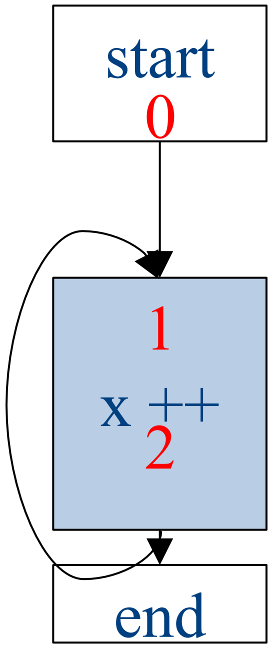
Iteration 2



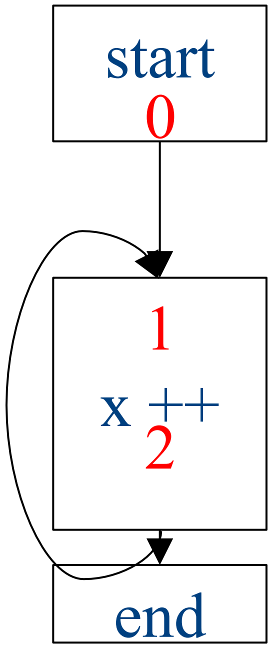
Iteration 2



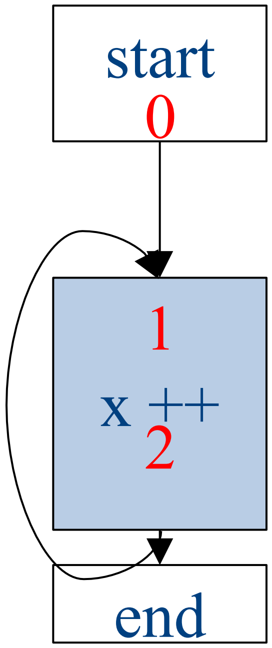
Iteration 2



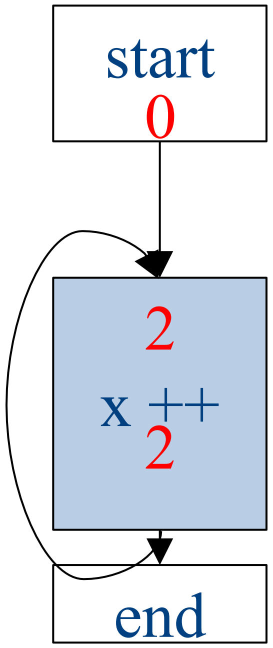
Choose a block



Iteration 3



Iteration 3



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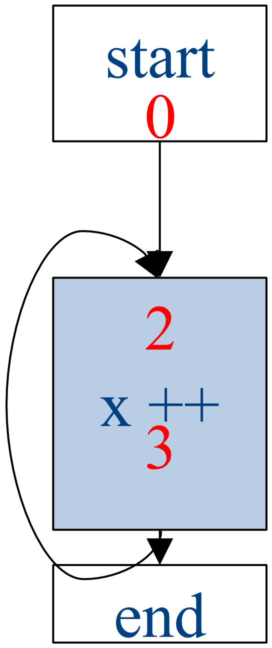
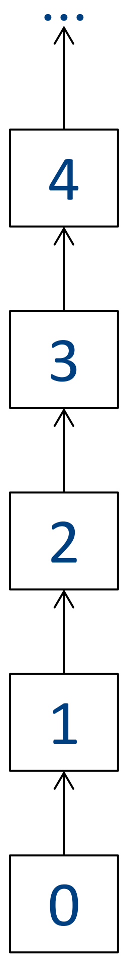


Why doesn't this terminate?

- Values can increase without bound
- Note that "increase" refers to the lattice ordering, not the ordering on the natural numbers
- The height of a semilattice is the length of the longest increasing sequence in that semilattice
- The dataflow framework is not guaranteed to terminate for semilattices of infinite height
- Note that a semilattice can be infinitely large but have finite height
  - e.g. constant propagation



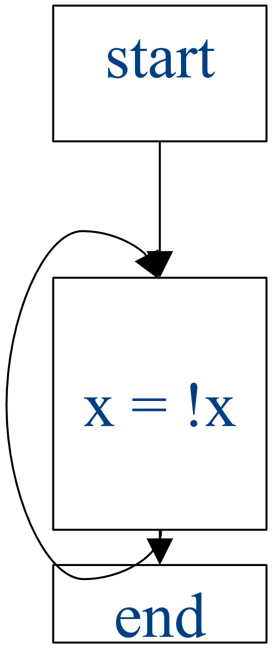
Height of a lattice

- An increasing chain is a sequence of elements $\perp \sqsubseteq a_1 \sqsubseteq a_2 \sqsubseteq \ldots \sqsubseteq a_k$
  - The length of such a chain is k
- The height of a lattice is the length of the maximal increasing chain
- For liveness with *n* program variables:
  - $\{\} \subseteq \{v_1\} \subseteq \{v_1,v_2\} \subseteq \ldots \subseteq \{v_1,\ldots,v_n\}$
- For available expressions it is the number of expressions of the form a=b op c
  - For *n* program variables and *m* operator types:mn^3

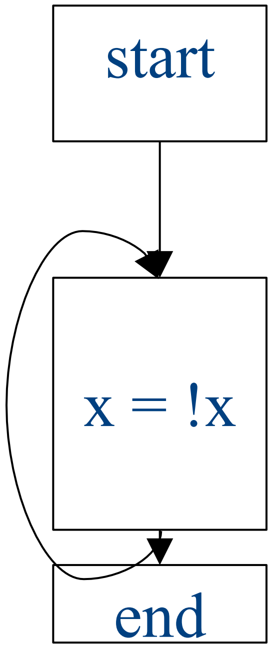
Another non-terminating analysis

- This analysis works on a finite-height semilattice, but will not terminate on certain CFGs:
- Direction: Forward
- Domain: Boolean values **`true`** and **`false`**
- Join operator: Logical OR
- Transfer function: Logical NOT
- Initial value: **`false`**

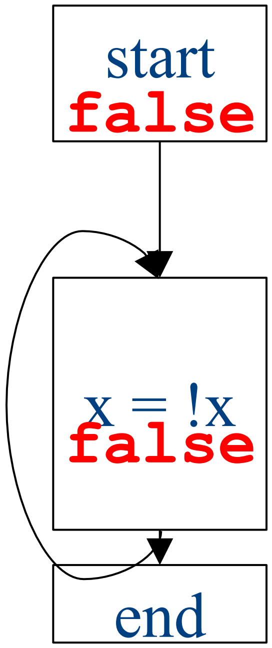
A non-terminating analysis



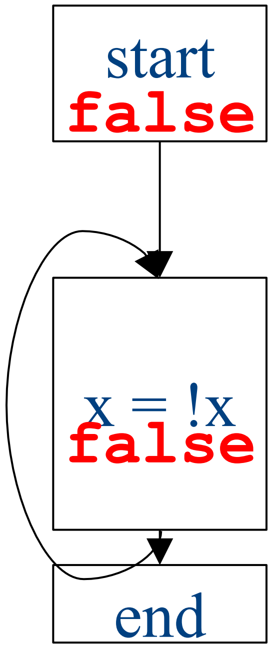
A non-terminating analysis



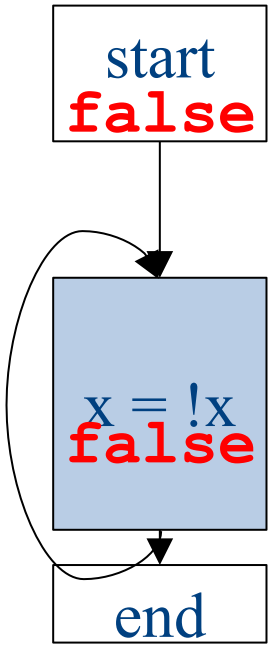
Initialization



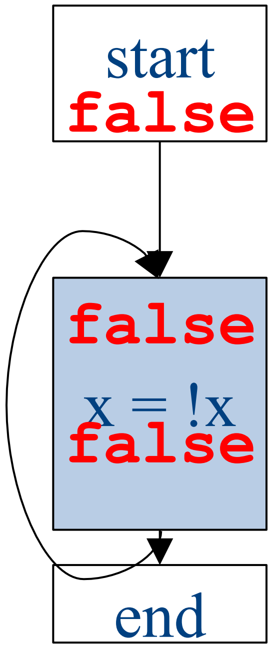
Fixed-point iteration



Choose a block



Iteration 1



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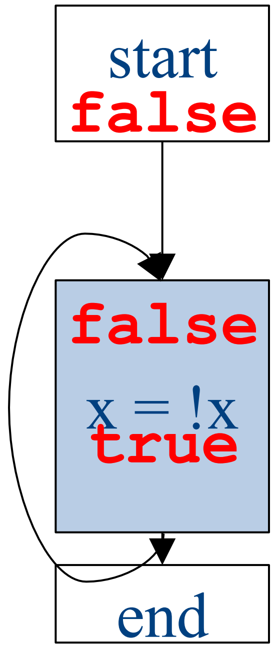
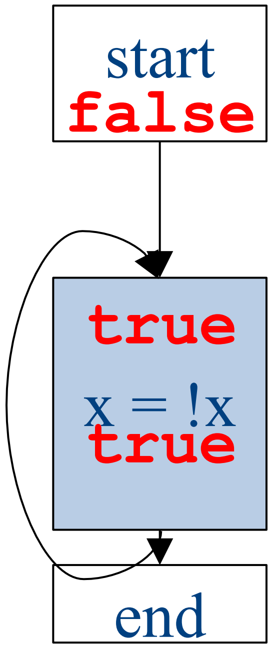
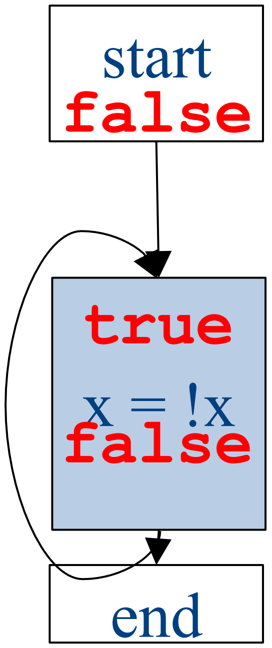


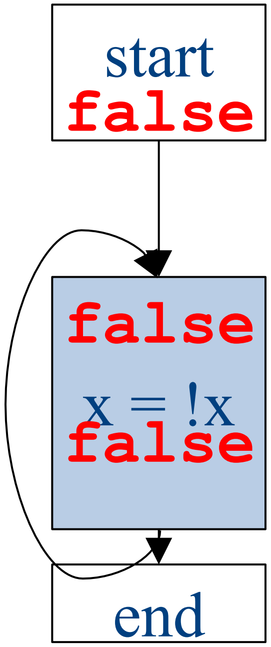
Iteration 2



Iteration 2



Iteration 3



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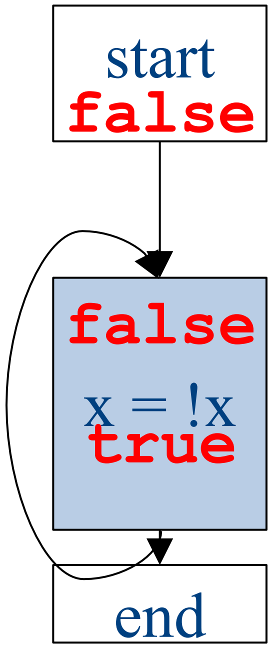
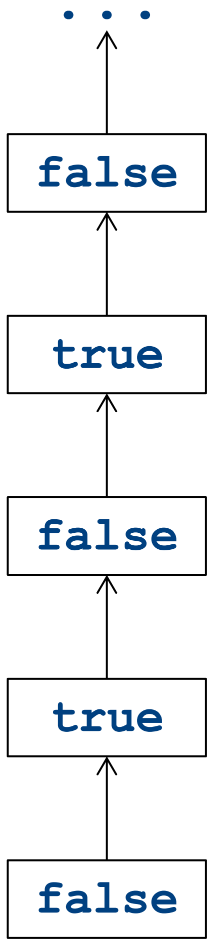


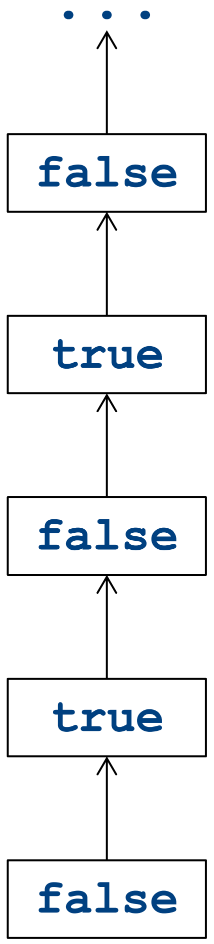
Why doesn't it terminate?

- Values can loop indefinitely
- Intuitively, the join operator keeps pulling values up
- If the transfer function can keep pushing values back down again, then the values might cycle forever



Why doesn't it terminate?

- Values can loop indefinitely
- Intuitively, the join operator keeps pulling values up
- If the transfer function can keep pushing values back down again, then the values might cycle forever
- How can we fix this?



Monotone transfer functions

- A transfer function *f* is monotone iff
  if $x \sqsubseteq y$, then $f(x) \sqsubseteq f(y)$
- Intuitively, if you know less information about a program point, you can't “gain back” more information about that program point
- Many transfer functions are monotone, including those for liveness and constant propagation
- Note: Monotonicity does **not** mean that $x \sqsubseteq f(x)$
  - (This is a different property called extensivity)

Liveness and monotonicity

- A transfer function f is monotone iff

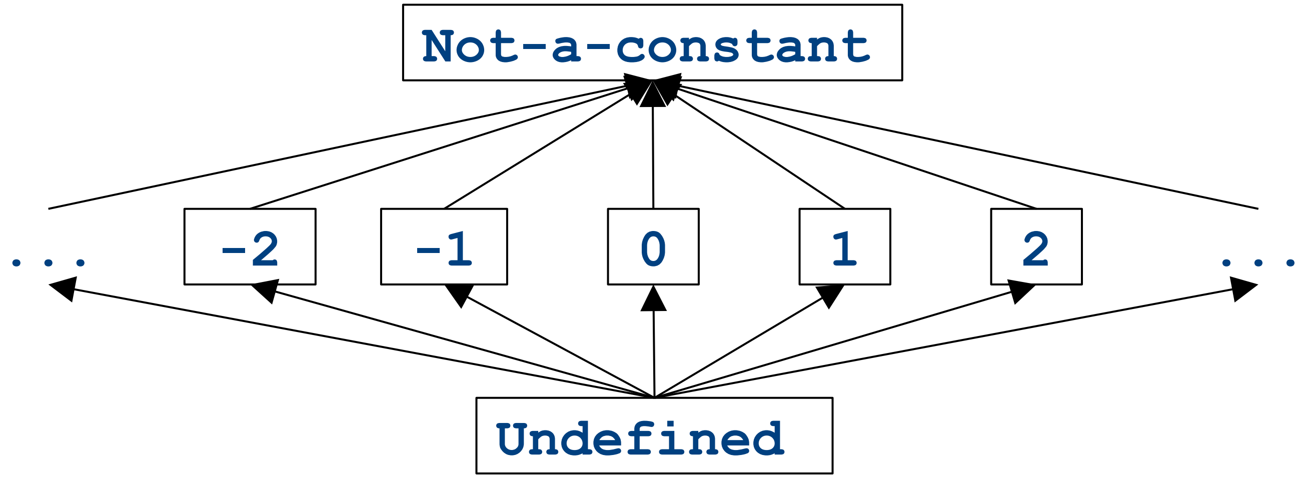
  if $x \sqsubseteq y$, then $f(x) \sqsubseteq f(y)$
- Recall our transfer function for **a = b + c** is
  - $f_{a = b + c}(V) = (V - \{a\}) \cup \{b, c\}$
- Recall that our join operator is set union and induces an ordering relationship

  $X \sqsubseteq Y$ iff $X \subseteq Y$
- Is this monotone?

Is constant propagation monotone?

- A transfer function *f* is monotone iff

  if $x \sqsubseteq y$, then $f(x) \sqsubseteq f(y)$
- Recall our transfer functions
  - $f_{\mathtt{x=k}}(V) = V[x \mapsto k]$ *(update V by mapping x to k)*
  - $f_{\mathtt{x=a+b}}(V) = V[x \mapsto \text{Not-a-Constant}]$ *(assign Not-a-Constant)*
- Is this monotone?



The grand result

- Theorem: A dataflow analysis with a **finite-height semilattice** and family of **monotone transfer functions** always terminates
- Proof sketch:
  - The join operator can only bring values up
  - Transfer functions can never lower values back down below where they were in the past (monotonicity)
  - Values cannot increase indefinitely (finite height)

An “optimality” result

- A transfer function *f* is distributive if
  $$f(a \sqcup b) = f(a) \sqcup f(\mathrm{b})$$
  for every domain elements *a* and *b*
- If all transfer functions are distributive then the fixed-point solution is the solution that would be computed by joining results from all (potentially infinite) control-flow paths
  - Join over all paths
- Optimal if we ignore program conditions

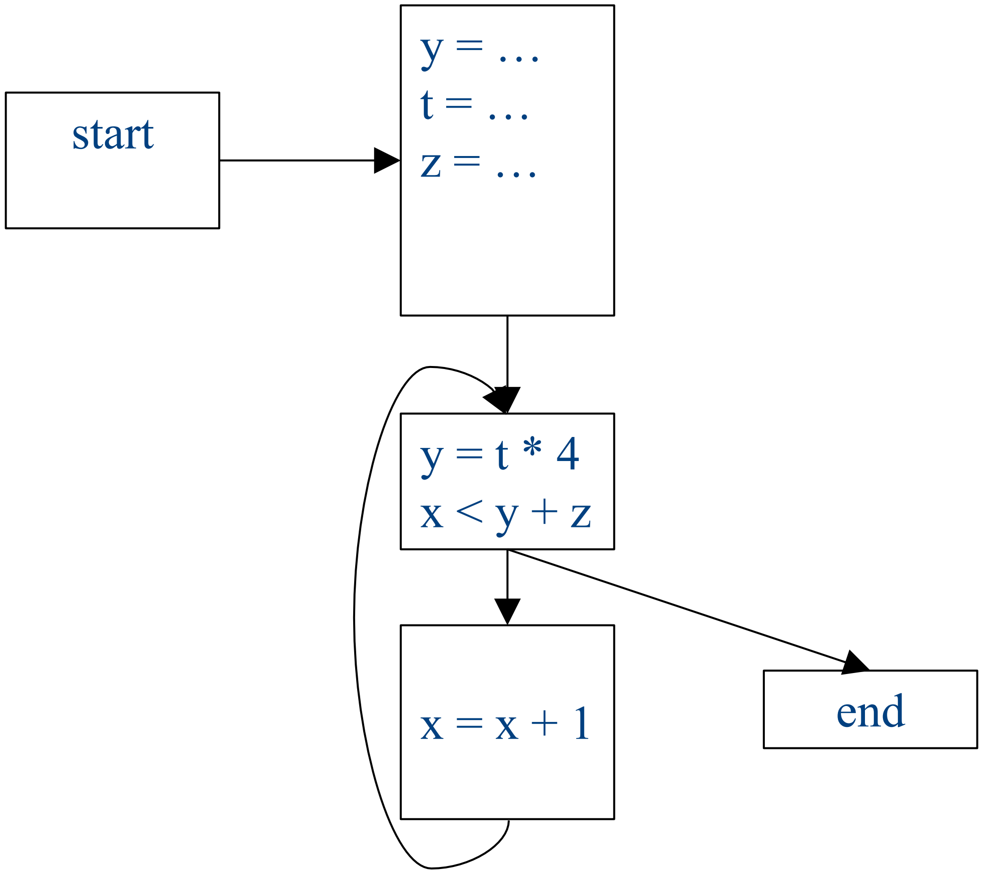
An “optimality” result

- A transfer function *f* is distributive if
  $$f(a \sqcup b) = f(a) \sqcup f(b)$$
  for every domain elements *a* and *b*
- If all transfer functions are distributive then the fixed-point solution is equal to the solution computed by joining results from all (potentially infinite) control-flow paths
  - Join over all paths
- Optimal if we pretend all control-flow paths can be executed by the program
- Which analyses use distributive functions?

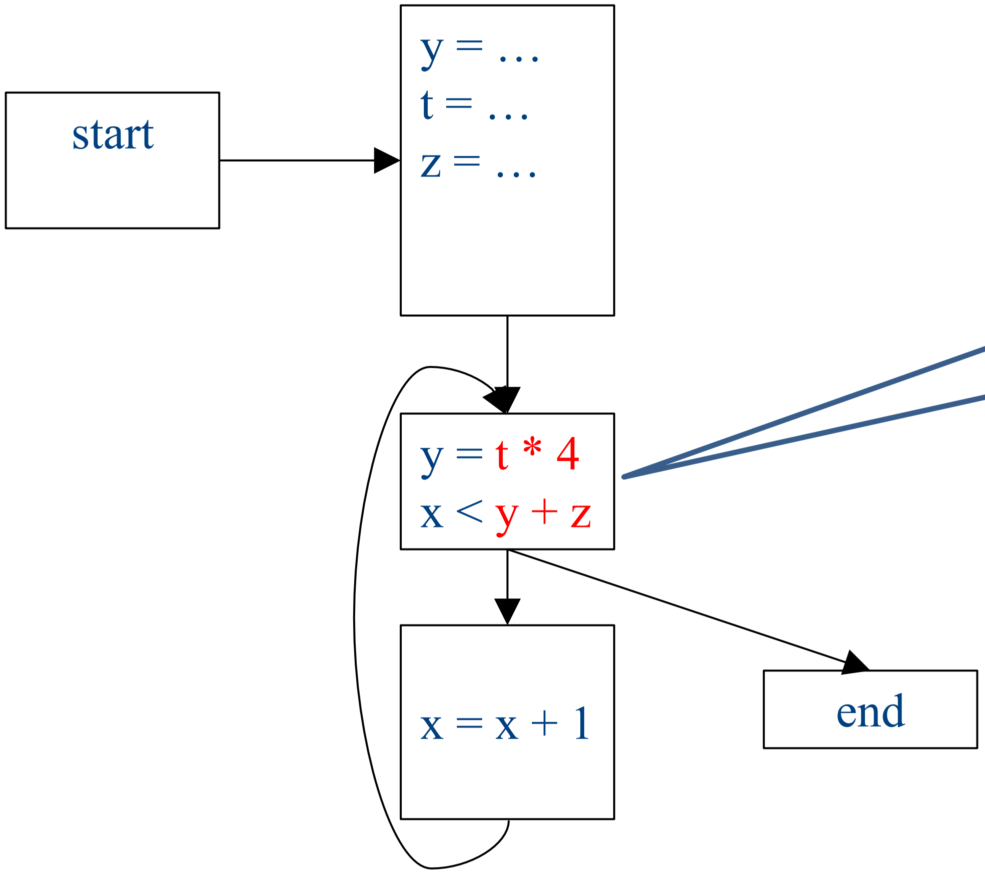
Loop optimizations

- Most of a program’s computations are done inside loops
  - Focus optimizations effort on loops
- The optimizations we’ve seen so far are independent of the control structure
- Some optimizations are specialized to loops
  - Loop-invariant code motion
  - (Strength reduction via induction variables)
- Require another type of analysis to find out where expressions get their values from
  - Reaching definitions
    - (Also useful for improving register allocation)

Loop invariant computation

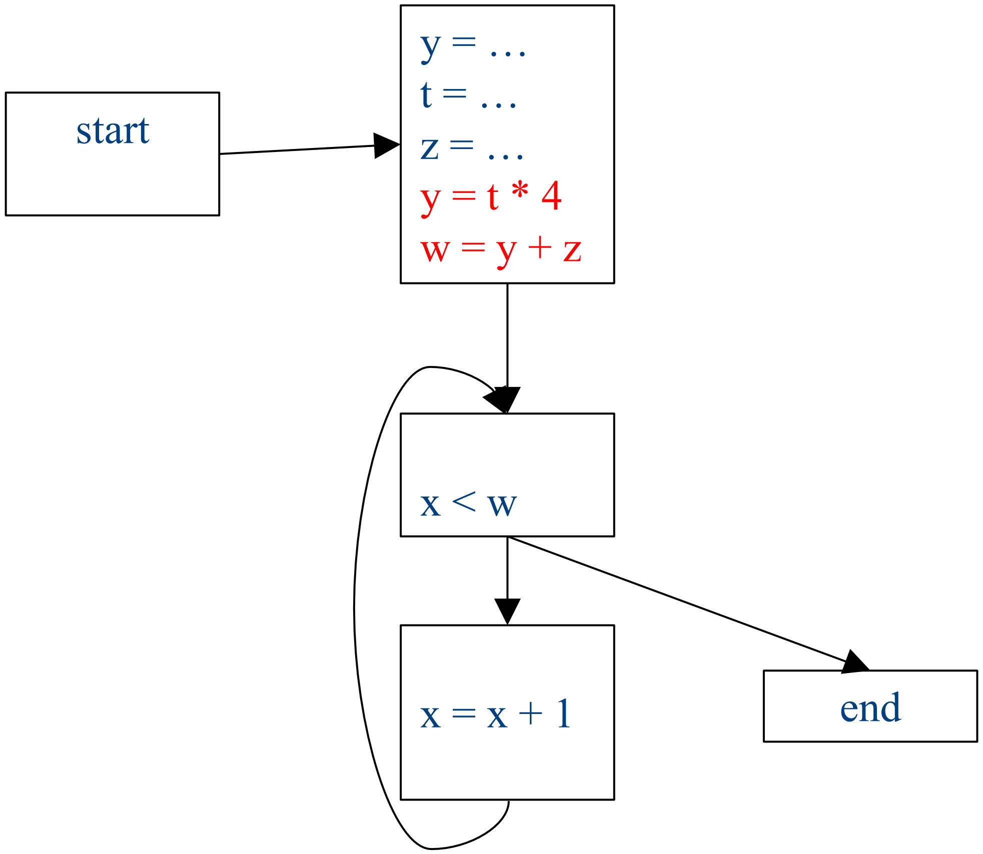


Loop invariant computation

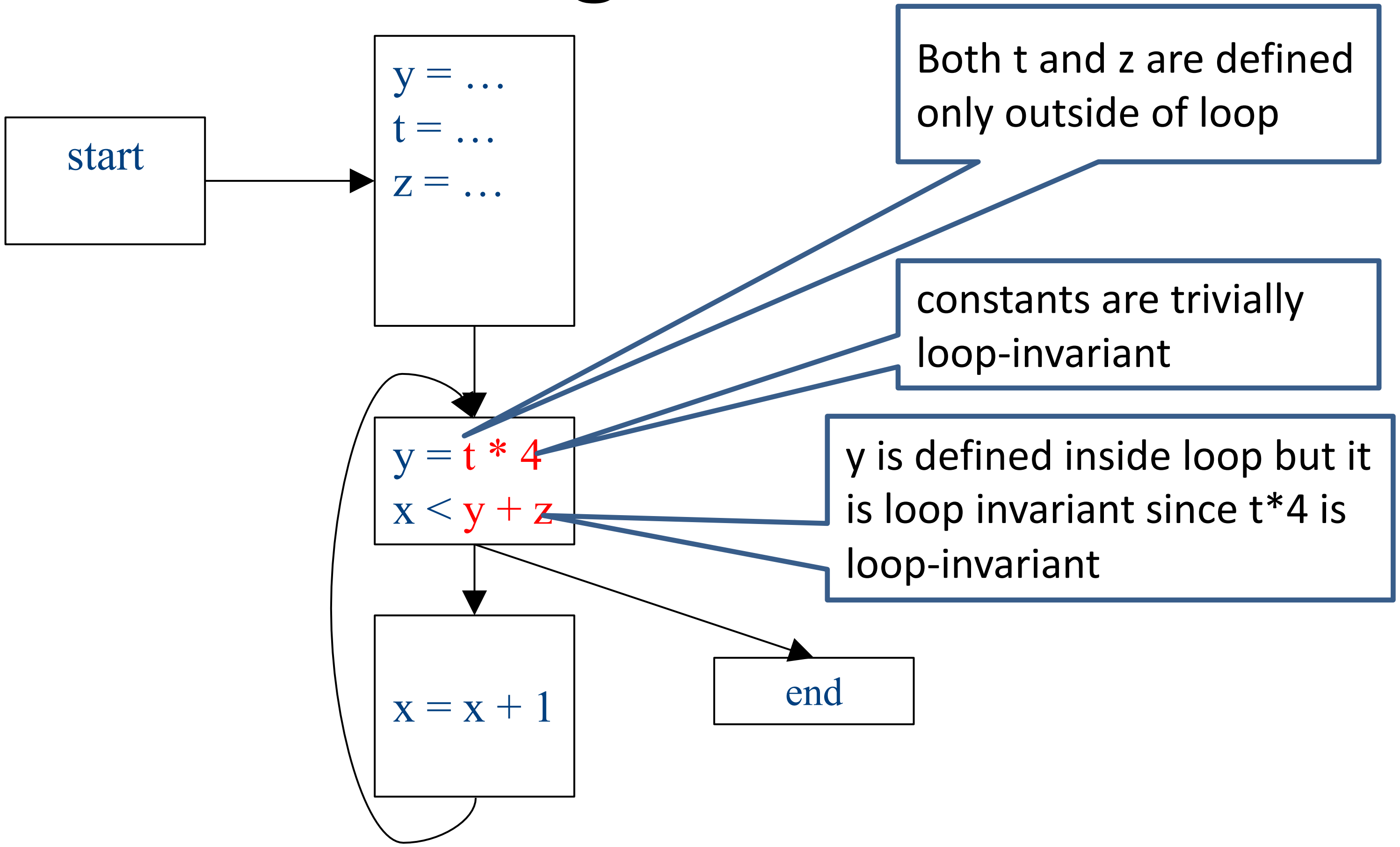


t*4 and y+z have same value on each iteration

Code hoisting

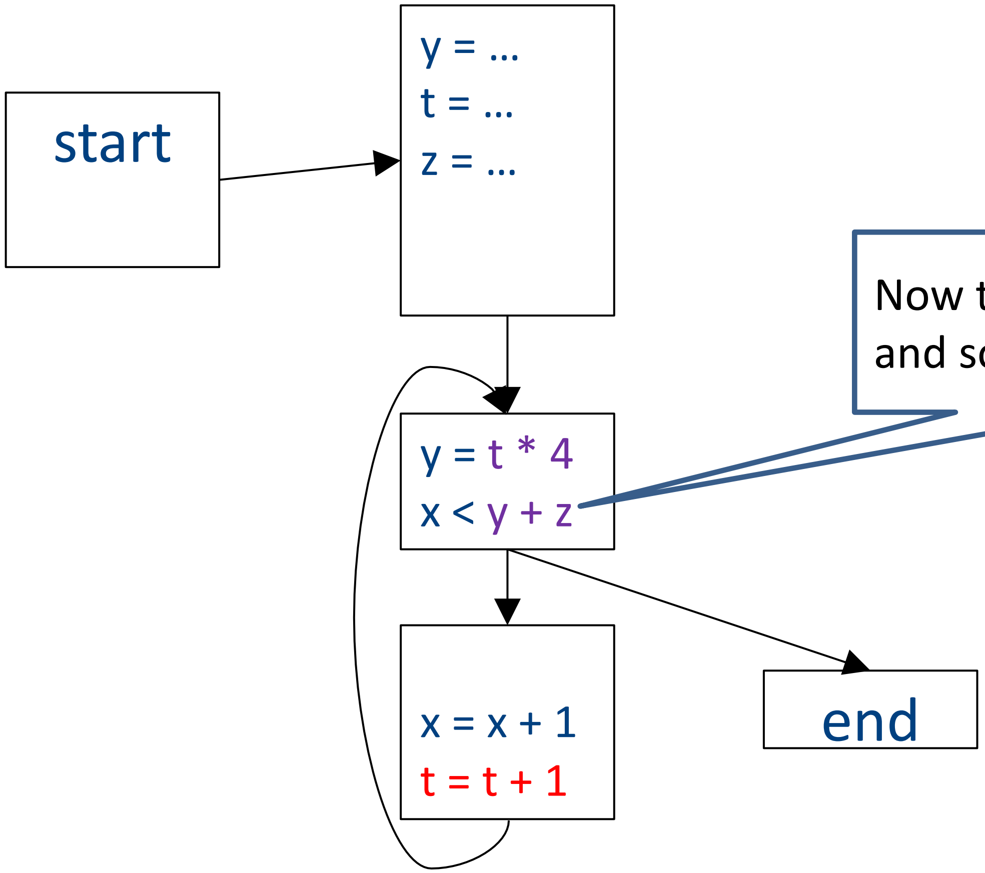


What reasoning did we use?



start
y = …
t = …
z = …
y = t * 4
x < y + z
x = x + 1
end
Both t and z are defined only outside of loop
constants are trivially loop-invariant
y is defined inside loop but it is loop invariant since t*4 is loop-invariant

What about now?



Now t is not loop-invariant and so are t*4 and y

Loop-invariant code motion

- *d*: t = a_1 op a_2
  - *d* is a program location
- a_1 op a_2 loop-invariant (for a loop *L*) if computes the same value in each iteration
  - Hard to know in general
- Conservative approximation
  - Each a_i is a constant, or
  - All definitions of a_i that reach *d* are outside *L*, or
  - Only one definition of of a_i reaches *d*, and is loop-invariant itself
- Transformation: hoist the loop-invariant code outside of the loop

Reaching definitions analysis

- A definition *d*: t = … reaches a program location if there is a path from the definition to the program location, along which the defined variable is never redefined

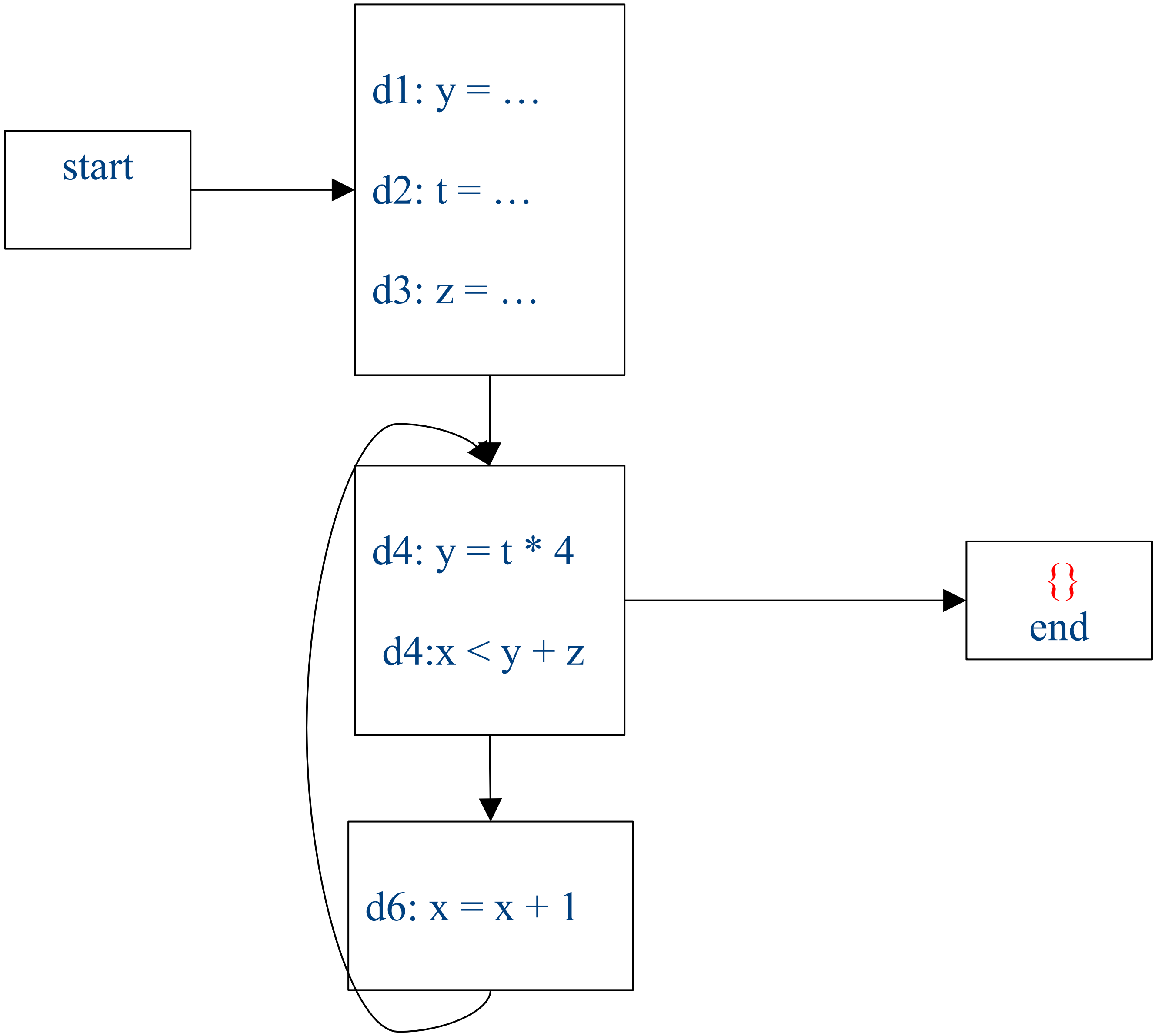
Reaching definitions analysis

- A definition *d*: t = … reaches a program location if there is a path from the definition to the program location, along which the defined variable is never redefined
- Direction: Forward
- Domain: sets of program locations that are definitions `
- Join operator: union
- Transfer function:

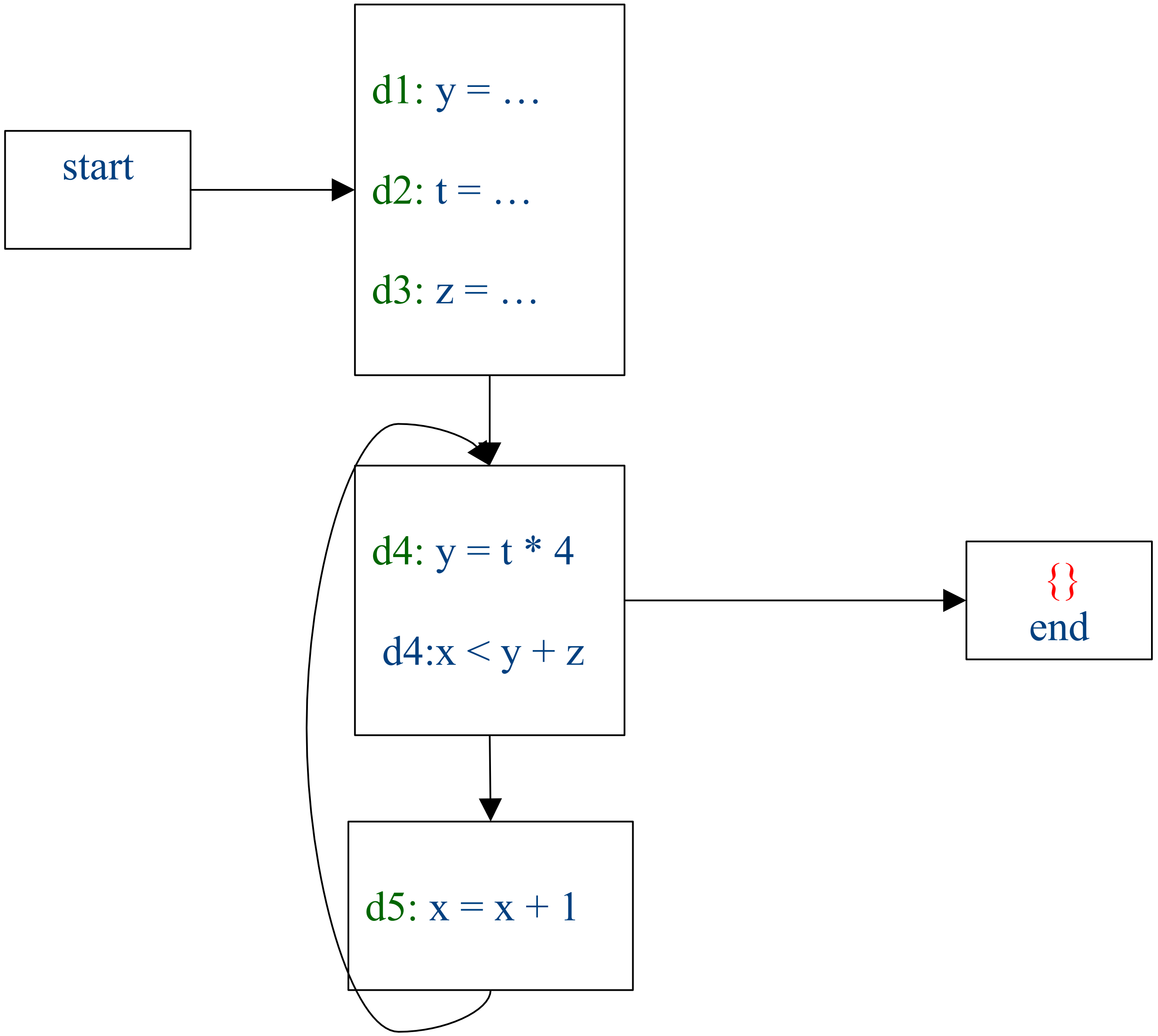
  $$f_{d:\ a=b\ op\ c}(RD) = (RD - defs(a)) \cup \{d\}$$
  $$f_{d:\ not\text{-}a\text{-}def}(RD) = RD$$

  – Where *defs*(*a*) is the set of locations defining *a* (statements of the form *a*=...)
- Initial value: {}

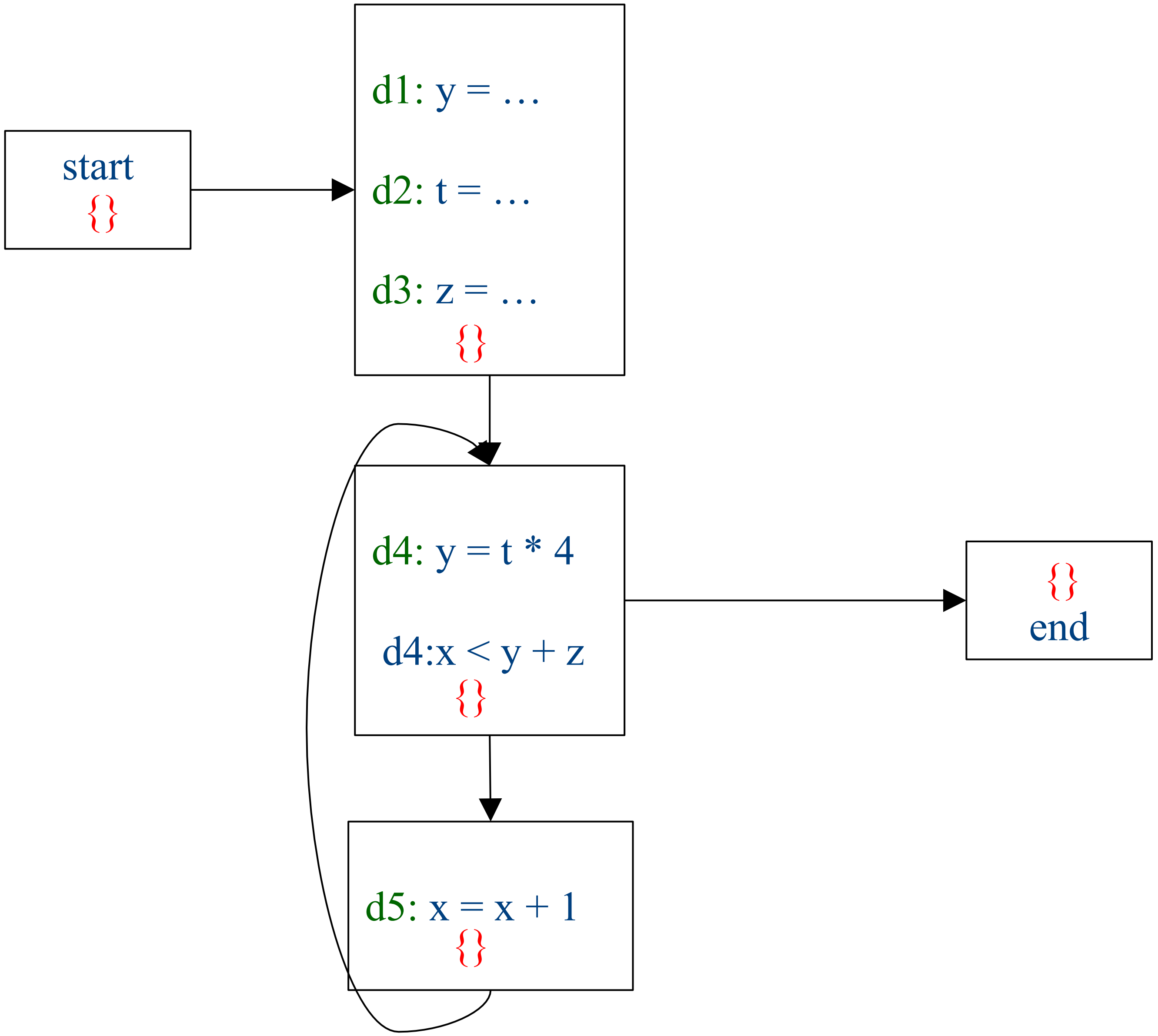
Reaching definitions analysis



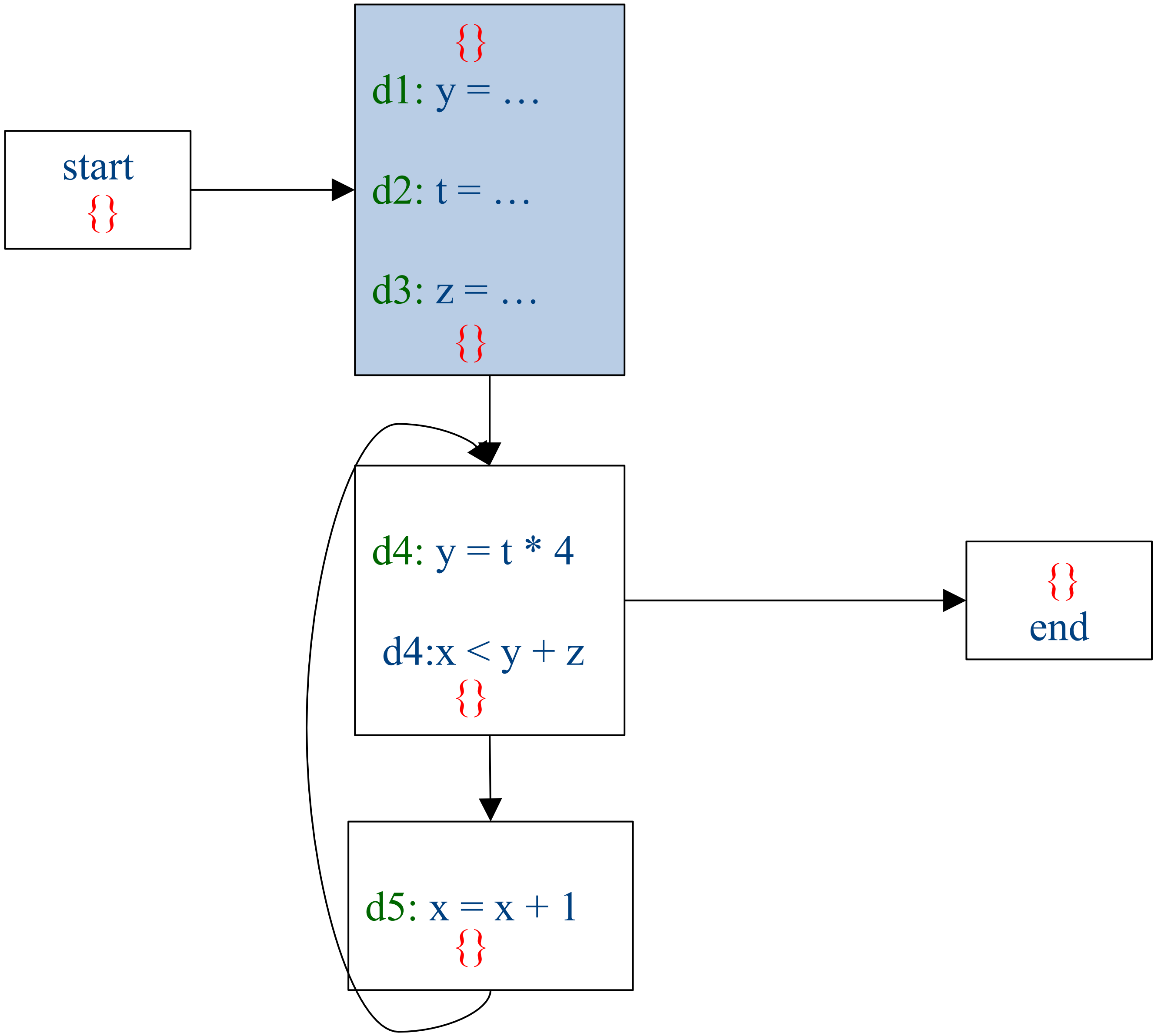
Reaching definitions analysis



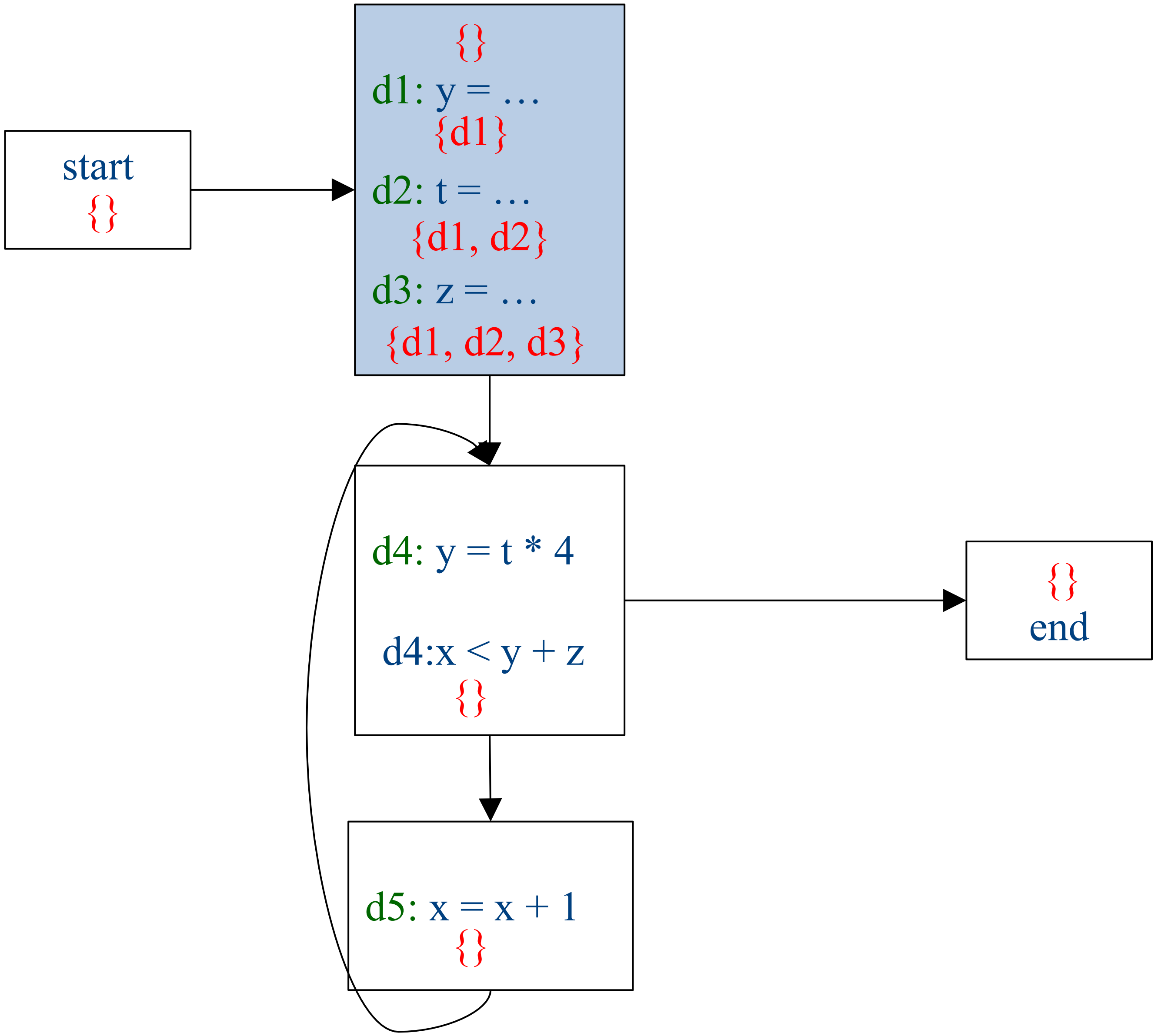
Initialization



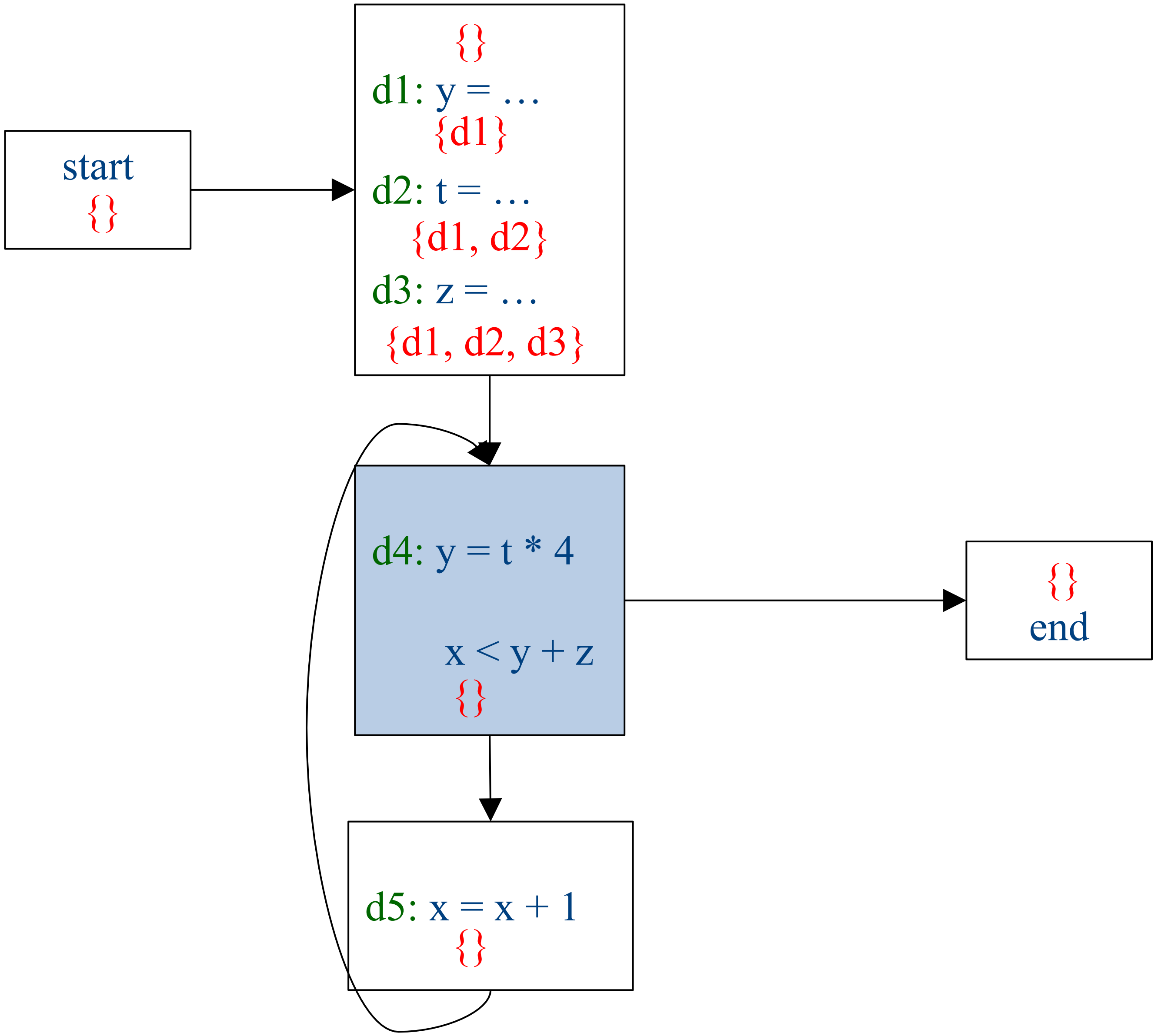
Iteration 1



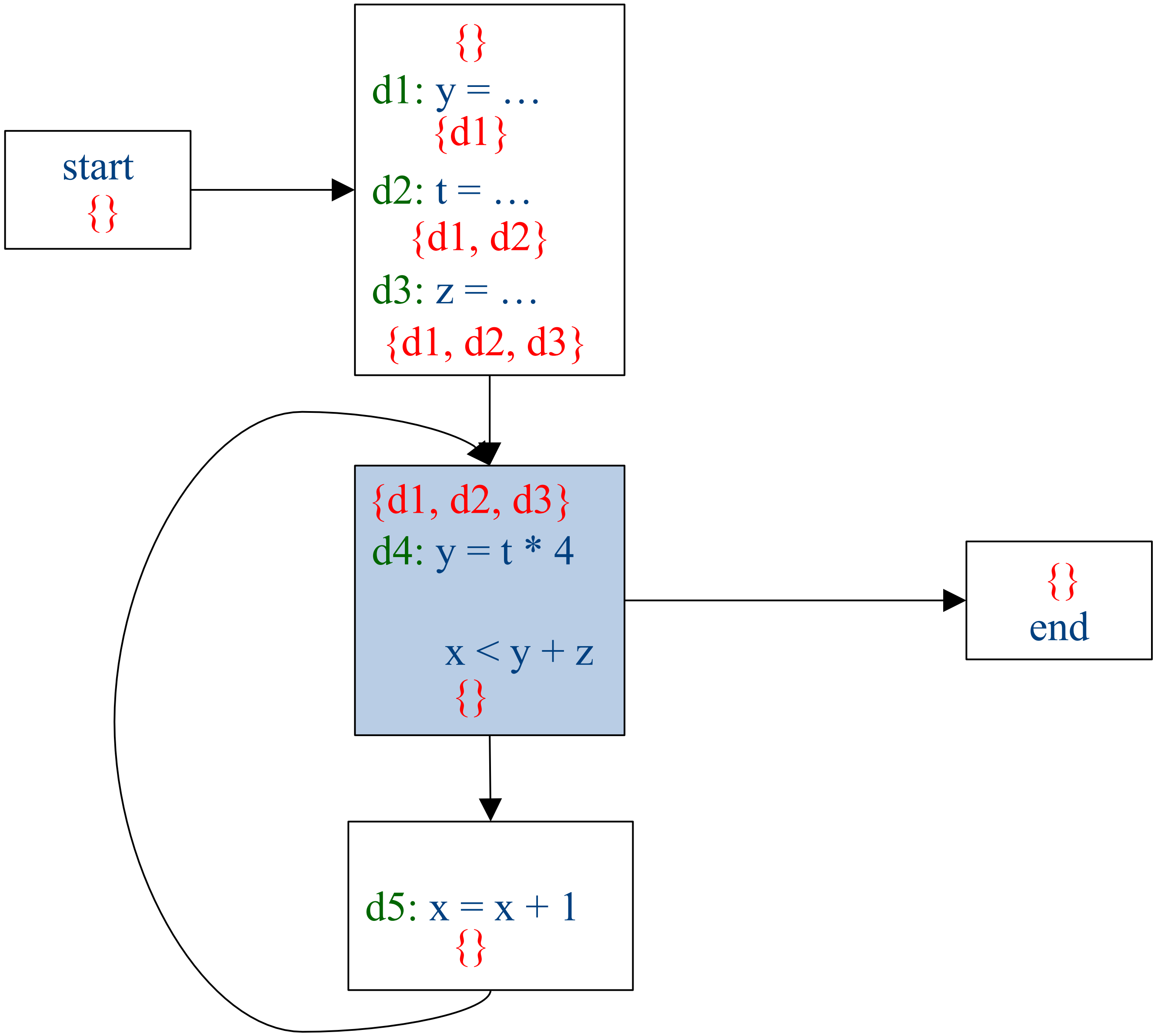
Iteration 1



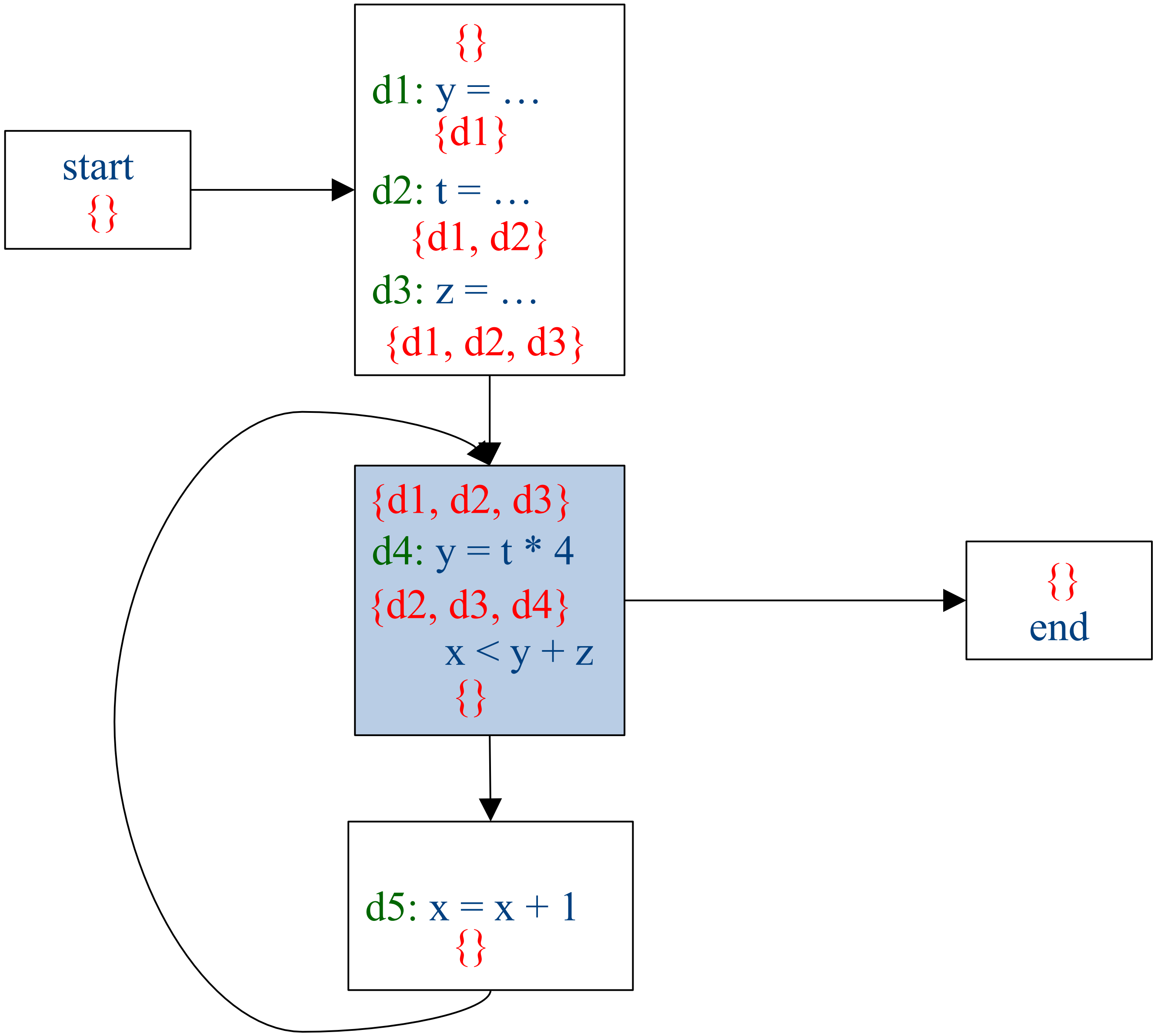
Iteration 2



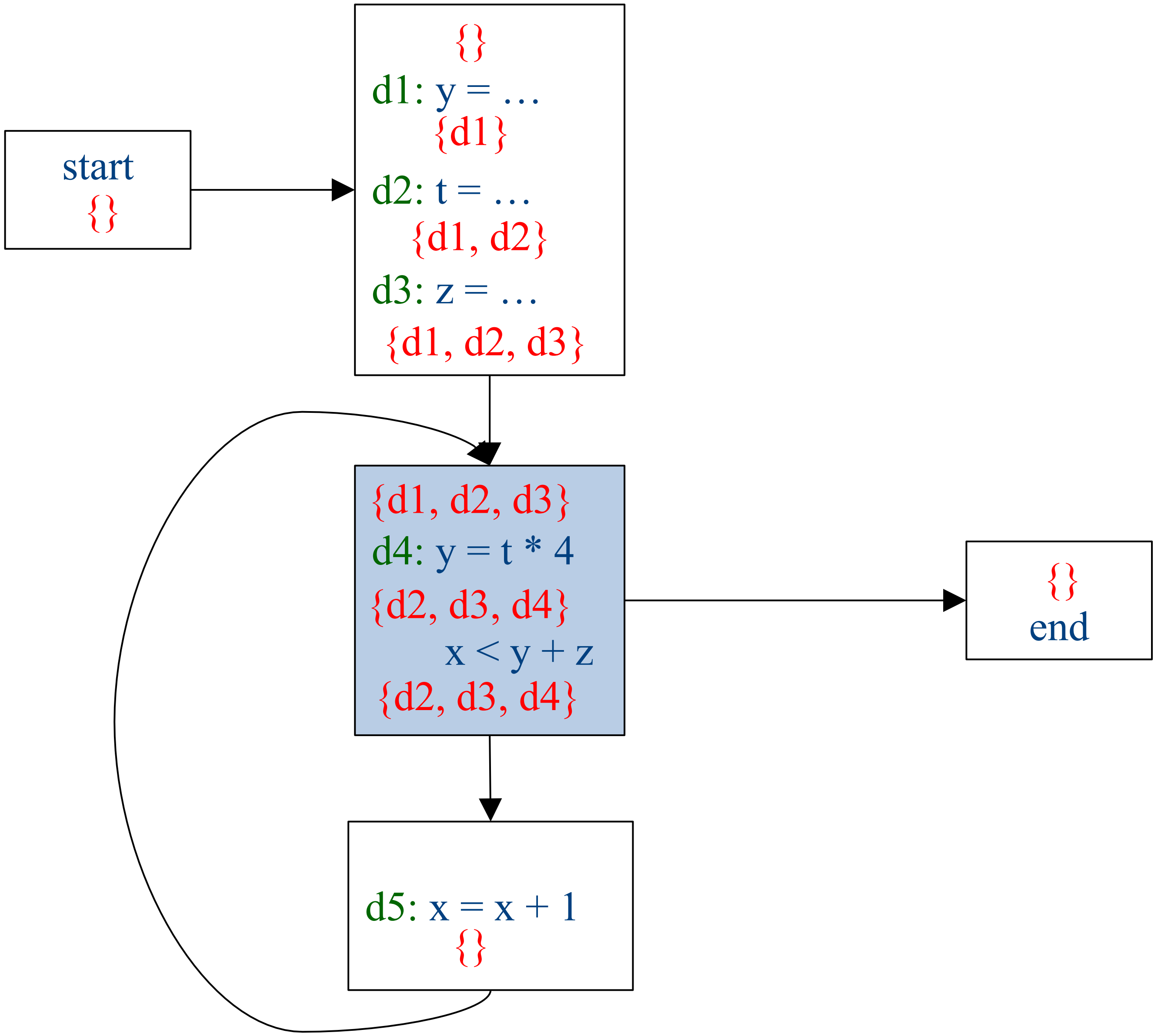
Iteration 2



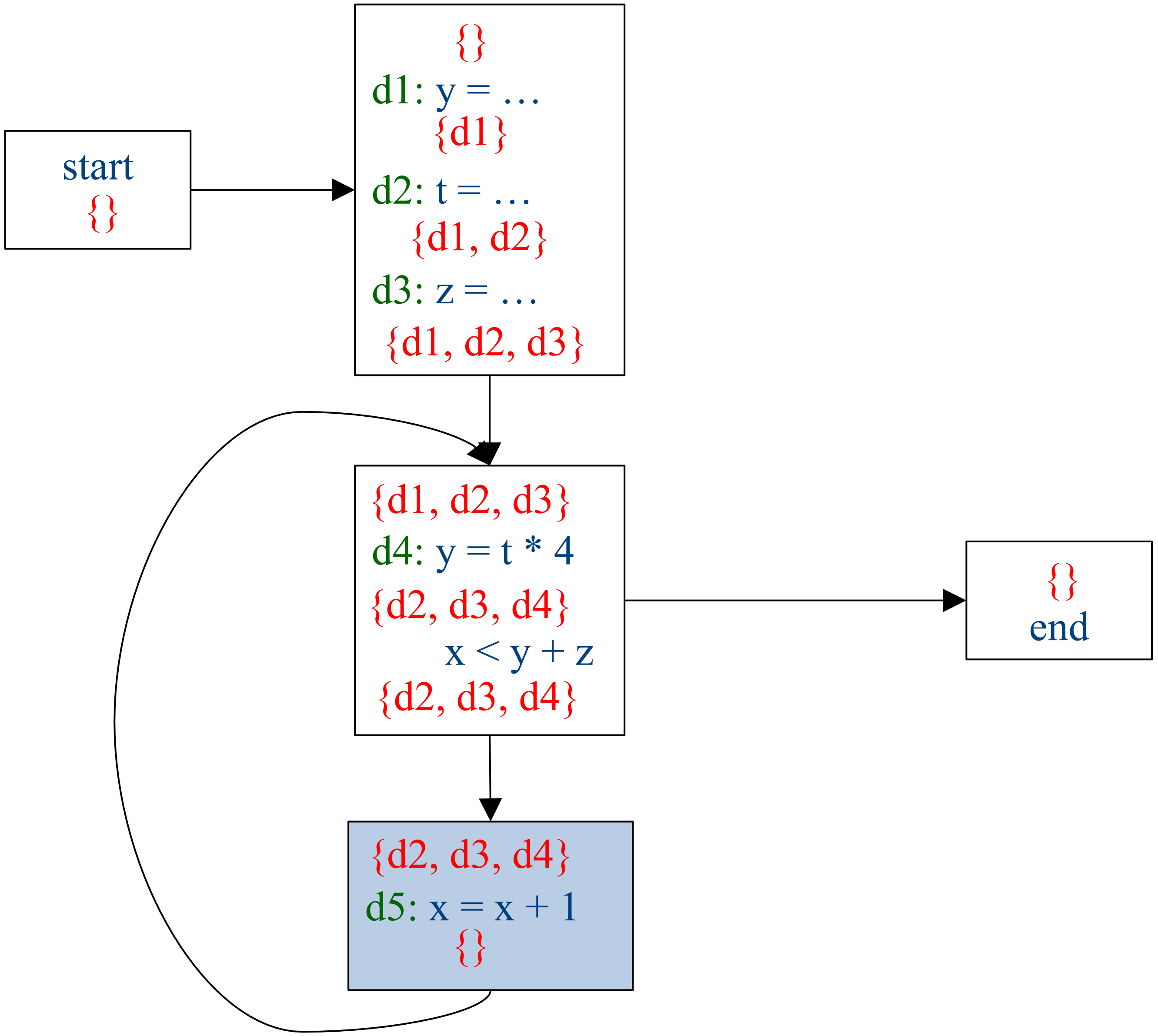
Iteration 2


start
{}
{}
d1: y = …
{d1}
d2: t = …
{d1, d2}
d3: z = …
{d1, d2, d3}
{d1, d2, d3}
d4: y = t * 4
{d2, d3, d4}
x < y + z
{}
{}
end
d5: x = x + 1
{}

Iteration 2

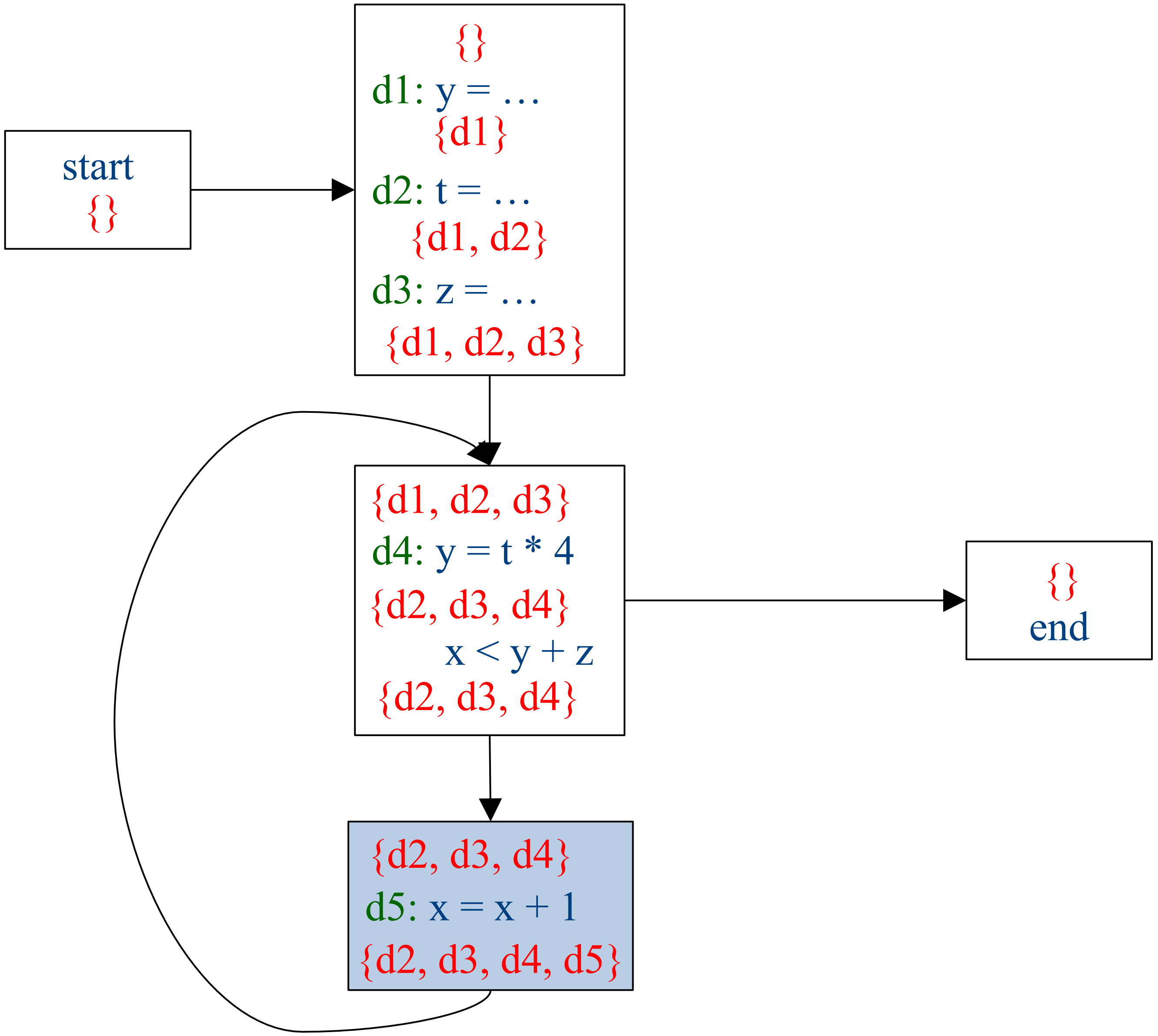

start
{}
{}
d1: y = …
{d1}
d2: t = …
{d1, d2}
d3: z = …
{d1, d2, d3}
{d1, d2, d3}
d4: y = t * 4
{d2, d3, d4}
x < y + z
{d2, d3, d4}
{}
end
d5: x = x + 1
{}

Iteration 3

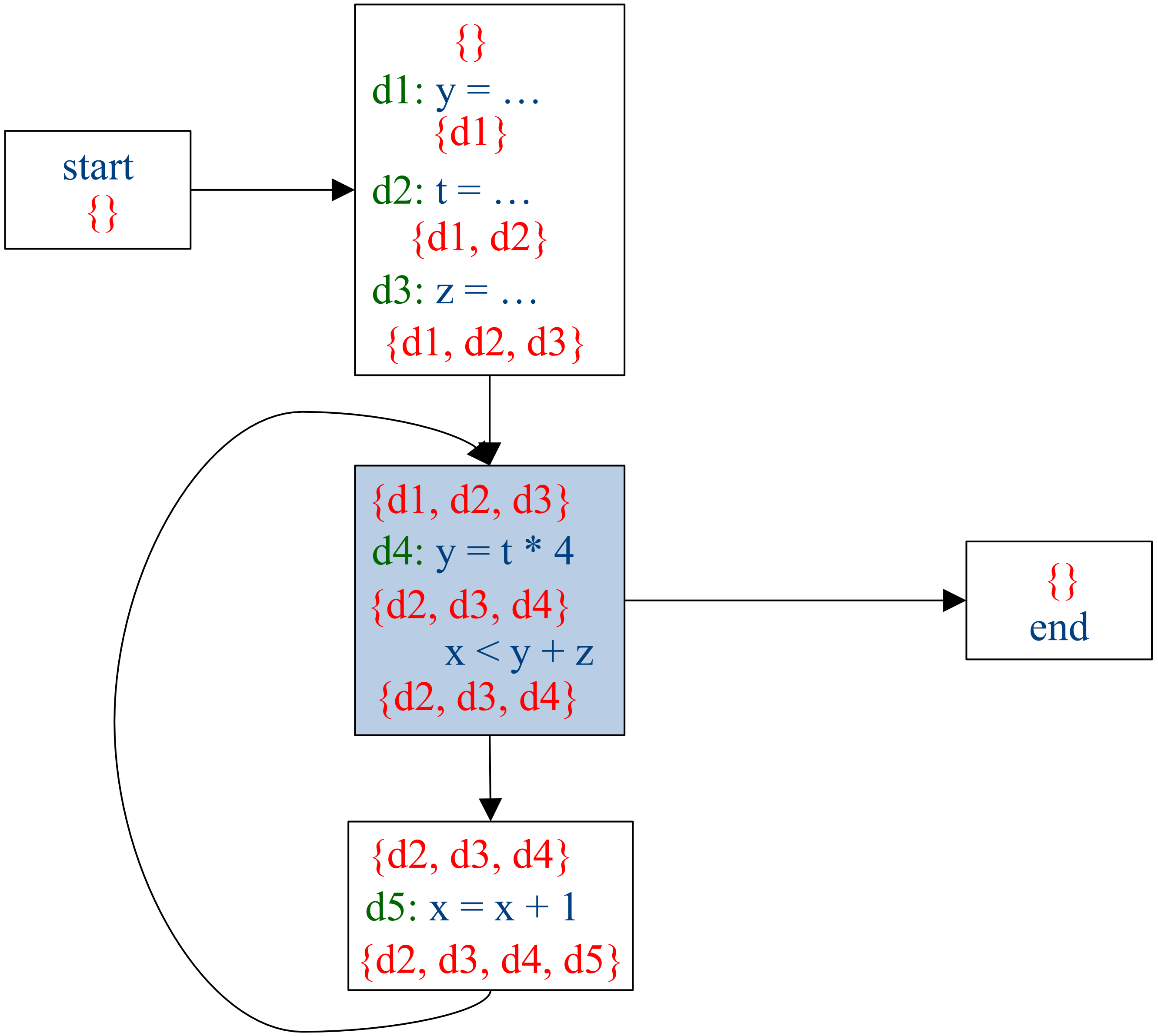


start
{}
{}
d1: y = …
{d1}
d2: t = …
{d1, d2}
d3: z = …
{d1, d2, d3}
{d1, d2, d3}
d4: y = t * 4
{d2, d3, d4}
x < y + z
{d2, d3, d4}
{}
end
{d2, d3, d4}
d5: x = x + 1
{}

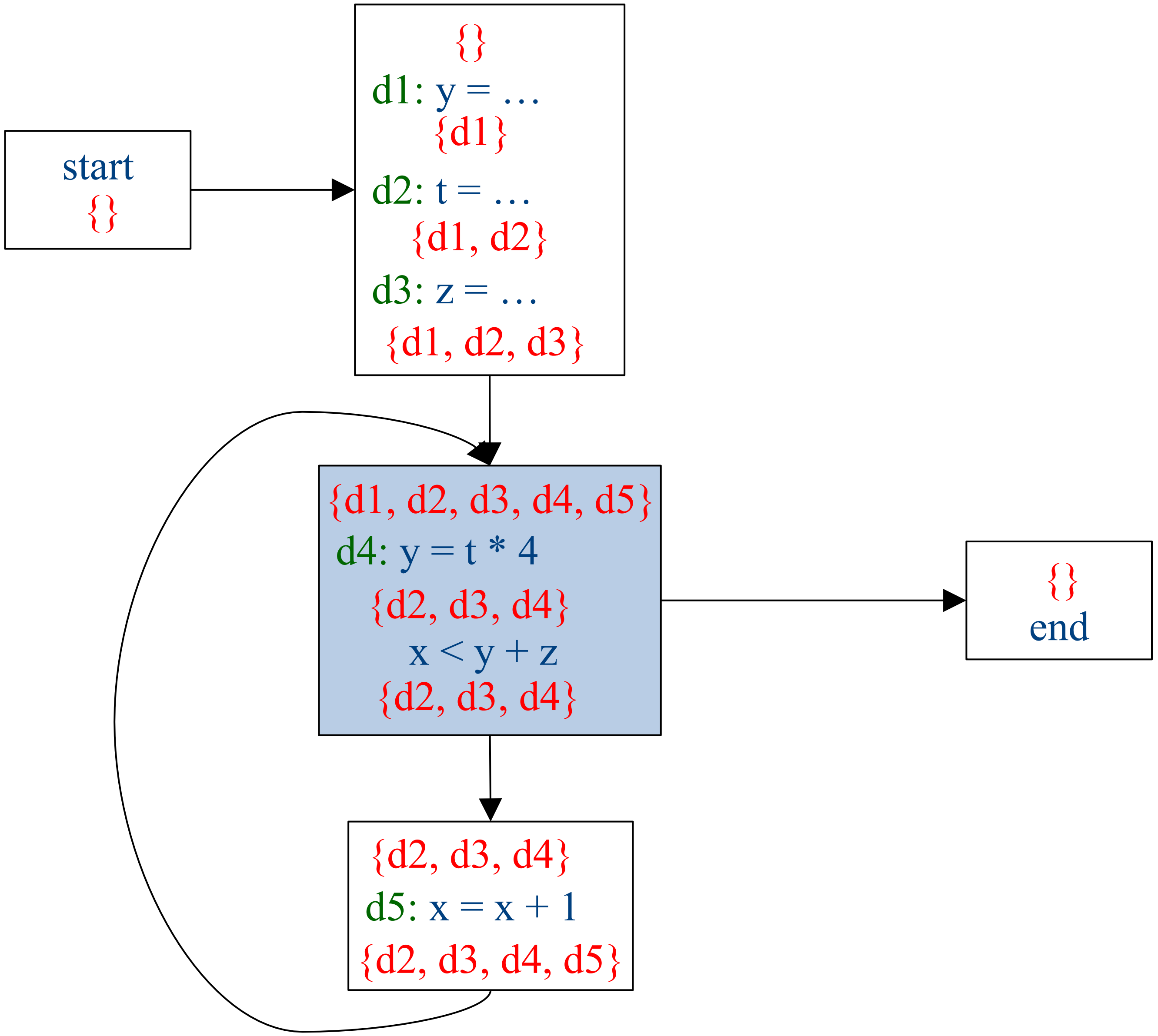
Iteration 3


start
{}
{}
d1: y = …
{d1}
d2: t = …
{d1, d2}
d3: z = …
{d1, d2, d3}
{d1, d2, d3}
d4: y = t * 4
{d2, d3, d4}
x < y + z
{d2, d3, d4}
{}
end
{d2, d3, d4}
d5: x = x + 1
{d2, d3, d4, d5}

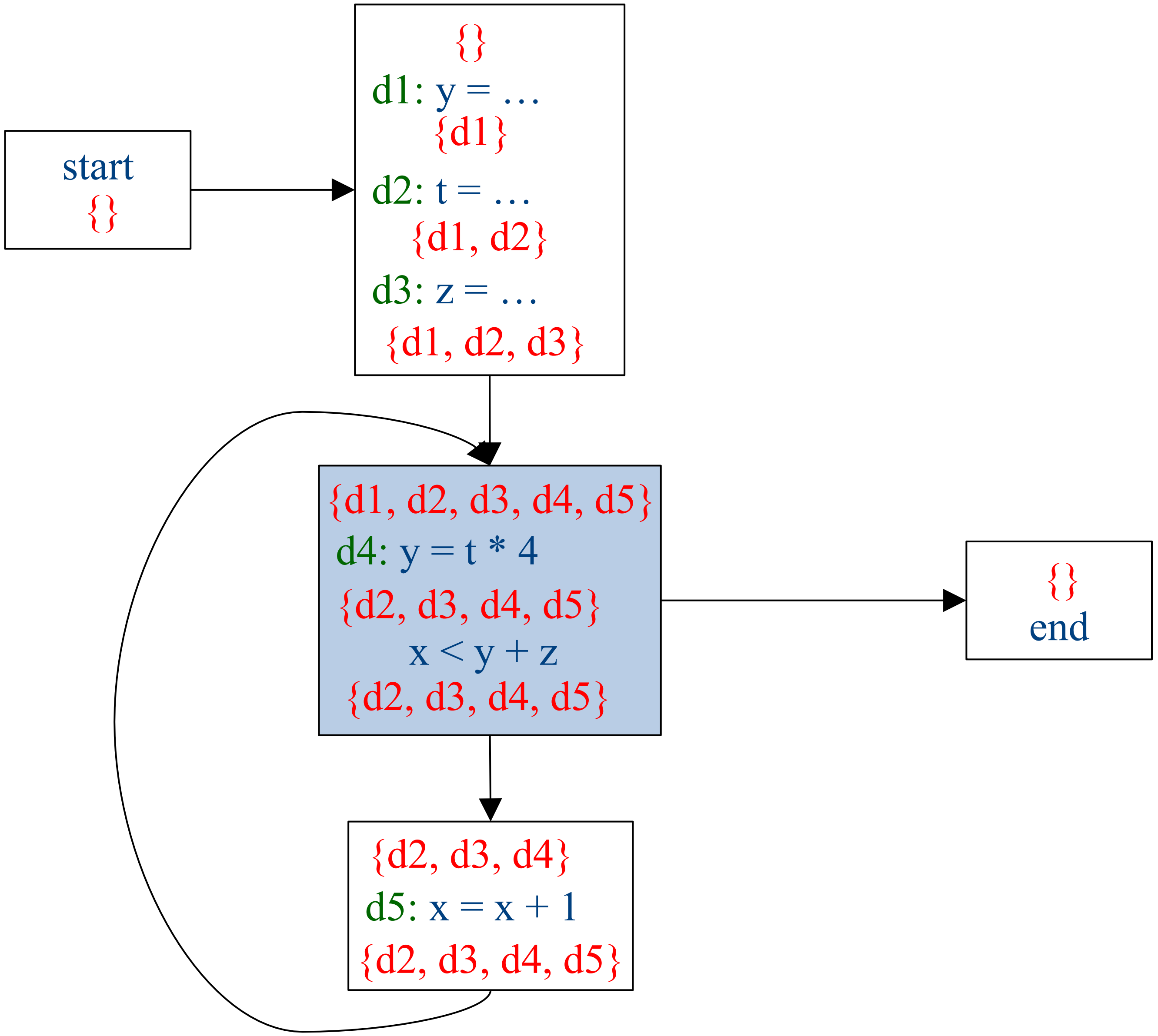
Iteration 4


start
{}
{}
d1: y = …
{d1}
d2: t = …
{d1, d2}
d3: z = …
{d1, d2, d3}
{d1, d2, d3}
d4: y = t * 4
{d2, d3, d4}
x < y + z
{d2, d3, d4}
{}
end
{d2, d3, d4}
d5: x = x + 1
{d2, d3, d4, d5}

Iteration 4

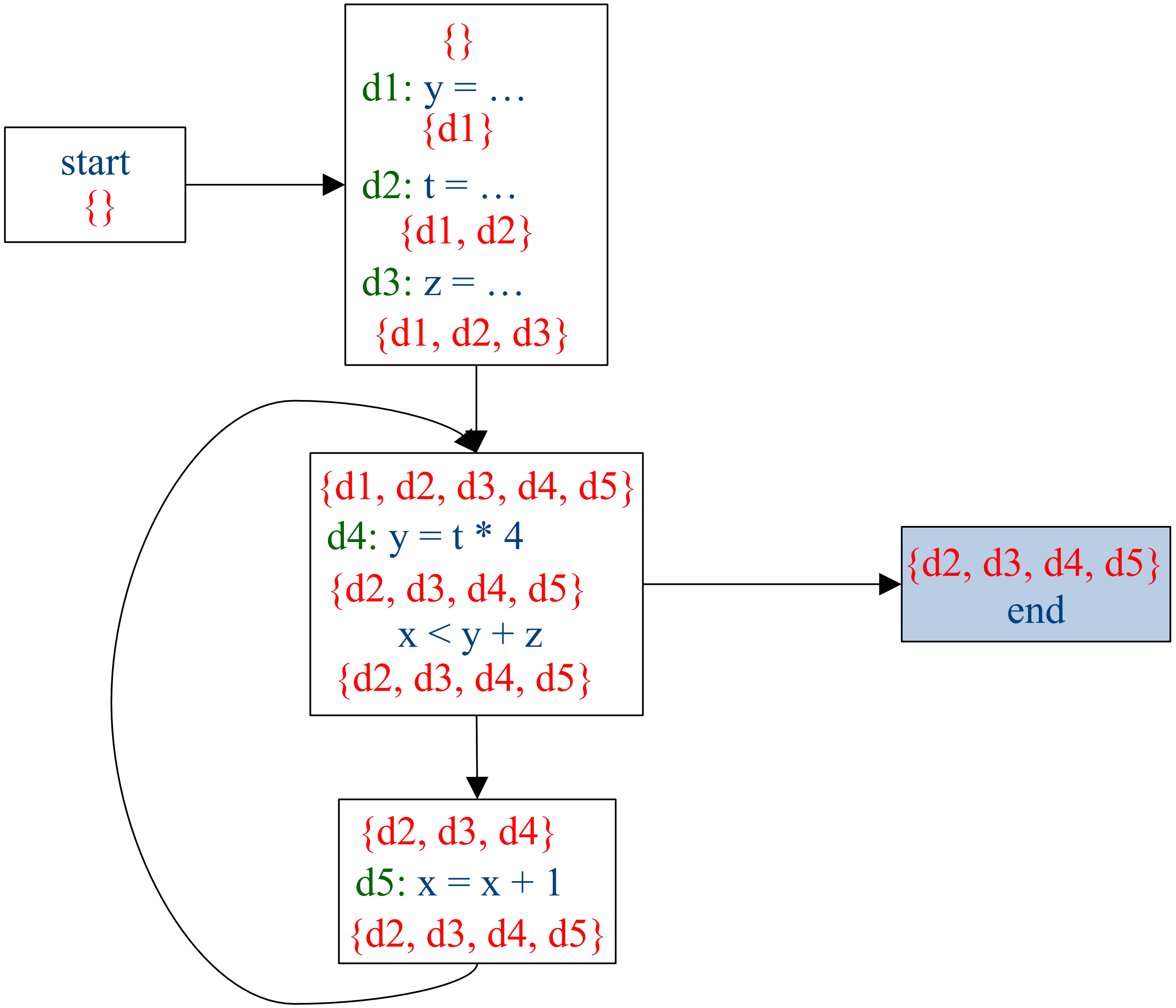

start
{}
{}
d1: y = …
{d1}
d2: t = …
{d1, d2}
d3: z = …
{d1, d2, d3}
{d1, d2, d3, d4, d5}
d4: y = t * 4
{d2, d3, d4}
x < y + z
{d2, d3, d4}
{}
end
{d2, d3, d4}
d5: x = x + 1
{d2, d3, d4, d5}

Iteration 4

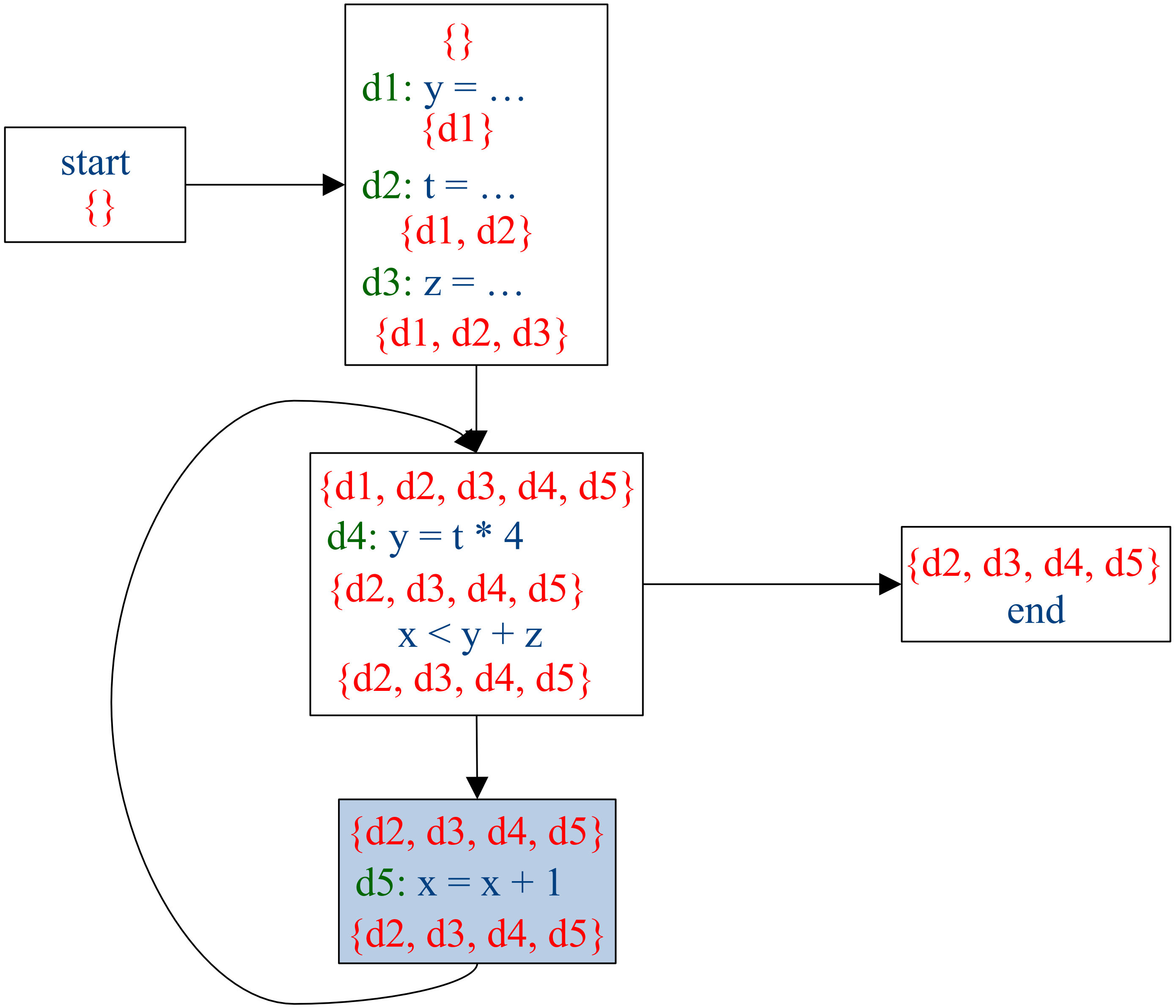


start
{}
{}
d1: y = …
{d1}
d2: t = …
{d1, d2}
d3: z = …
{d1, d2, d3}
{d1, d2, d3, d4, d5}
d4: y = t * 4
{d2, d3, d4, d5}
x < y + z
{d2, d3, d4, d5}
{}
end
{d2, d3, d4}
d5: x = x + 1
{d2, d3, d4, d5}

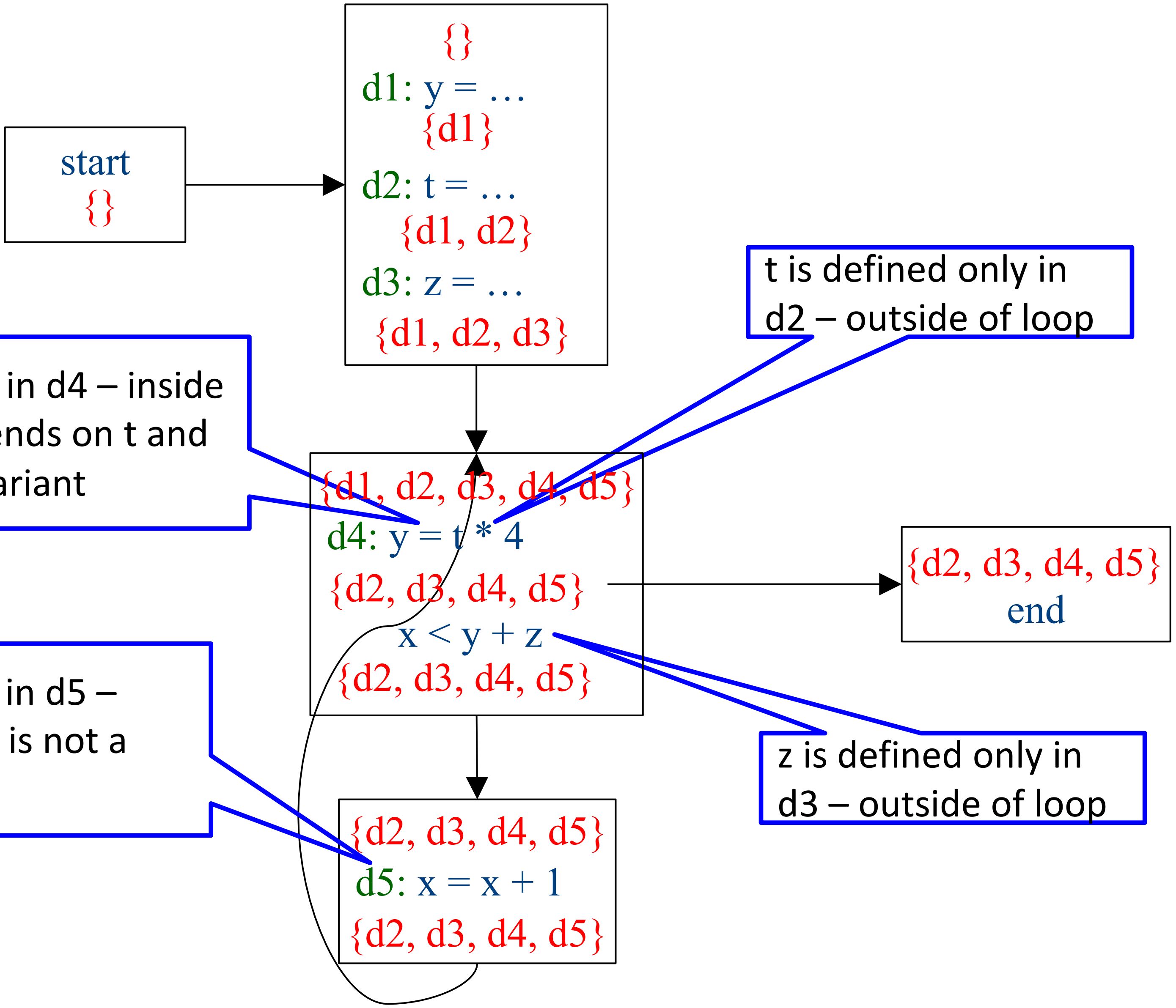
Iteration 5


start
{}
{}
d1: y = …
{d1}
d2: t = …
{d1, d2}
d3: z = …
{d1, d2, d3}
{d1, d2, d3, d4, d5}
d4: y = t * 4
{d2, d3, d4, d5}
x < y + z
{d2, d3, d4, d5}
{d2, d3, d4}
d5: x = x + 1
{d2, d3, d4, d5}
{d2, d3, d4, d5}
end

Iteration 6


start
{}
{}
d1: y = …
{d1}
d2: t = …
{d1, d2}
d3: z = …
{d1, d2, d3}
{d1, d2, d3, d4, d5}
d4: y = t * 4
{d2, d3, d4, d5}
x < y + z
{d2, d3, d4, d5}
{d2, d3, d4, d5}
end
{d2, d3, d4, d5}
d5: x = x + 1
{d2, d3, d4, d5}

Which expressions are loop invariant?



y is defined only in d4 – inside of loop but depends on t and 4, both loop-invariant

t is defined only in d2 – outside of loop

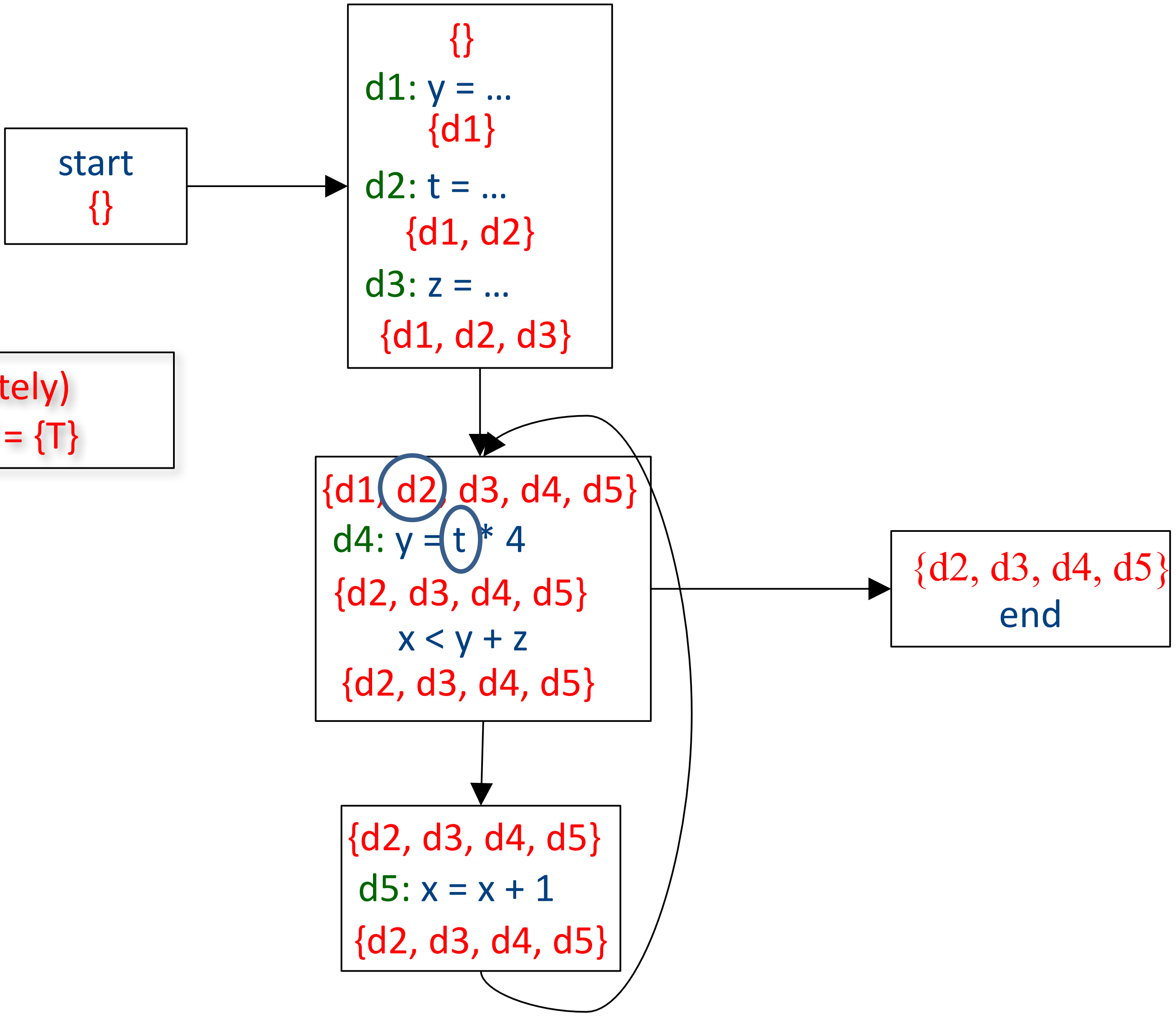
x is defined only in d5 – inside of loop so is not a loop-invariant

z is defined only in d3 – outside of loop

Inferring loop-invariant expressions

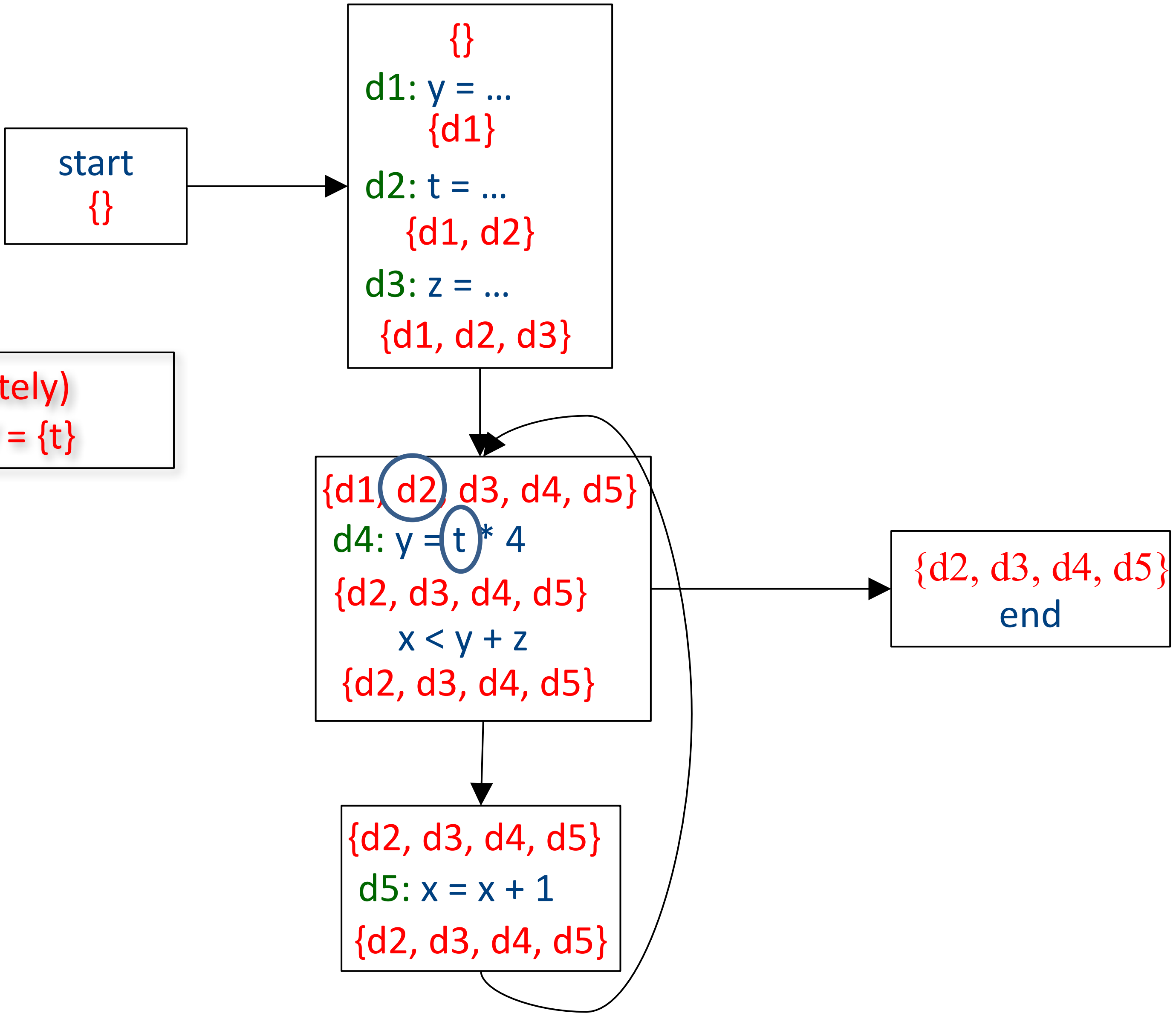
- For a statement *s* of the form t = a_1 op a_2
- A variable a_i is immediately loop-invariant if all reaching definitions IN[*s*]={d_1,…,d_k} for a_i are outside of the loop
- LOOP-INV = immediately loop-invariant variables and constants
  LOOP-INV = LOOP-INV ▶ {x | d: x = a_1 op a_2, d is in the loop, and both a_1 and a_2 are in LOOP-INV}
  - Iterate until fixed-point
- An expression is loop-invariant if all operands are loop-invariants

Computing LOOP-INV



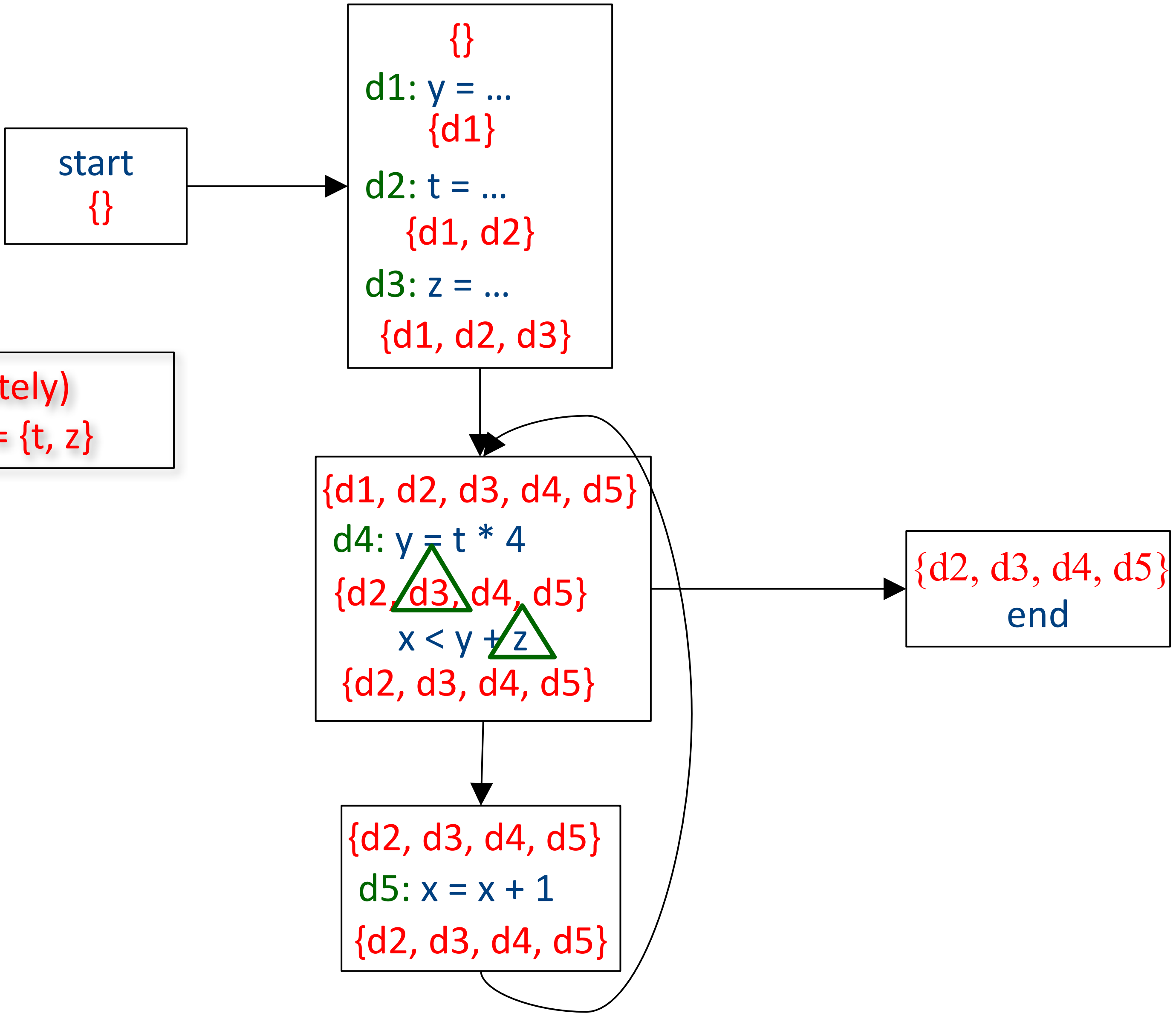
(immediately)
LOOP-INV = {T}

Computing LOOP-INV



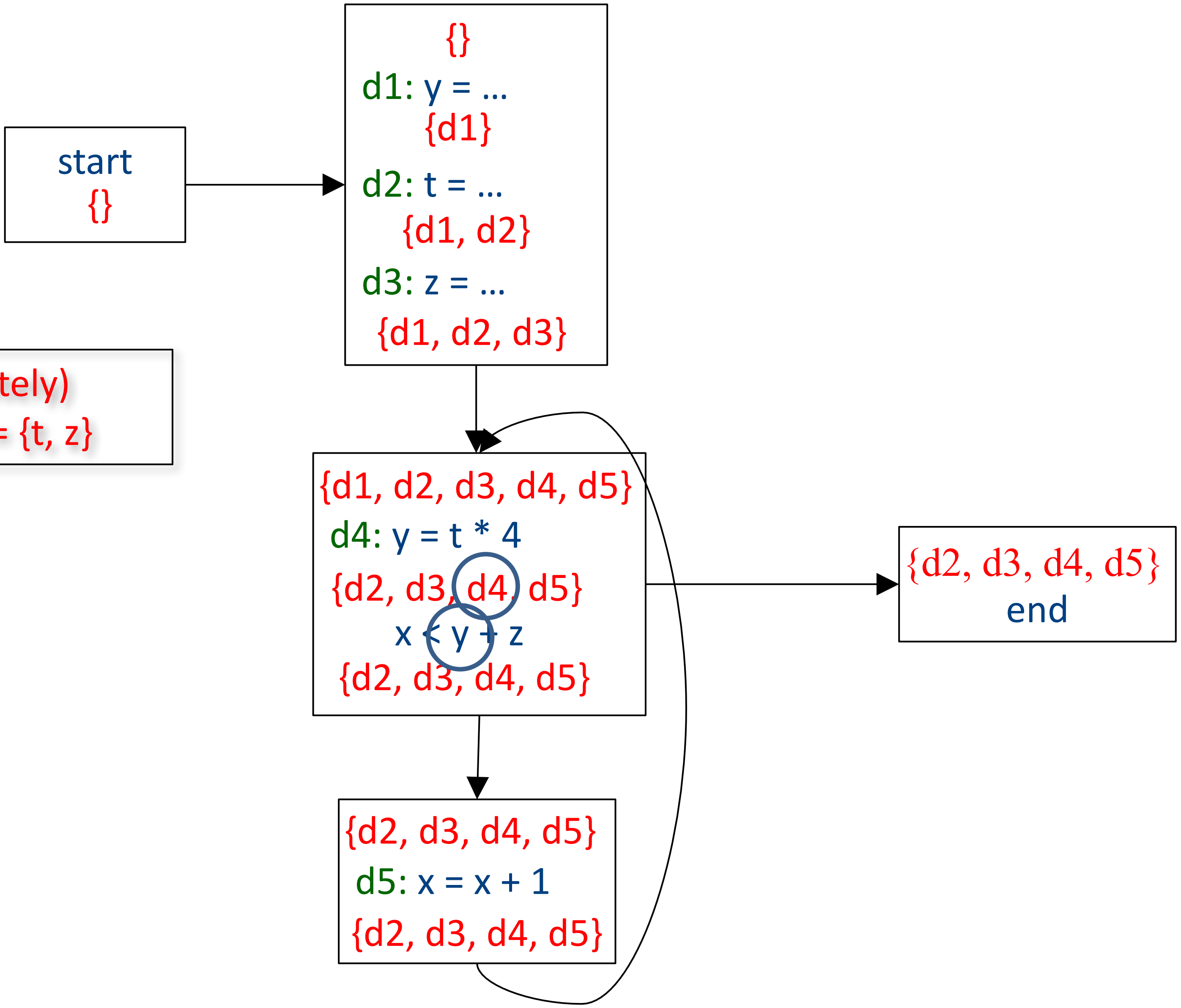
(immediately)
LOOP-INV = {t}

Computing LOOP-INV



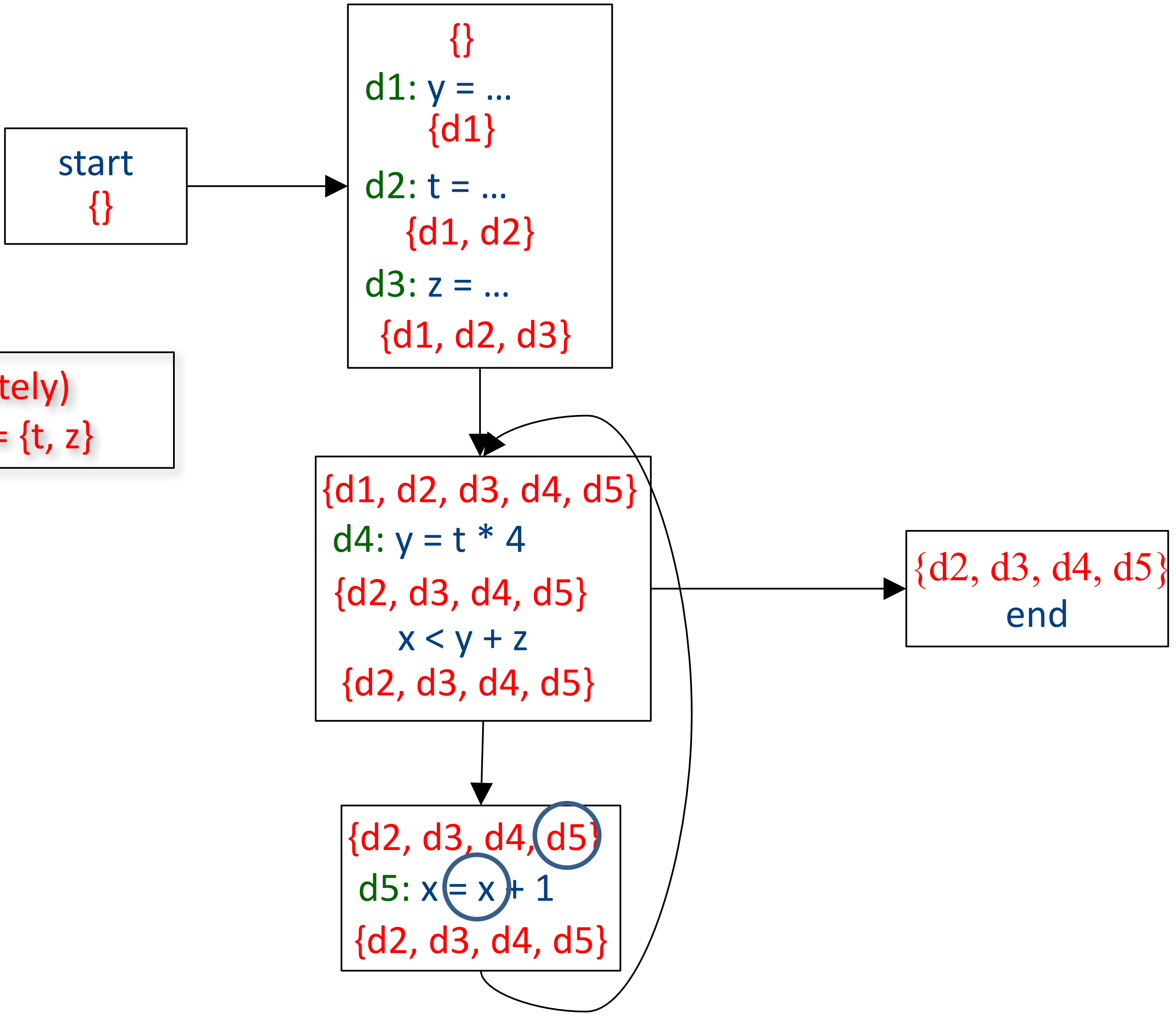
(immediately)
LOOP-INV = {t, z}

Computing LOOP-INV



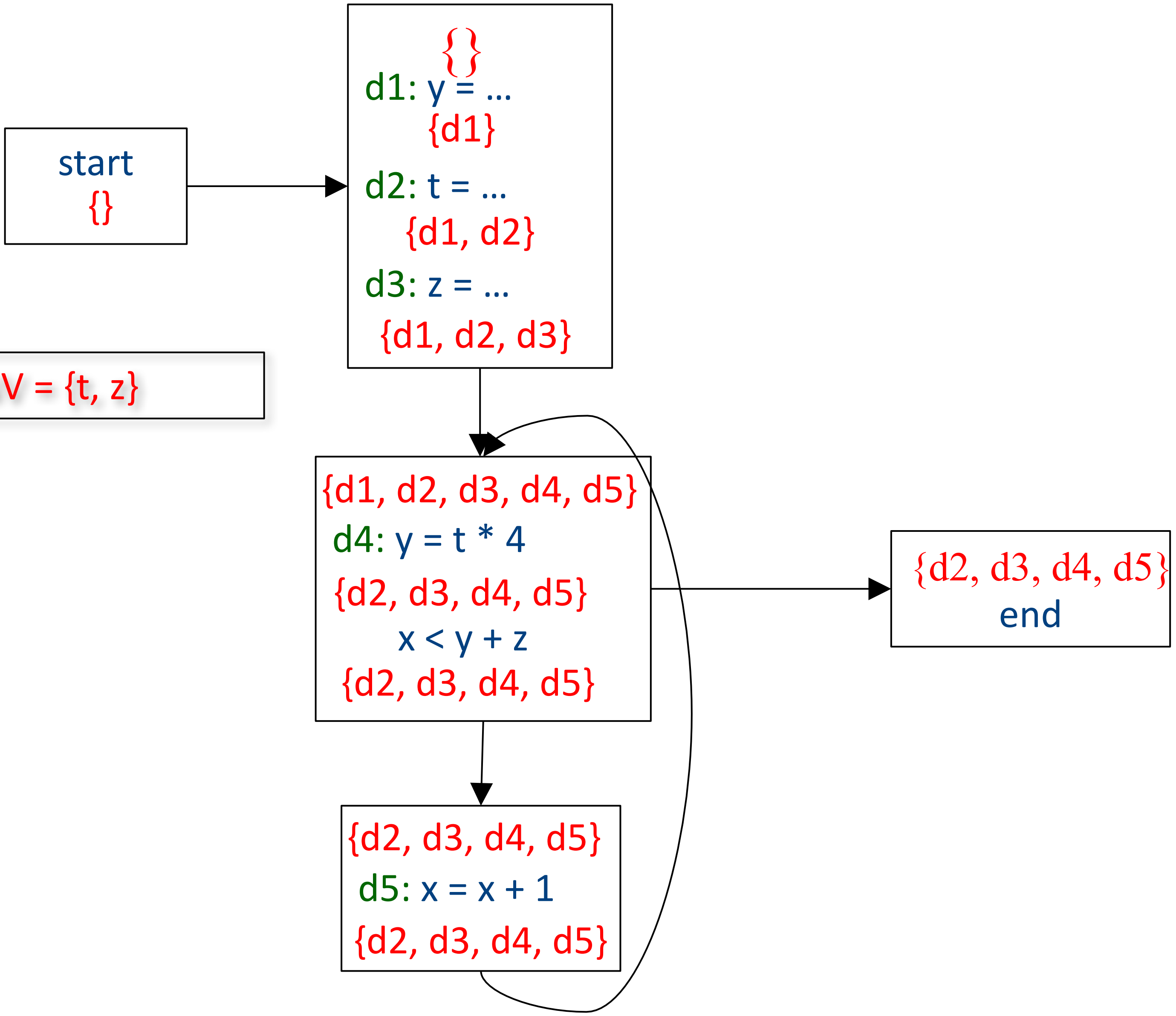
(immediately)
LOOP-INV = {t, z}

Computing LOOP-INV



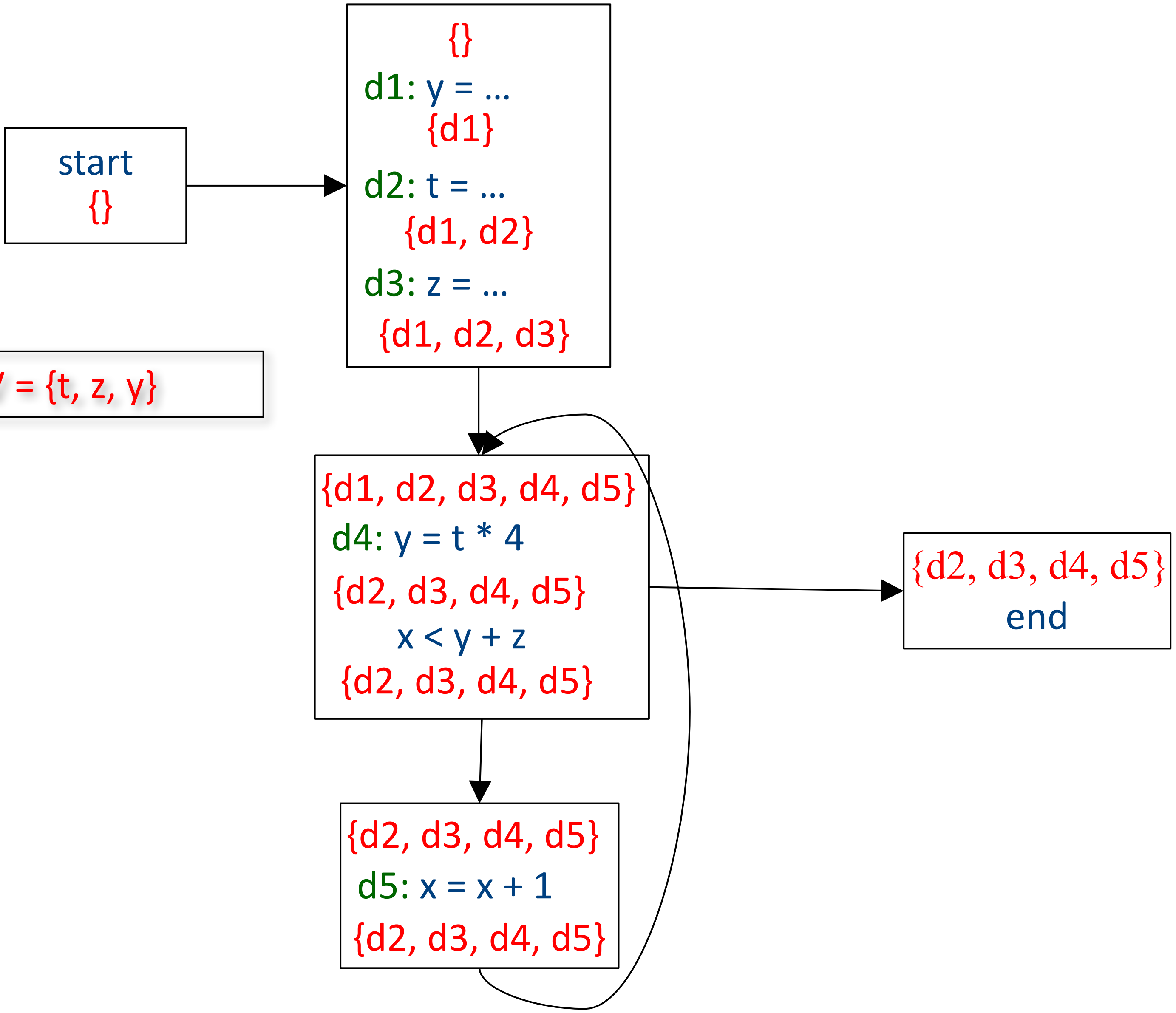
(immediately)
LOOP-INV = {t, z}

Computing LOOP-INV



LOOP-INV = {t, z}

Computing LOOP-INV



LOOP-INV = {t, z, y}

Induction variables

j is a linear function of the induction variable with multiplier 4

```
while (i < x) {
    j = a + 4 * i
    a[j] = j
    i = i + 1
}
```

i is incremented by a loop-invariant expression on each iteration – this is called an induction variable

Strength-reduction

Prepare initial value

```
j = a + 4 * i
while (i < x) {
    j = j + 4
    a[j] = j
    i = i + 1
}
```

Increment by multiplier

Compilation

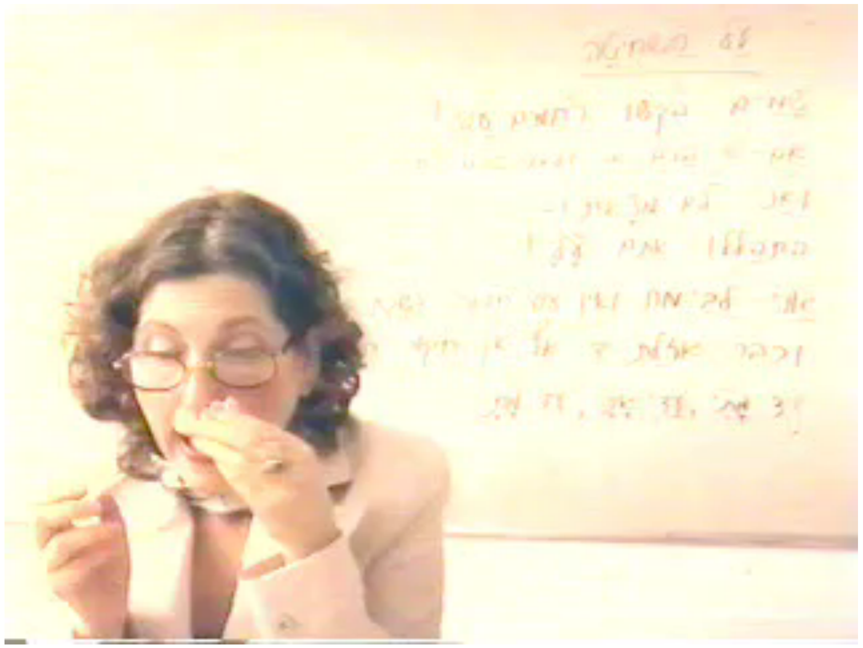
0368-3133

Lecture 10b

Register Allocation

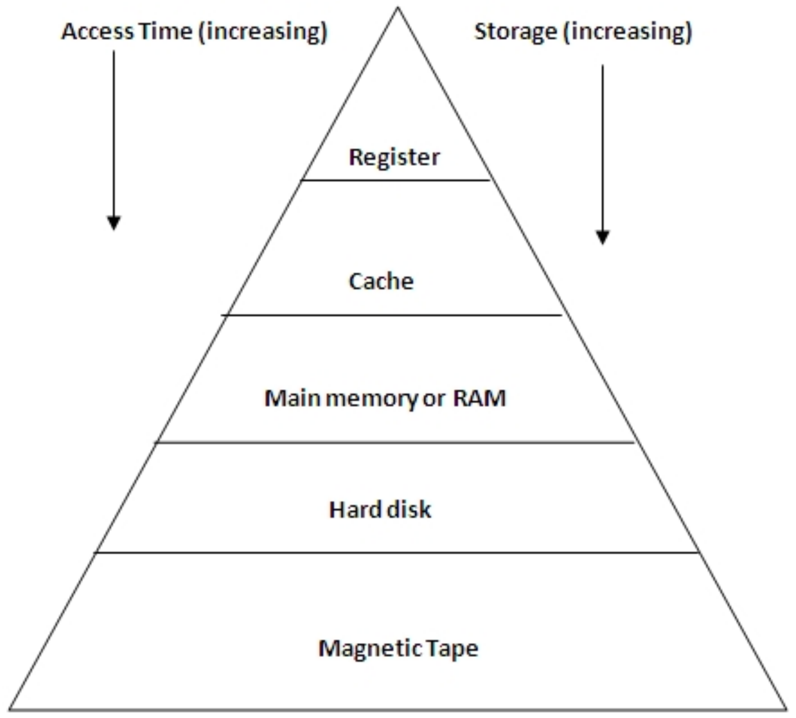
Noam Rinetzky

What is a Compiler?



Registers

- **Dedicated memory** locations that
  - can be accessed quickly,
  - can have computations performed on them, and



Registers

- **Dedicated memory** locations that
  - can be accessed quickly,
  - can have computations performed on them, and

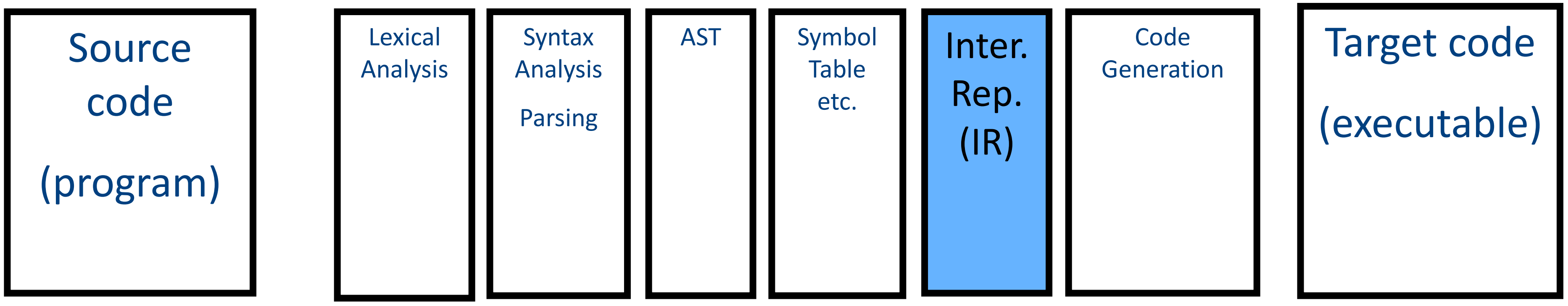
- Usages
  - Operands of instructions
  - Store temporary results
  - Can (should) be used as loop indexes due to frequent arithmetic operation
  - Used to manage administrative info
    - e.g., runtime stack

Register allocation

- Number of registers is **limited**

- Need to **allocate** them in a clever way
  - Using registers intelligently is a critical step in any compiler
    - A good register allocator can generate code orders of magnitude better than a bad register allocator

Register Allocation: IR



Simple approach

- Straightforward solution:
  - Allocate each variable in activation record
  - At each instruction, bring values needed into registers, perform operation, then store result to memory

```
x = y + z
```

```
mov 16(%ebp), %eax
mov 20(%ebp), %ebx
add %ebx, %eax
mov %eax, 24(%ebp)
```

- Problem: program execution very inefficient– moving data back and forth between memory and registers

Simple code generation

- assume machine instructions of the form
- `LD reg, mem`
- `ST mem, reg`
- `OP reg,reg,reg` (*)

- assume that we have all registers available for our use
  - Ignore registers allocated for stack management
  - Treat all registers as general-purpose

Simple code generation

- assume machine instructions of the form
- `LD reg, mem`
- `ST mem, reg`
- `OP reg,reg,reg` (*)

Fixed number of Registers!

Register allocation

- In **TAC**, there is an unlimited number of variables (temporaries)
- On a physical machine there is a small number of registers:
  - **x86** has **4** general-purpose registers and a number of specialized registers
  - **MIPS** has **24** general-purpose registers and **8** special-purpose registers

- Register allocation is the process of assigning variables to registers and managing data transfer in and out of registers

simple code generation

- assume machine instructions of the form
- `LD reg, mem`
- `ST mem, reg`
- `OP reg,reg,reg` (*)

Fixed number of Registers!

- We will assume that we have all registers available for any usage
  - Ignore registers allocated for stack management
  - Treat all registers as general-purpose

Plan

- Goal: Reduce number of temporaries (registers)
  - Machine-agnostic optimizations
    - Assume unbounded number of registers
  - Machine-dependent optimization
    - Use at most K registers
    - K is machine dependent

Sethi-Ullman translation

- Algorithm by Ravi Sethi and Jeffrey D. Ullman to emit optimal TAC
  - Minimizes number of temporaries for a **single expression**

Generating Compound Expressions

- Use registers to store temporaries
  - Why can we do it?

- Maintain a counter for temporaries in c
- Initially: c = 0
- **cgen**(e_1 *op* e_2) = {

```
    Let A = cgen(e_1)
    c = c + 1
    Let B = cgen(e_2)
    c = c + 1
    Emit( _tc = A op B; ) // _tc is a register
    Return _tc
}
```

Improving **cgen** for expressions

- Observation – naïve translation needlessly generates temporaries for leaf expressions
- Observation – temporaries used exactly once
  - Once a temporary has been read it can be reused for another sub-expression
- **cgen**(e_1 *op* e_2) = {

```
  Let _t1 = cgen(e_1)
  Let _t2 = cgen(e_2)
  Emit( _t1 =_t1 op _t2; )
  Return _t1
}
```

- Temporaries **cgen**(e_1) can be reused in **cgen**(e_2)

Register Allocation

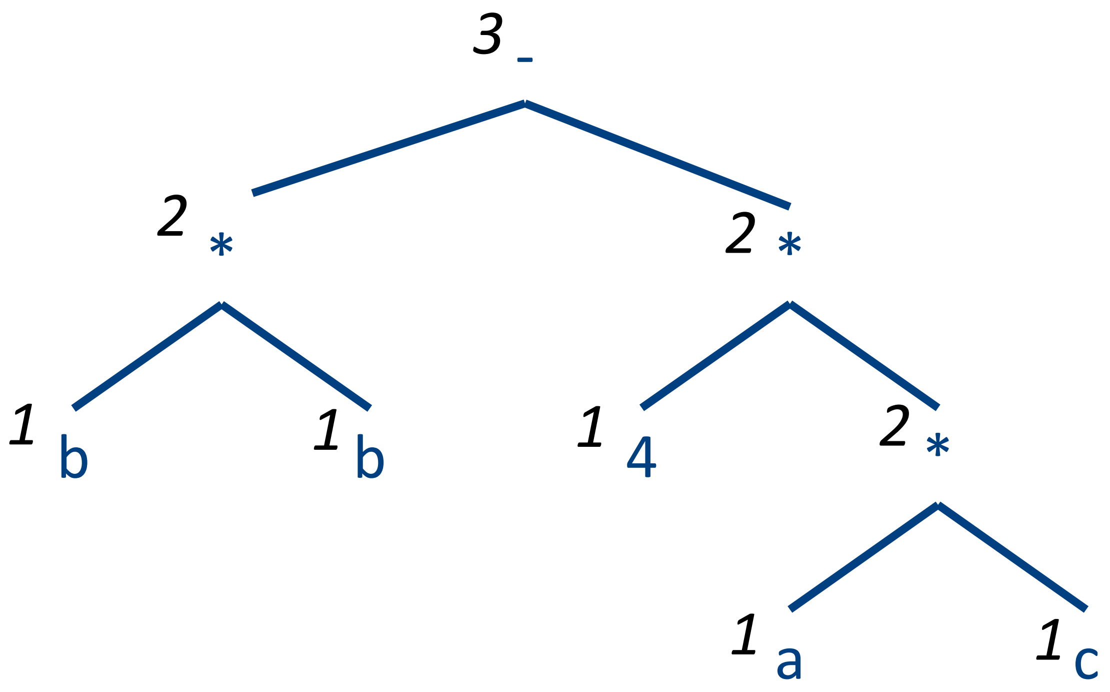
- Machine-agnostic optimizations
    - Assume unbounded number of registers
  - Expression trees
  - Basic blocks

- Machine-dependent optimization
    - K registers
    - Some have special purposes
  - Control flow graphs (whole program)

Sethi-Ullman translation

- Algorithm by Ravi Sethi and Jeffrey D. Ullman to emit optimal TAC
  - Minimizes number of temporaries for a **single expression**

Example (optimized): b*b-4*a*c



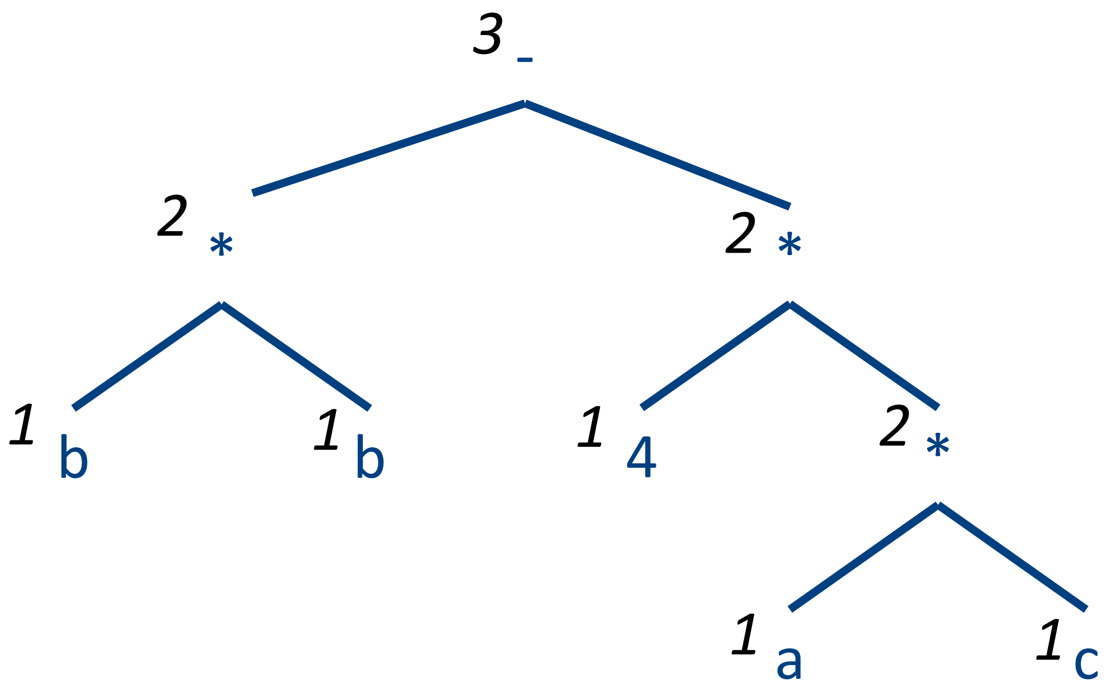
Generalizations

- More than two arguments for operators
  - Function calls
- Multiple effected registers
  - Multiplication
- Spilling
  - Need more registers than available
- Register/memory operations

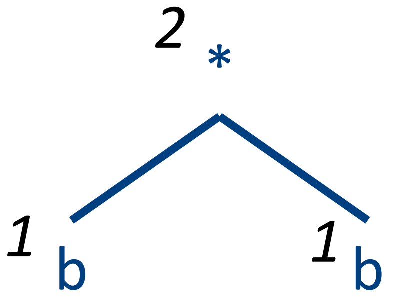
Simple **Spilling** Method

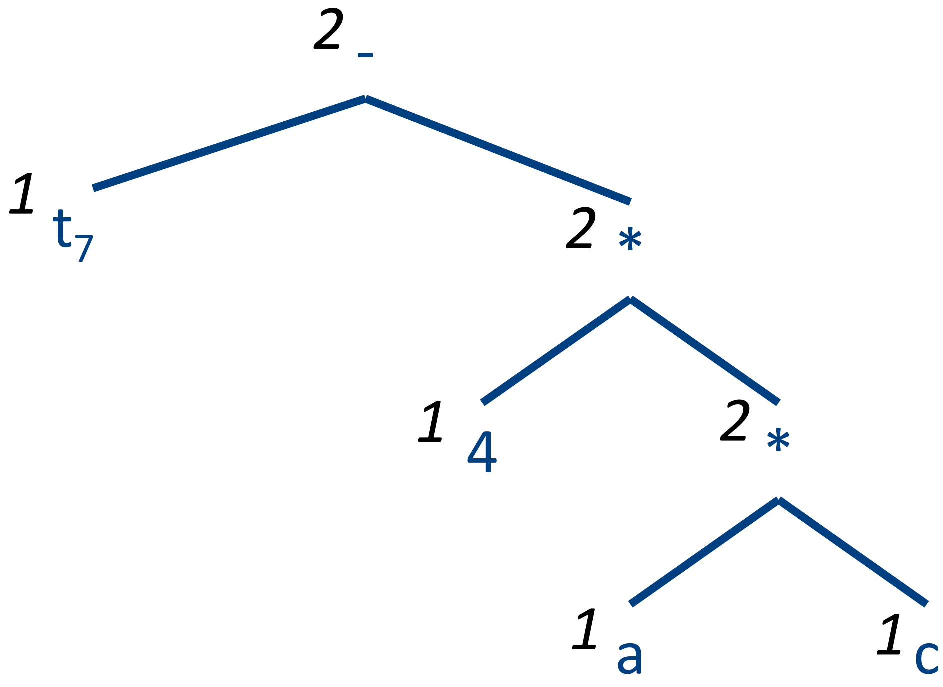
- Heavy tree – Needs more registers than available
- A “heavy” tree contains a “heavy” subtree whose dependents are “light”
- Simple spilling
  - Generate code for the light tree
  - Spill the content into memory and replace subtree by temporary
  - Generate code for the resultant tree

Example (optimized): x:=b*b-4*a*c



Example (spilled): x := b*b-4*a*c

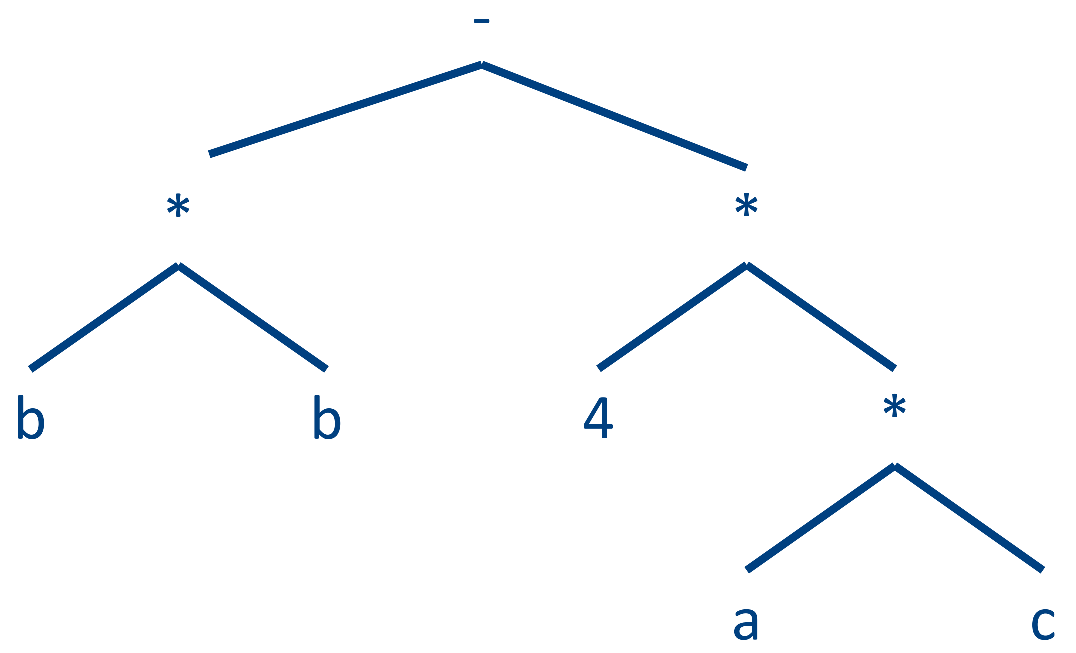




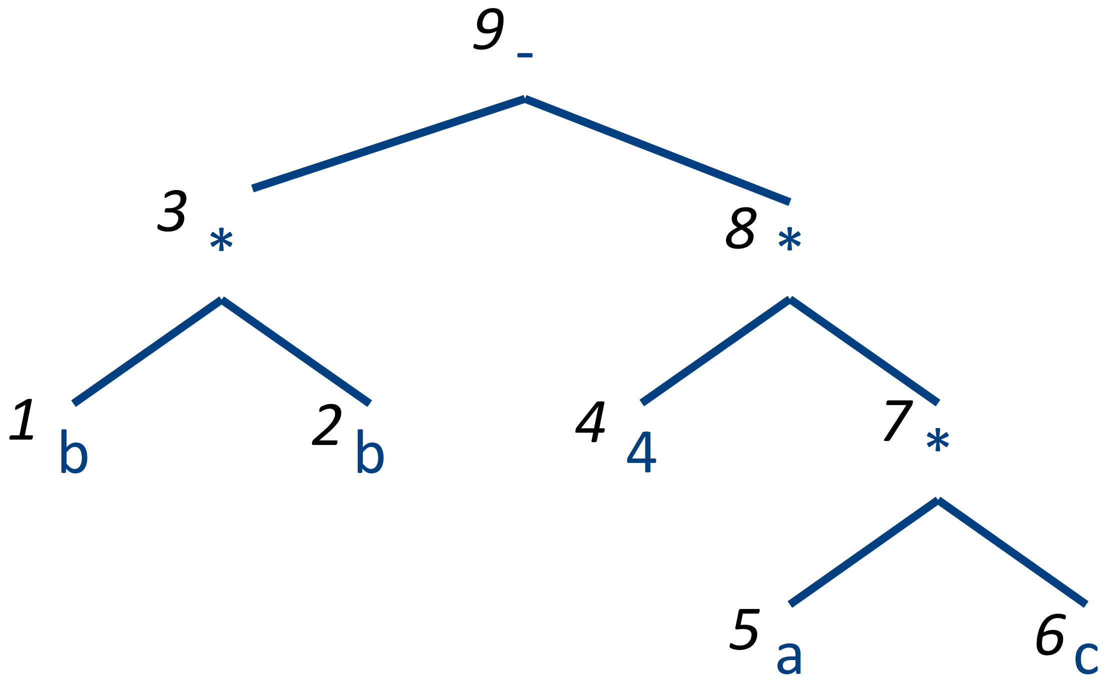
```
t7 := b * b
```

```
x := t7 - 4 * a * c
```

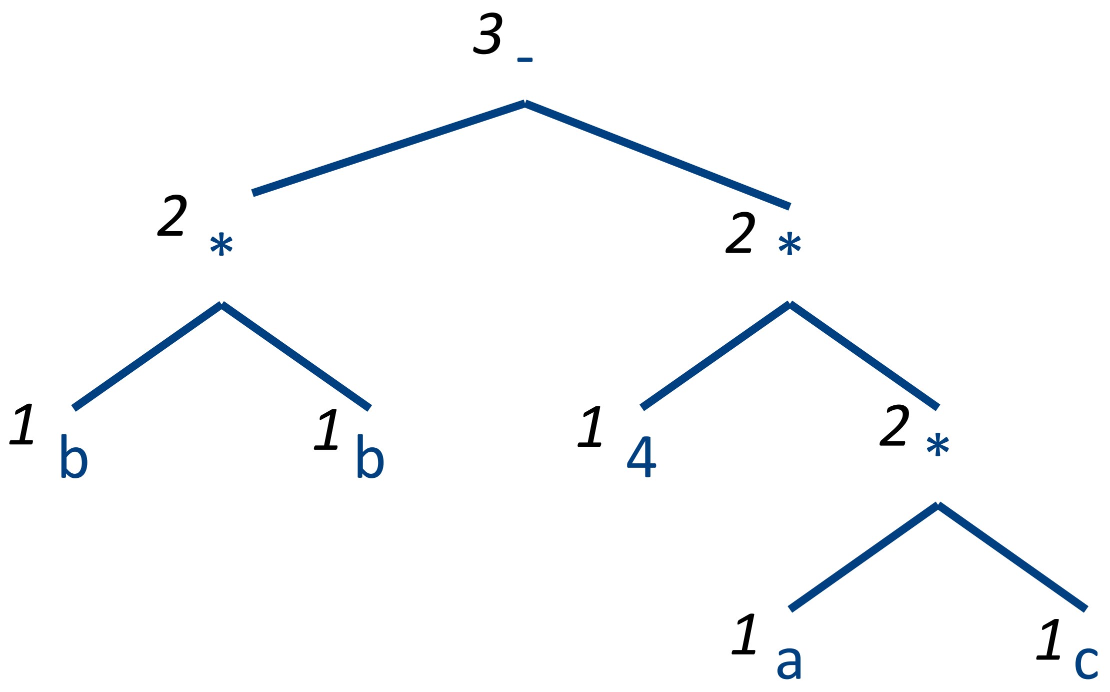
Example: b*b-4*a*c



Example (simple): b*b-4*a*c



Example (optimized): b*b-4*a*c



Spilling

- Even an optimal register allocator can require more registers than available
- Need to generate code for every correct program
- The compiler can save temporary results
  - Spill registers into temporaries
  - Load when needed
- Many heuristics exist

Simple Spilling Method

- Heavy tree – Needs more registers than available
- A `heavy’ tree contains a `heavy’ subtree whose dependents are ‘light’
- Generate code for the light tree
- Spill the content into memory and replace subtree by temporary
- Generate code for the resultant tree

Spilling

- Even an optimal register allocator can require more registers than available
- Need to generate code for every correct program
- The compiler can save temporary results
  - Spill registers into temporaries
  - Load when needed
- Many heuristics exist

Simple approach

- Straightforward solution:
  - Allocate each variable in activation record
  - At each instruction, bring values needed into registers, perform operation, then store result to memory

```
x = y + z
```

$\Rightarrow$

```
mov 16(%ebp), %eax
mov 20(%ebp), %ebx
add %ebx, %eax
mov %eax, 24(%ebx)
```

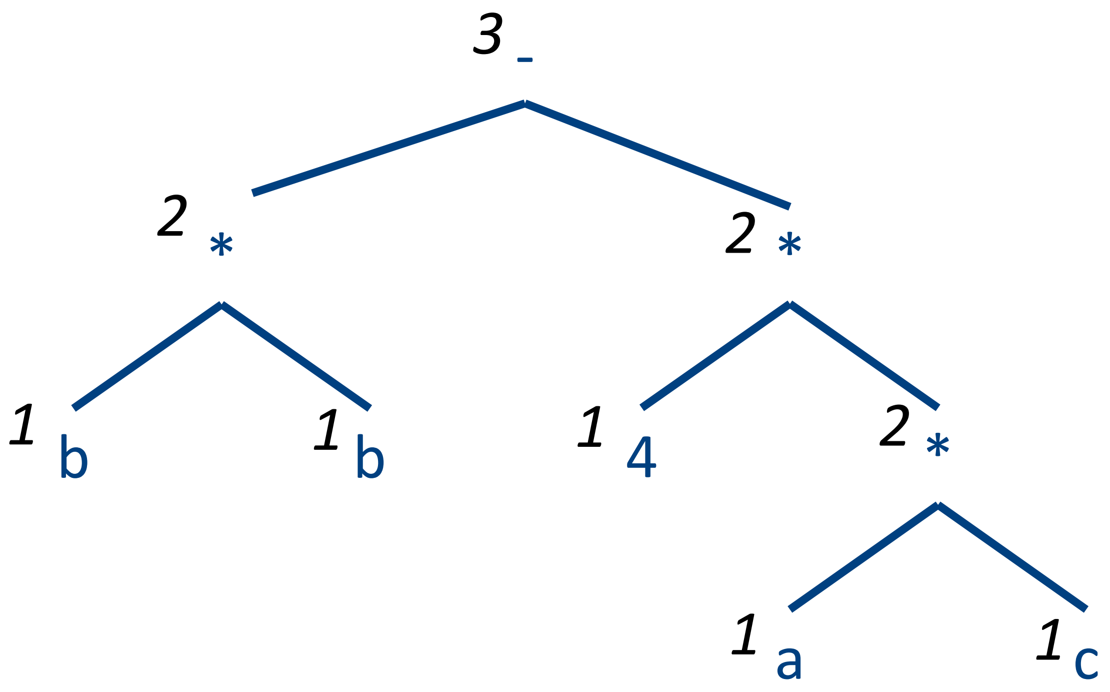
- Problem: program execution very inefficient– moving data back and forth between memory and registers

Register Allocation

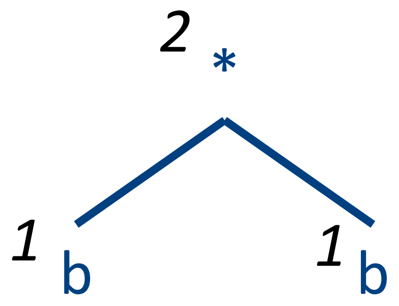
- Machine-agnostic optimizations
    - Assume unbounded number of registers
  - Expression trees (tree-local)
  - Basic blocks (block-local)

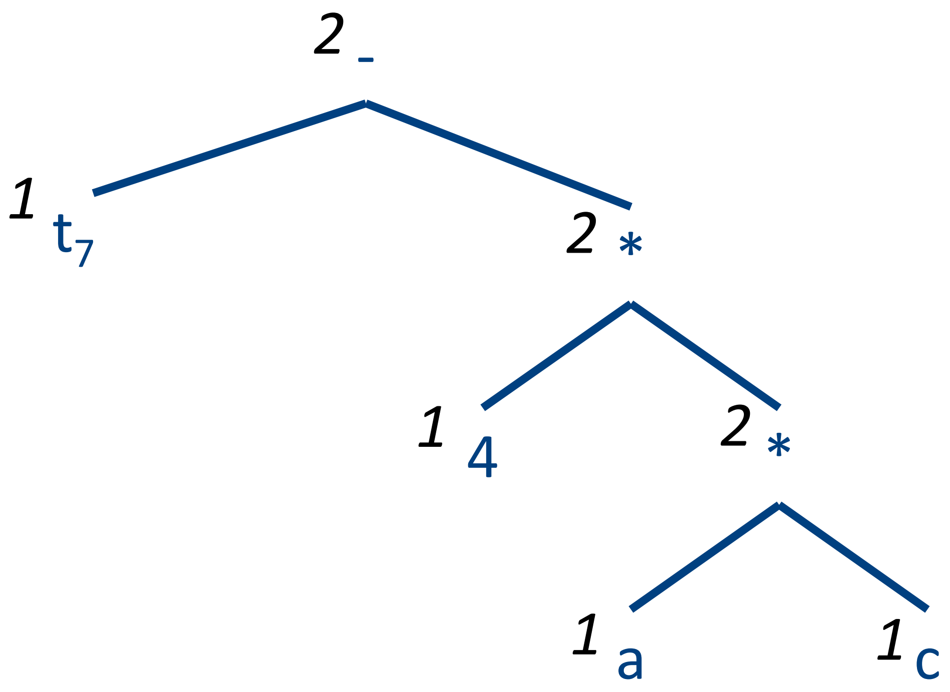
- Machine-dependent optimization
    - K registers
    - Some have special purposes
  - Control flow graphs (global register allocation)

Example (optimized): b*b-4*a*c



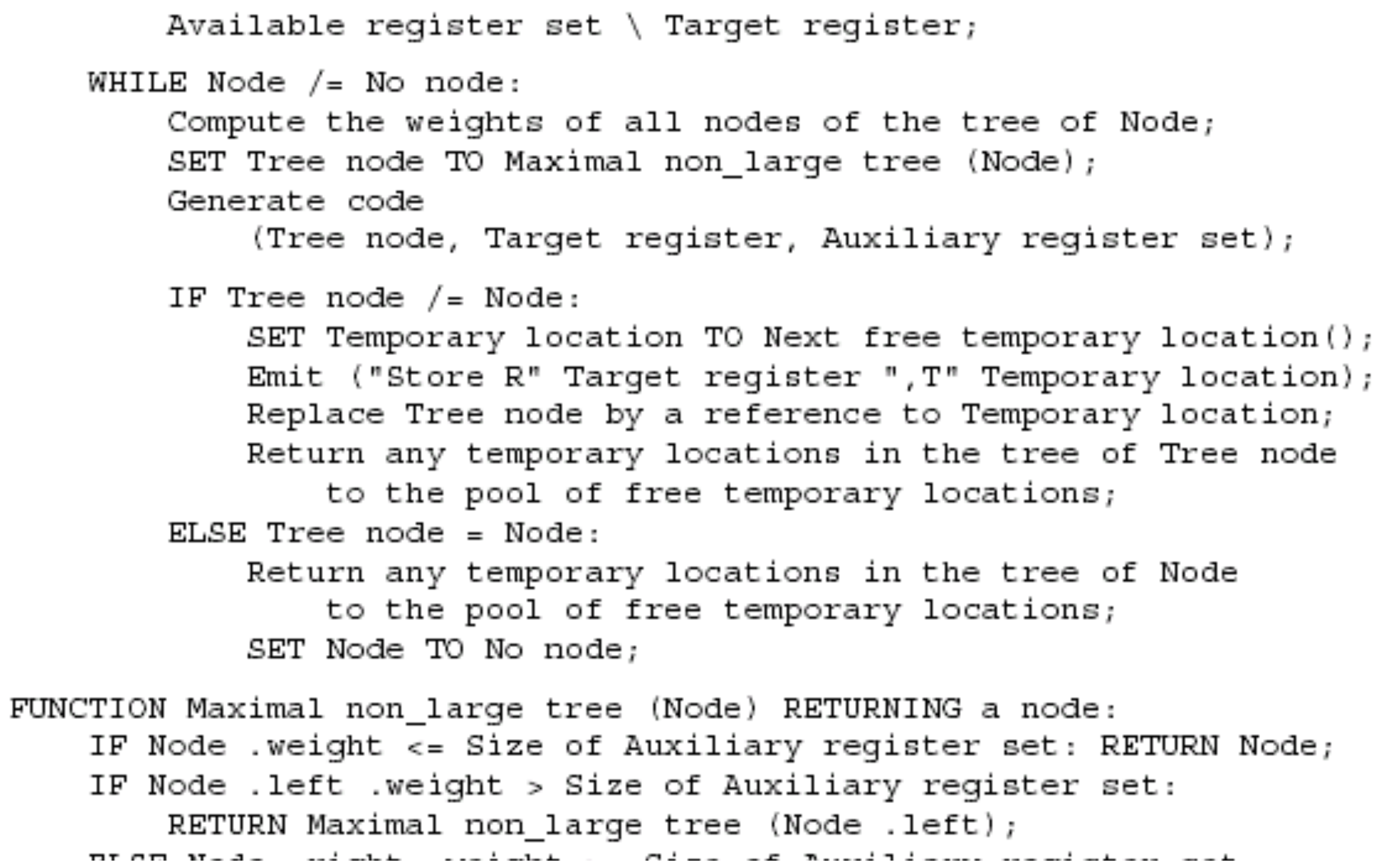
Example (spilled): x := b*b-4*a*c





```
t7 := b * b                x := t7 - 4 * a * c
```

Simple Spilling Method



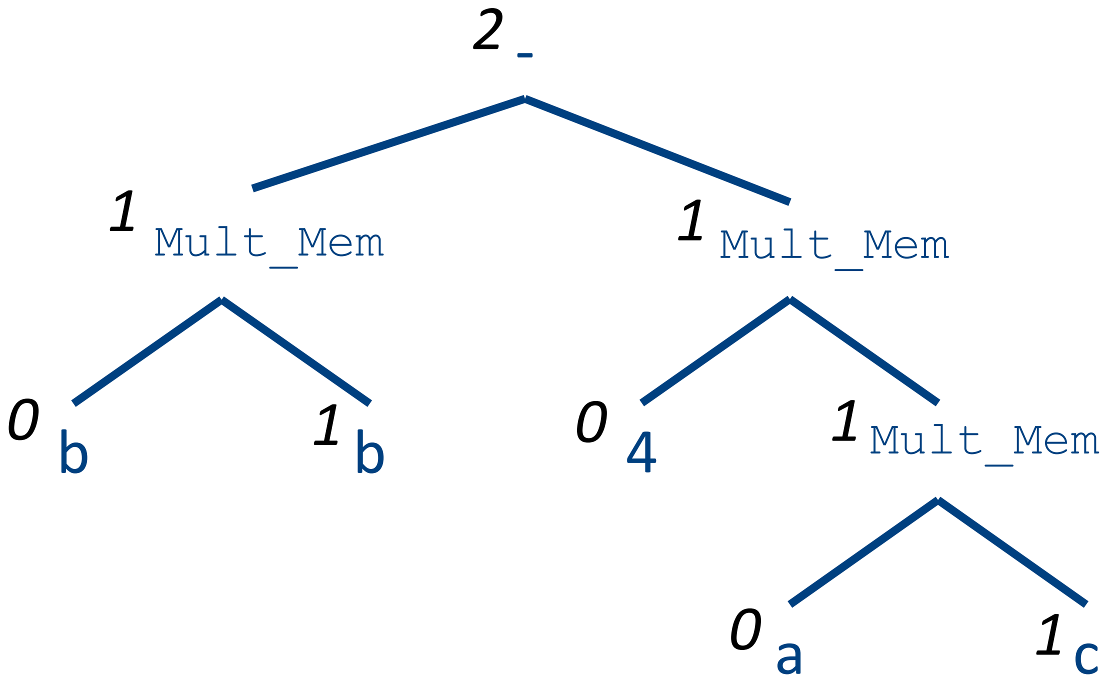
```
        Available register set \ Target register;

    WHILE Node /= No node:
        Compute the weights of all nodes of the tree of Node;
        SET Tree node TO Maximal non_large tree (Node);
        Generate code
            (Tree node, Target register, Auxiliary register set);

        IF Tree node /= Node:
            SET Temporary location TO Next free temporary location();
            Emit ("Store R" Target register ",T" Temporary location);
            Replace Tree node by a reference to Temporary location;
            Return any temporary locations in the tree of Tree node
                to the pool of free temporary locations;
        ELSE Tree node = Node:
            Return any temporary locations in the tree of Node
                to the pool of free temporary locations;
            SET Node TO No node;

FUNCTION Maximal non_large tree (Node) RETURNING a node:
    IF Node .weight <= Size of Auxiliary register set: RETURN Node;
    IF Node .left .weight > Size of Auxiliary register set:
        RETURN Maximal non_large tree (Node .left);
    ELSE Node .right .weight >= Size of Auxiliary register set
```

Register Memory Operations

- Add_Mem X, R1
- Mult_Mem X, R1
- No need for registers to store right operands

Hidden Registers

Example: b*b-4*a*c



Can We do Better?

- Yes: Increase view of code
  - Simultaneously allocate registers for multiple expressions

- But: Lose per expression optimality
  - Works well in practice

Register Allocation

- Machine-agnostic optimizations
    - Assume unbounded number of registers
  - – Expression trees
  - – Basic blocks

- Machine-dependent optimization
    - K registers
    - Some have special purposes
  - – Control flow graphs (whole program)

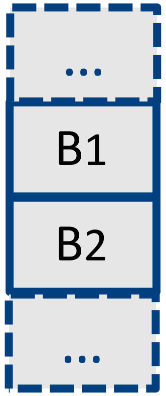
Basic Blocks

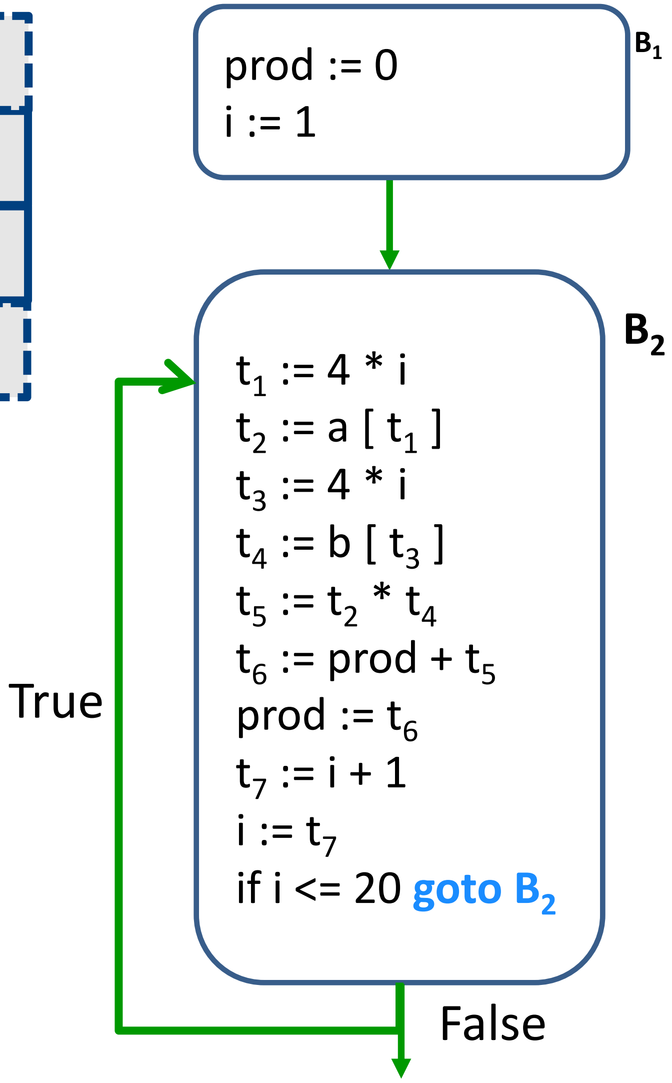
- **basic block** is a sequence of instructions with
  - **single entry** (to first instruction), no jumps to the middle of the block
  - **single exit** (last instruction)
  - code execute as a sequence from first instruction to last instruction without any jumps
- edge from one basic block B1 to another block B2 when the last statement of B1 may jump to B2

control flow graph

- A directed graph G=(V,E)
- nodes V = basic blocks
- edges E = control flow
  - (B1,B2) $\in$ E when control from B1 flows to B2

- Leaders-based construction
  - Target of jump instructions
  - Instructions following jumps





control flow graph

- A directed graph G=(V,E)
- nodes V = basic blocks
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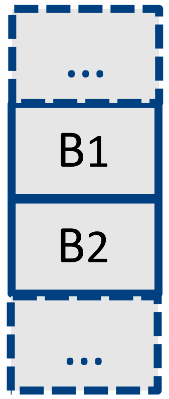
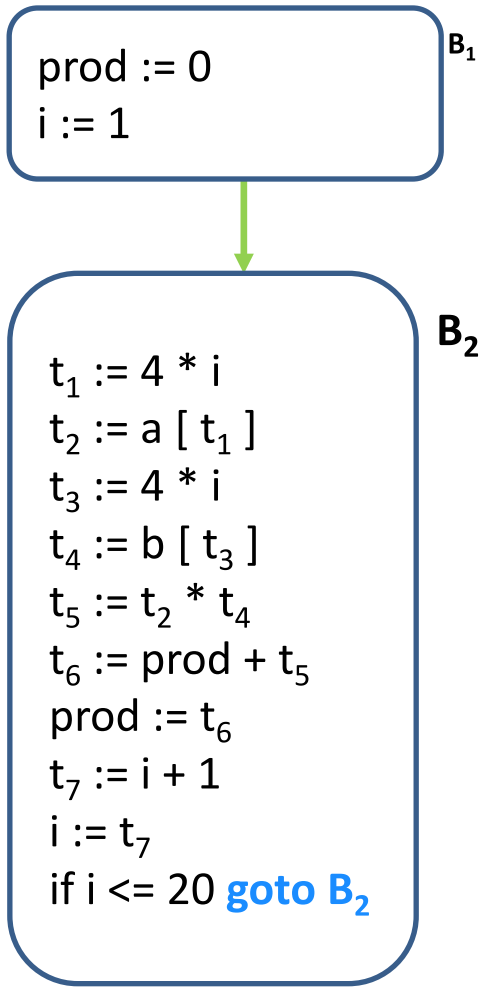
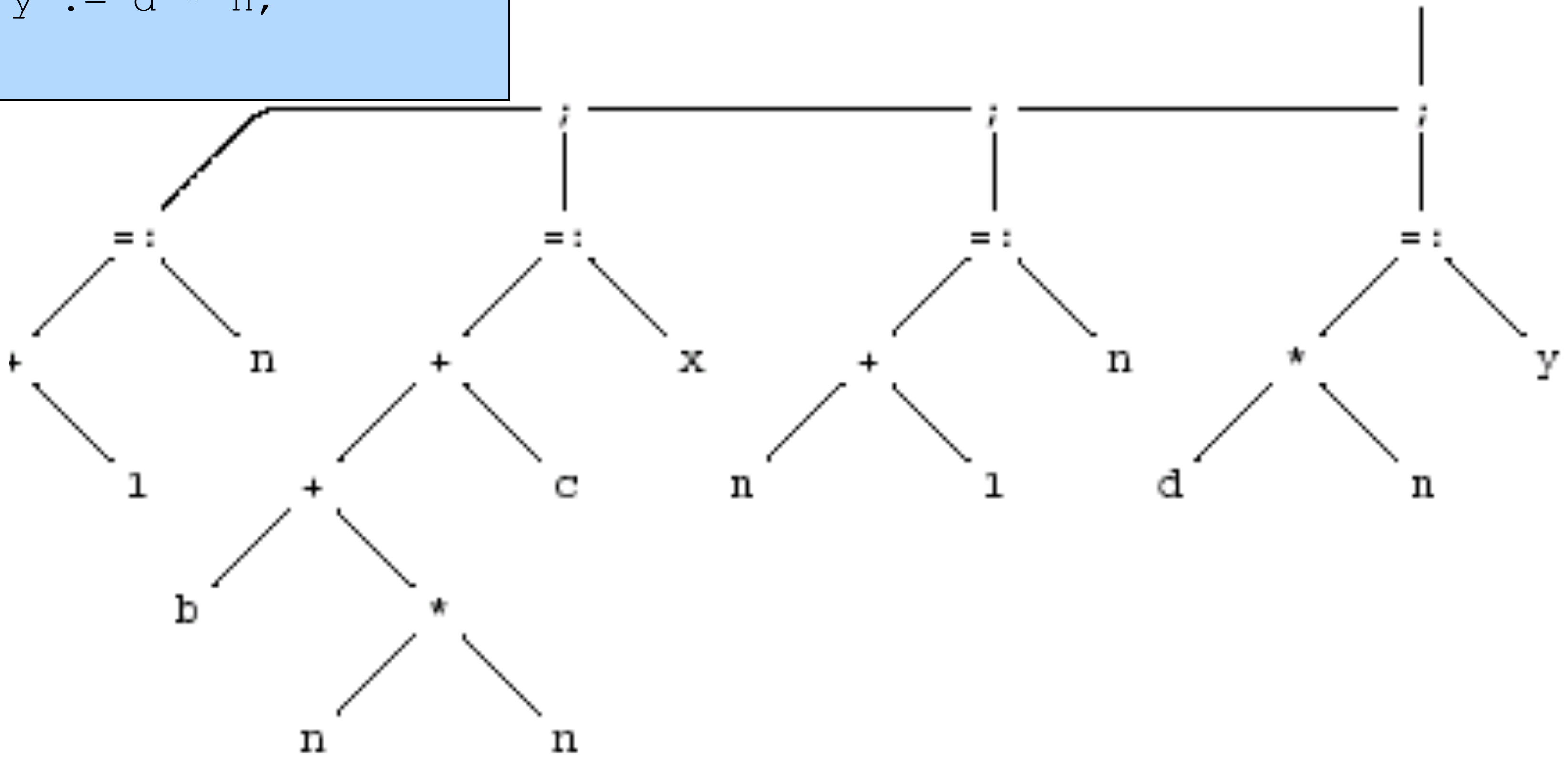




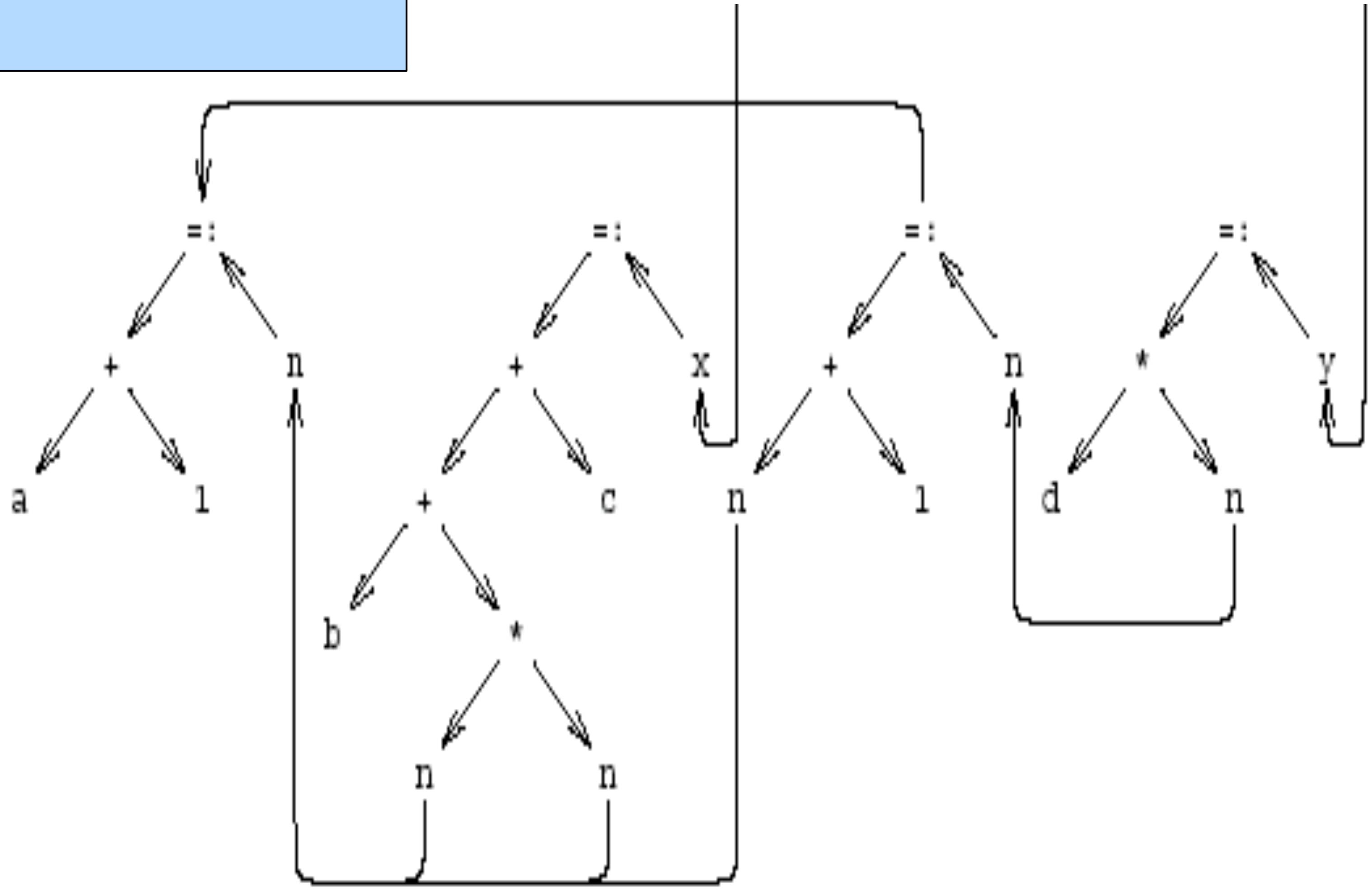
AST for a Basic Block

```
{
    int n;
    n := a + 1;
    x := b + n * n + c;
    n := n + 1;
    y := d * n;
}
```



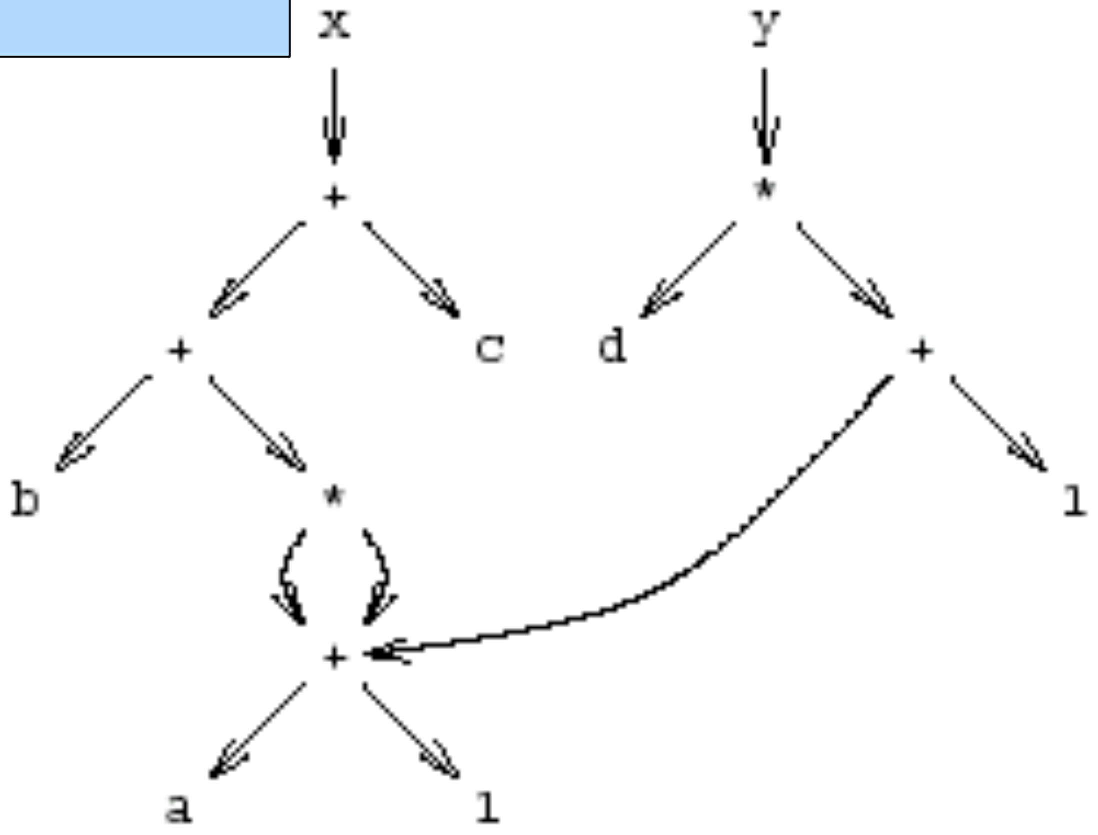
Dependency graph

```
{
    int n;
    n := a + 1;
    x := b + n * n + c;
    n := n + 1;
    y := d * n;
}
```

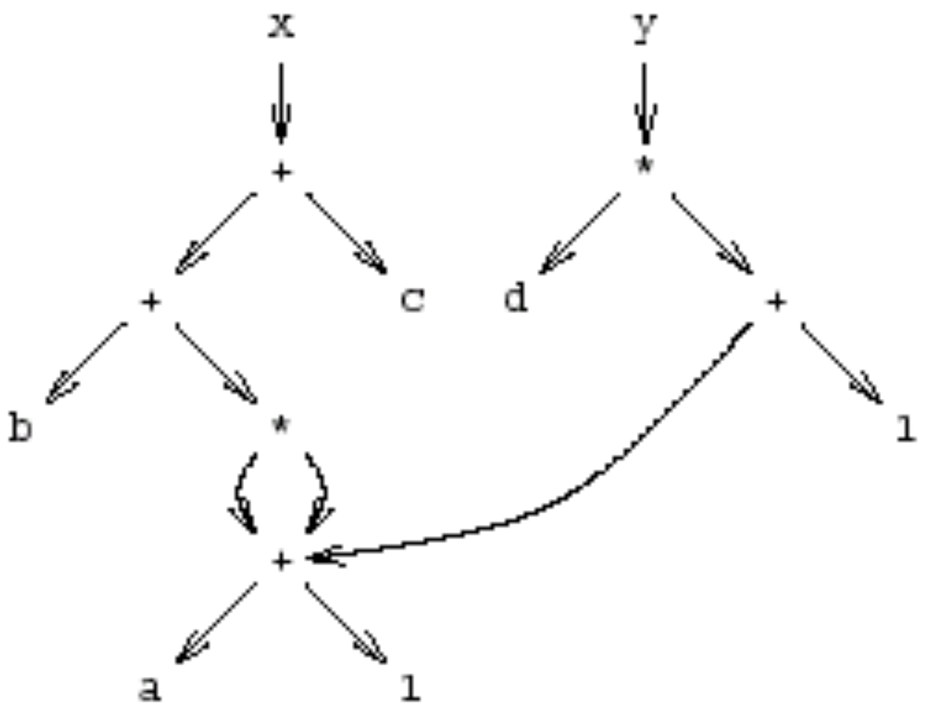


```
{
    int n;
    n := a + 1;
    x := b + n * n + c;
    n := n + 1;
    y := d * n;
}
```

Simplified Data Dependency Graph



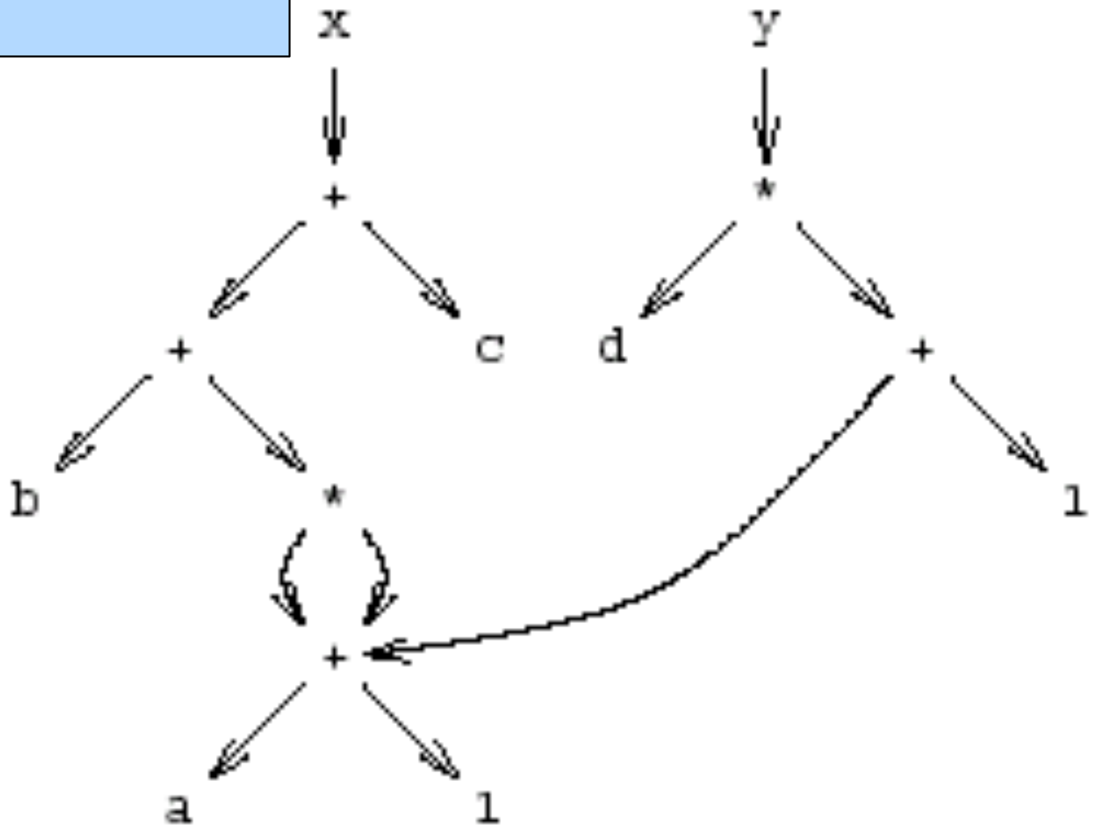
Pseudo Register Target Code



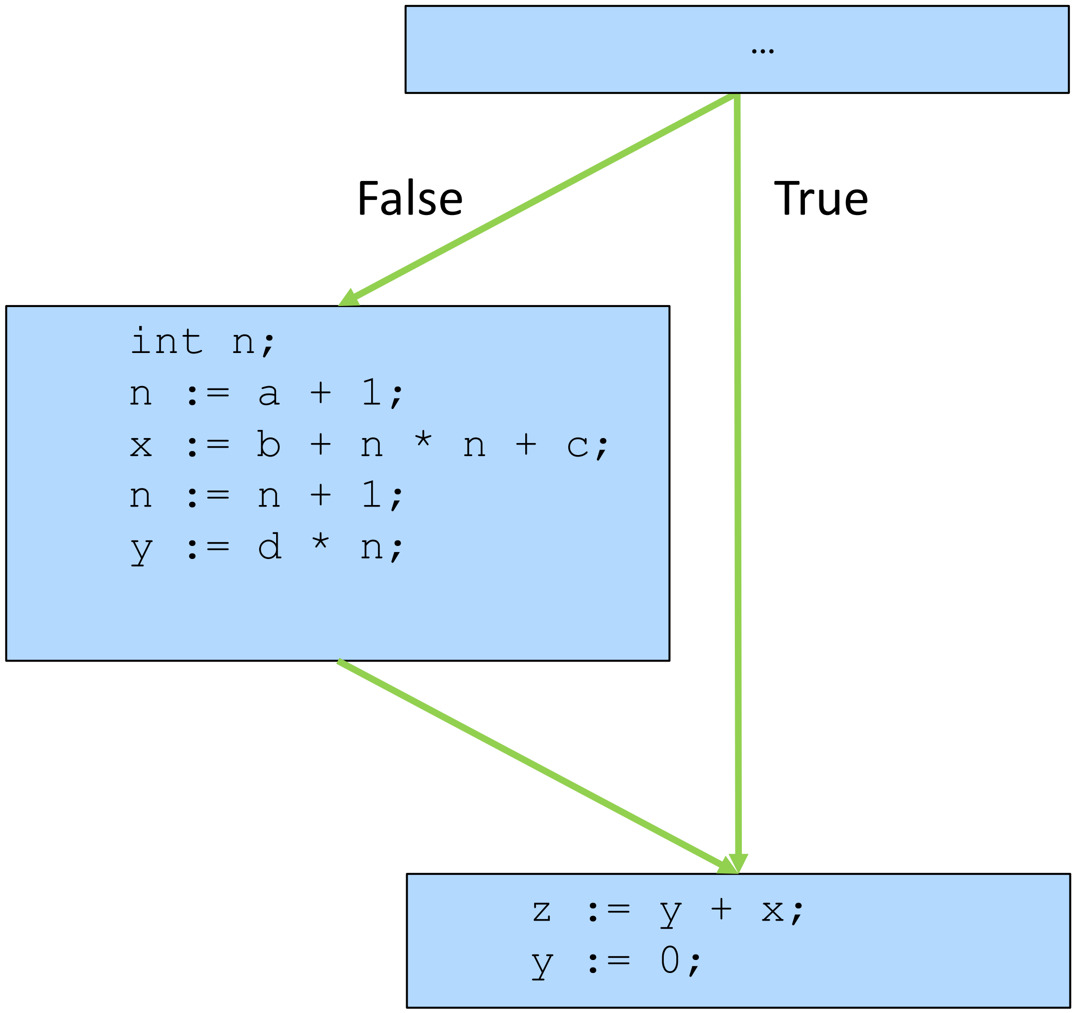
```
Load_Mem     a,R1
Add_Const    1,R1
Load_Reg     R1,X1

Load_Reg     X1,R1
Mult_Reg     X1,R1
Add_Mem      b,R1
Add_Mem      c,R1
Store_Reg    R1,x

Load_Reg     X1,R1
Add_Const    1,R1
Mult_Mem     d,R1
Store_Reg    R1,y
```

```
{
    int n;
    n := a + 1;
    x := b + n * n + c;
    n := n + 1;
    y := d * n;
}
```

Question: Why "`y`"?



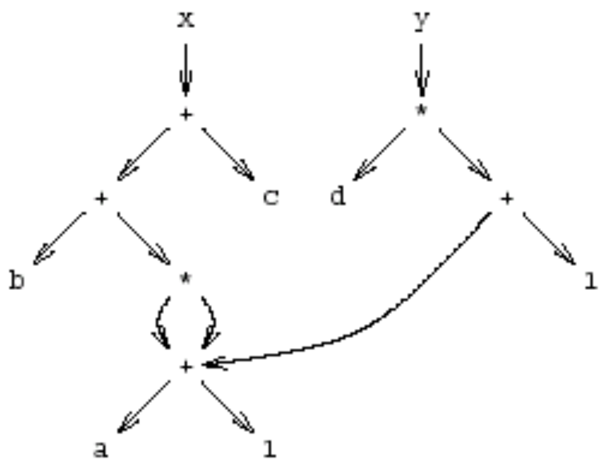
Question: Why "`y`"?



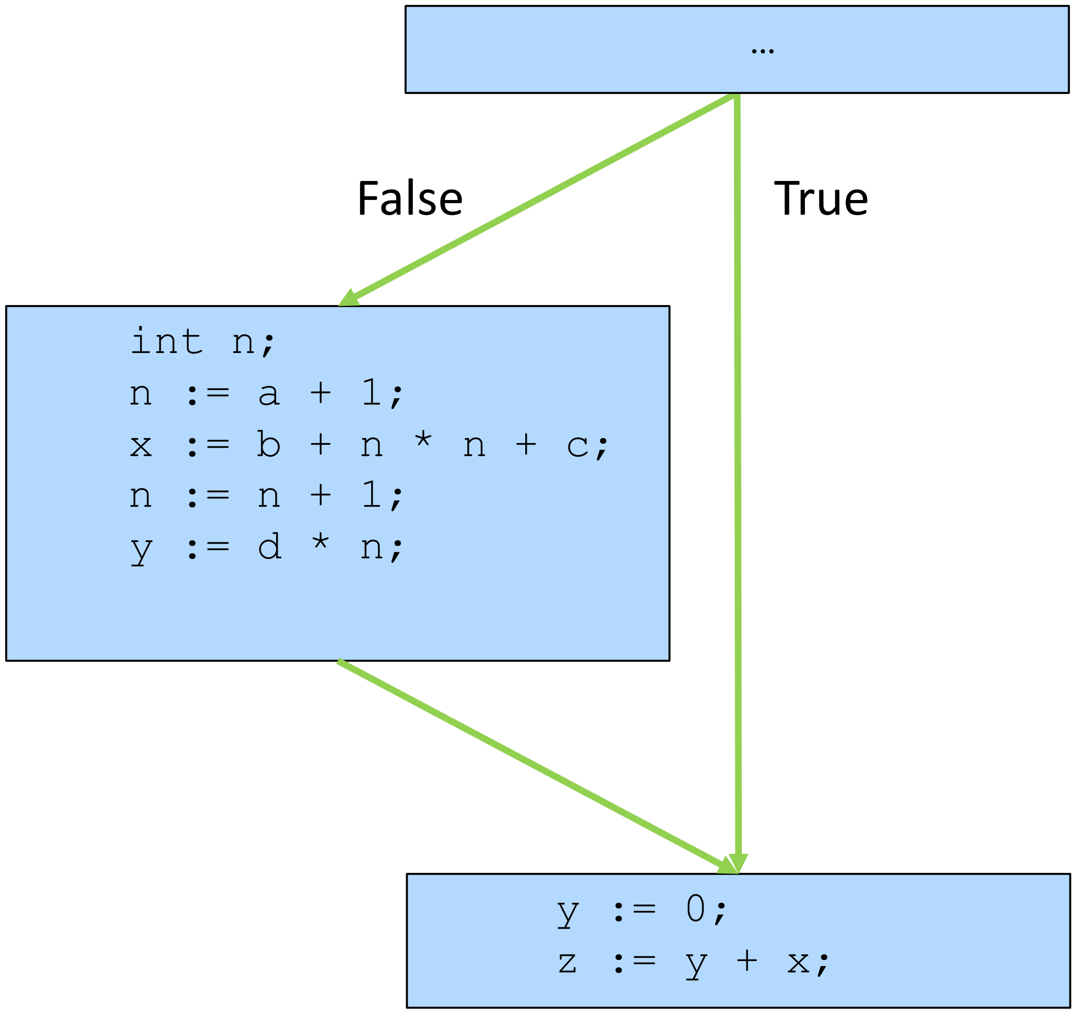
```
…
```

```
int n;
n := a + 1;
x := b + n * n + c;
n := n + 1;
y := d * n;
```

```
z := y + x;
y := 0;
```

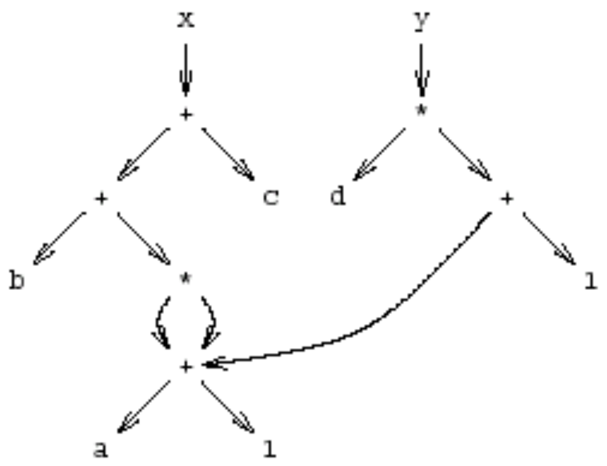


Question: Why "y"?

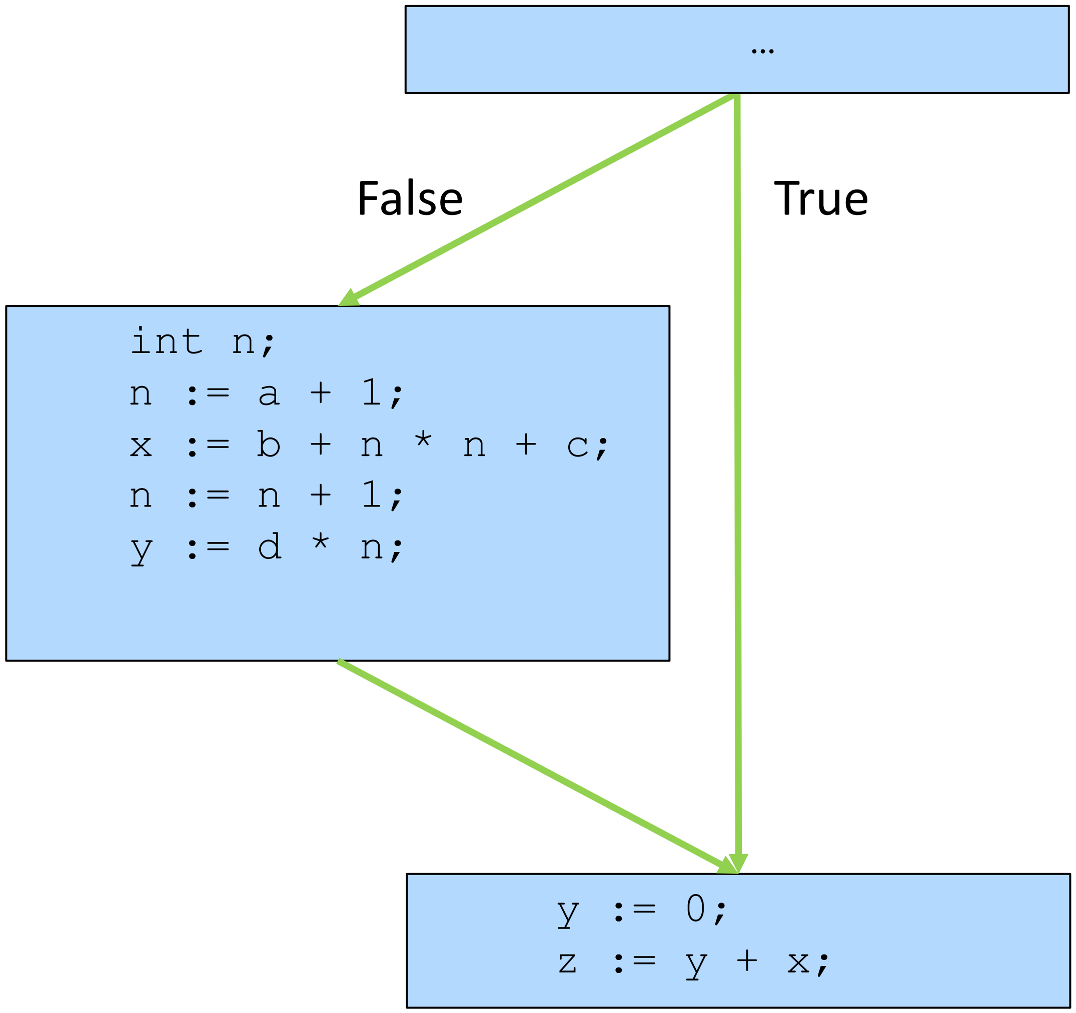


```
int n;
n := a + 1;
x := b + n * n + c;
n := n + 1;
y := d * n;
```

```
y := 0;
z := y + x;
```

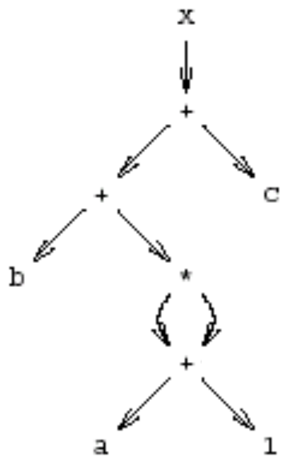


Question: Why "y"?

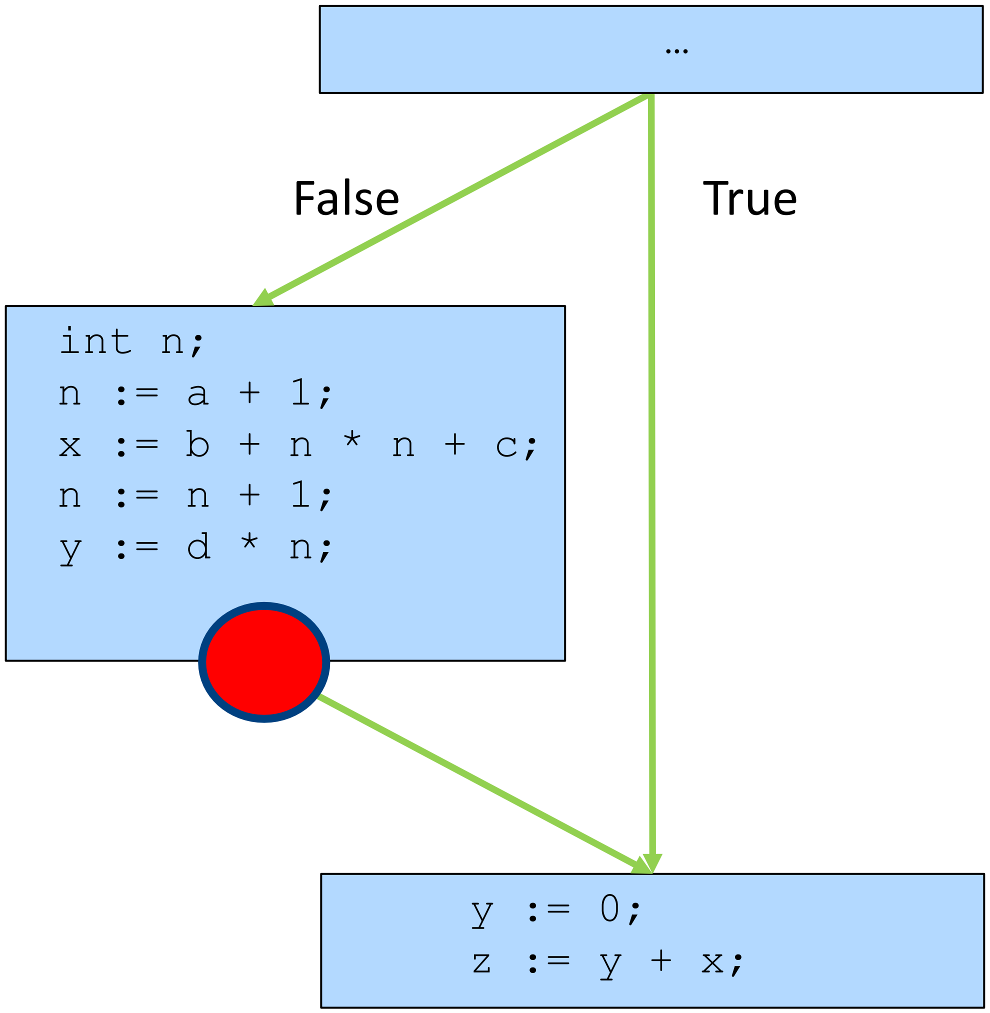


```
int n;
n := a + 1;
x := b + n * n + c;
n := n + 1;
y := d * n;
```

```
y := 0;
z := y + x;
```

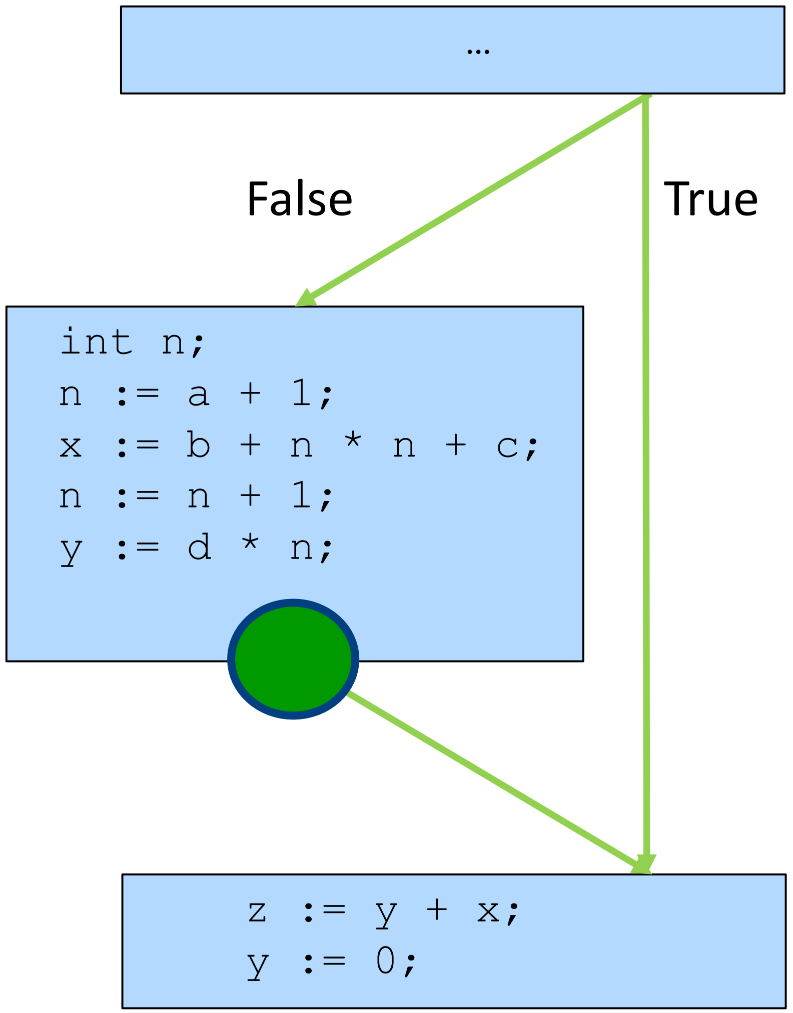


y, dead or alive?



```
int n;
n := a + 1;
x := b + n * n + c;
n := n + 1;
y := d * n;
```

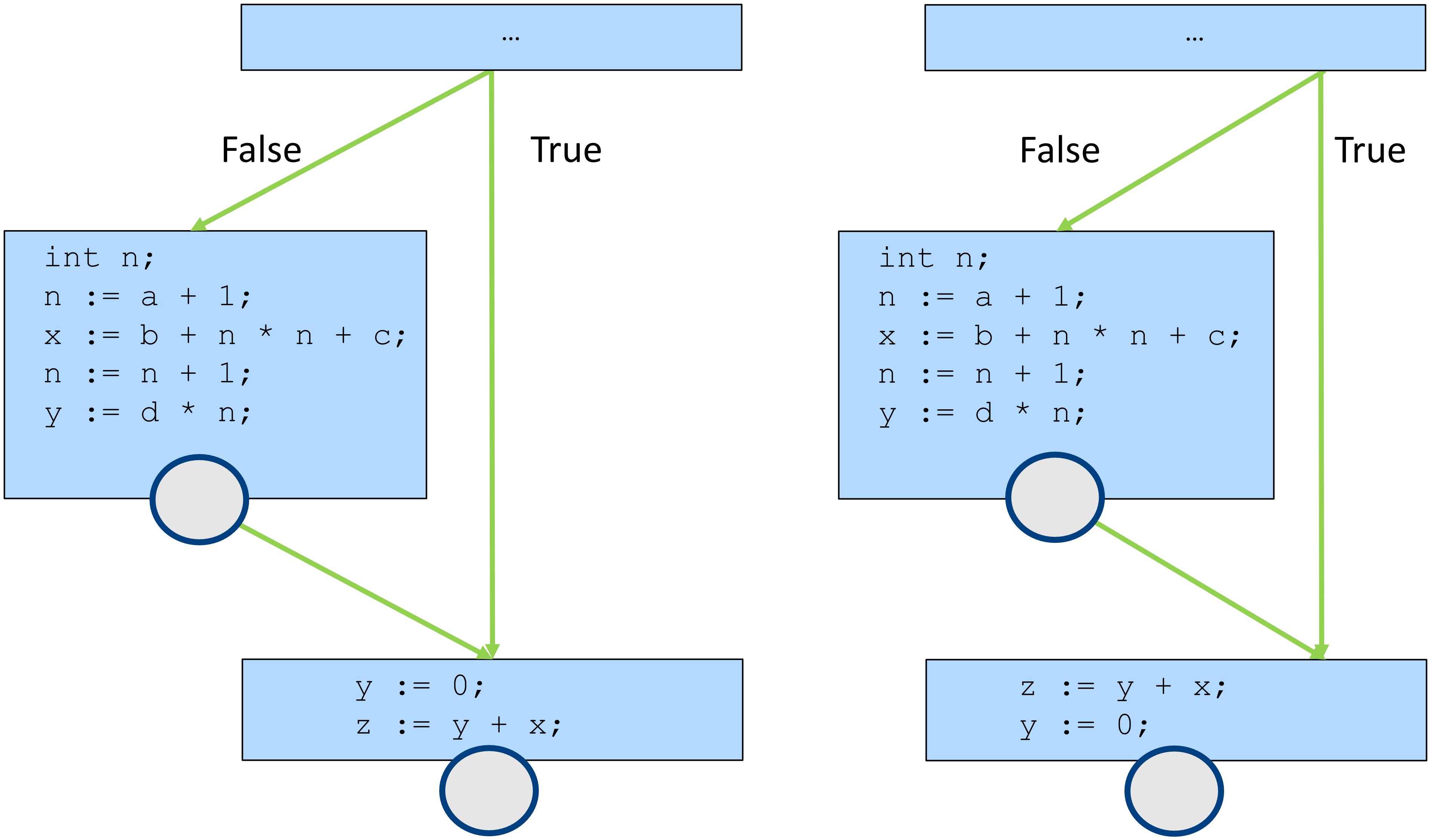
```
y := 0;
z := y + x;
```



```
int n;
n := a + 1;
x := b + n * n + c;
n := n + 1;
y := d * n;
```

```
z := y + x;
y := 0;
```

`x`, dead or alive?



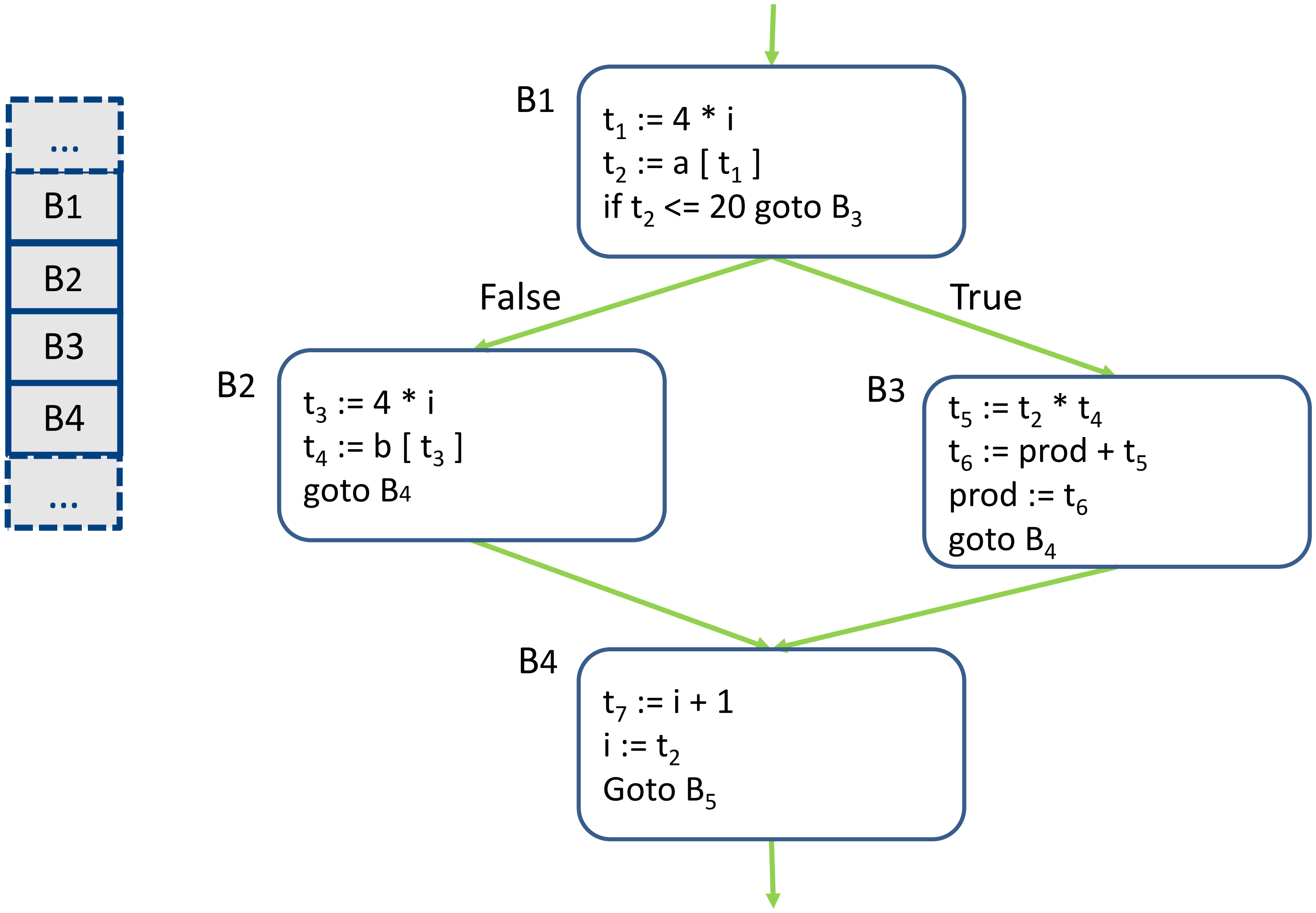
```
int n;
n := a + 1;
x := b + n * n + c;
n := n + 1;
y := d * n;
```

```
y := 0;
z := y + x;
```

```
int n;
n := a + 1;
x := b + n * n + c;
n := n + 1;
y := d * n;
```

```
z := y + x;
y := 0;
```

Another Example



Creating Basic Blocks

- **Input**: A sequence of three-address statements
- **Output**: A list of basic blocks with each three-address statement in exactly one block
- **Method**
  - Determine the set of **leaders** (first statement of a block)
    - The first statement is a leader
    - Any statement that is the target of a jump is a leader
    - Any statement that immediately follows a jump is a leader
  - For each leader, its basic block consists of the leader and all statements up to but not including the next leader or the end of the program

example

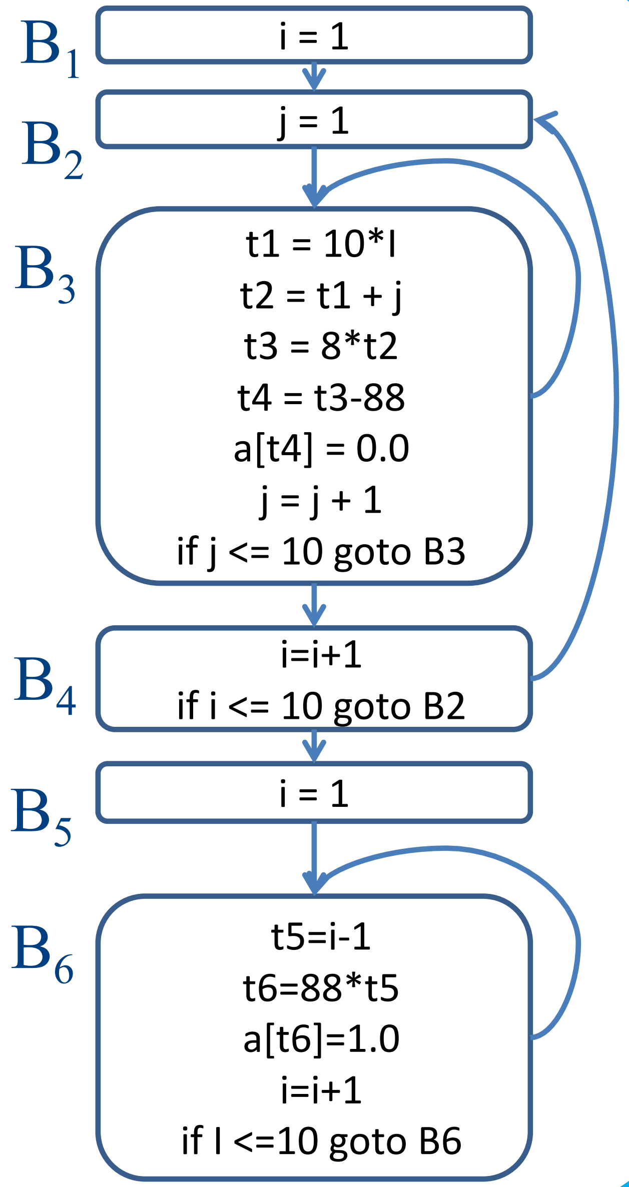
source

```
for i from 1 to 10
do
 for j from 1 to 10
 do
   a[i, j] = 0.0;
for i from 1 to 10
do
 a[i, i] = 1.0;
```

IR

```
1)    i = 1
2)    j =1
3)    t1 = 10*I
4)    t2 = t1 + j
5)    t3 = 8*t2
6)    t4 = t3-88
7)    a[t4] = 0.0
8)    j = j + 1
9)    if j <= 10 goto (3)
10)   i=i+1
11)   if i <= 10 goto (2)
12)   i=1
13)   t5=i-1
14)   t6=88*t5
15)   a[t6]=1.0
16)   i=i+1
17)   if I <=10 goto (13)
```

CFG



Example: Code Block

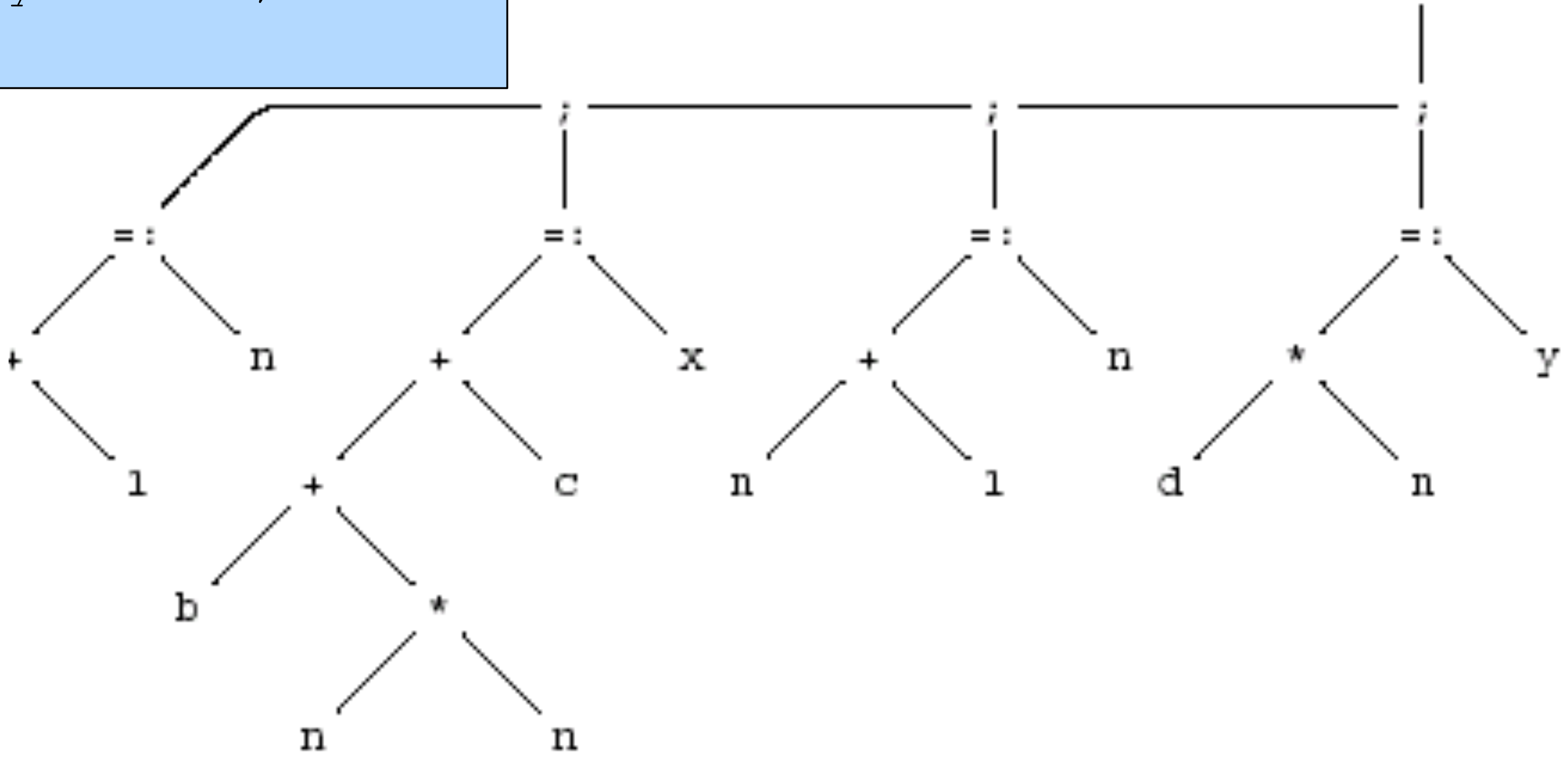
```
{
    int n;
    n := a + 1;
    x := b + n * n + c;
    n := n + 1;
    y := d * n;
}
```

Example: Basic Block

```
    n := a + 1;
    x := b + n * n + c;
    n := n + 1;
    y := d * n;
```

AST of the Example

```
{
    int n;
    n := a + 1;
    x := b + n * n + c;
    n := n + 1;
    y := d * n;
}
```



Optimized Code (gcc)

```
{
  int n;
  n := a + 1;
  x := b + n * n + c;
  n := n + 1;
  y := d * n;
}
```

```
Load_Mem     a,R1
Add_Const    1,R1
Load_Reg     R1,R2

Mult_Reg     R1,R2
Add_Mem      b,R2
Add_Mem      c,R2
Store_Reg    R2,x

Add_Const    1,R1
Mult_Mem     d,R1
Store_Reg    R1,y
```

Register Allocation for B.B.

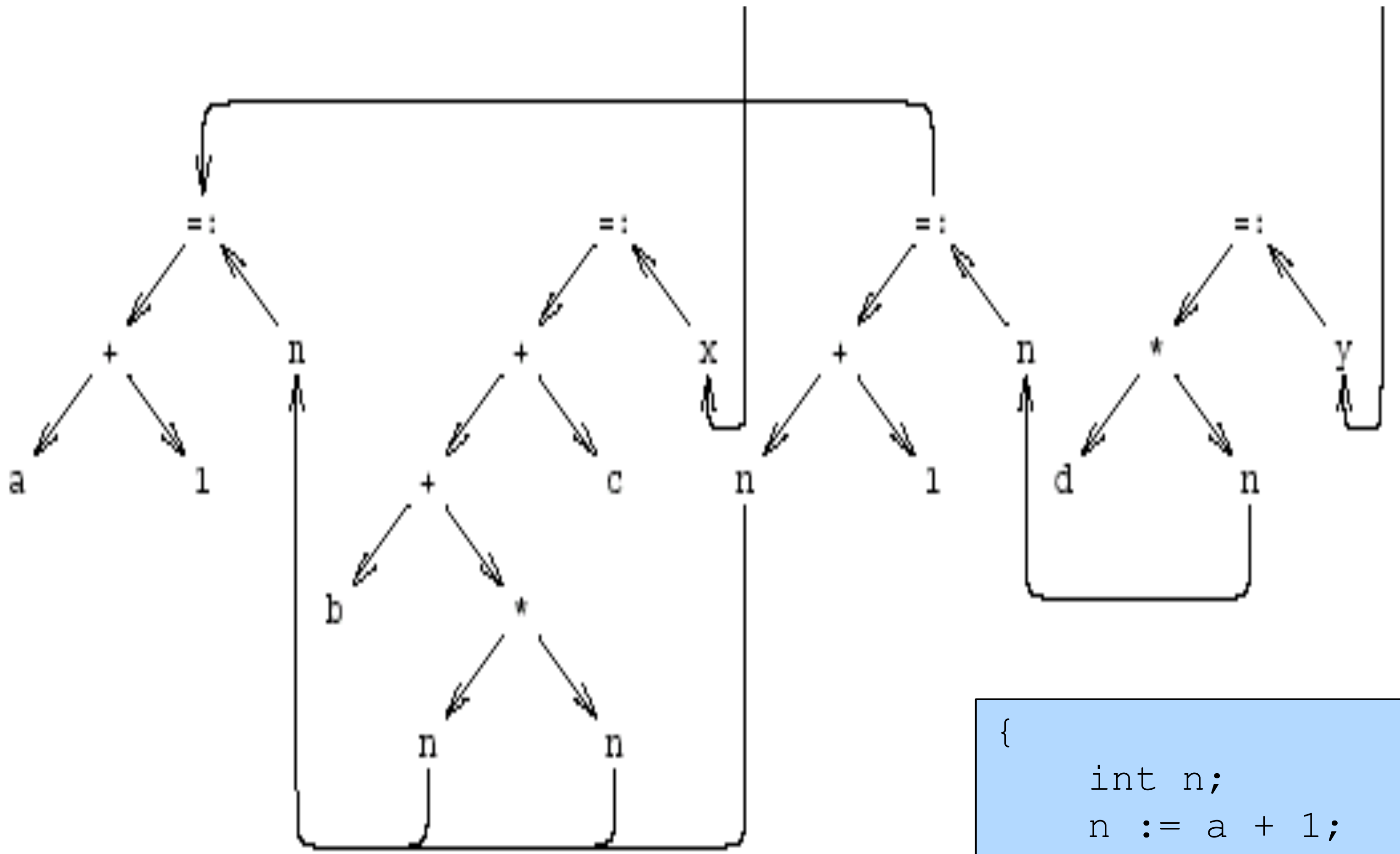
- Dependency graphs for basic blocks
- Transformations on dependency graphs
- From dependency graphs into code
  - Instruction selection
    - linearizations of dependency graphs
  - Register allocation
    - At the basic block level

Dependency graphs

- TAC imposes an order of execution
  - But the compiler can reorder assignments as long as the program results are not changed

- Define a partial order on assignments
  - $a < b \Leftrightarrow$ a must be executed before b
  - Represented as a directed graph
    - Nodes are assignments
    - Edges represent dependency
  - Acyclic for basic blocks

Running Example



```
{
    int n;
    n := a + 1;
    x := b + n * n + c;
    n := n + 1;
    y := d * n;
}
```

Sources of dependency

- Data flow inside expressions
  - Operator depends on operands
  - Assignment depends on assigned expressions
- Data flow between statements
  - From assignments to their use
  - Pointers complicate dependencies

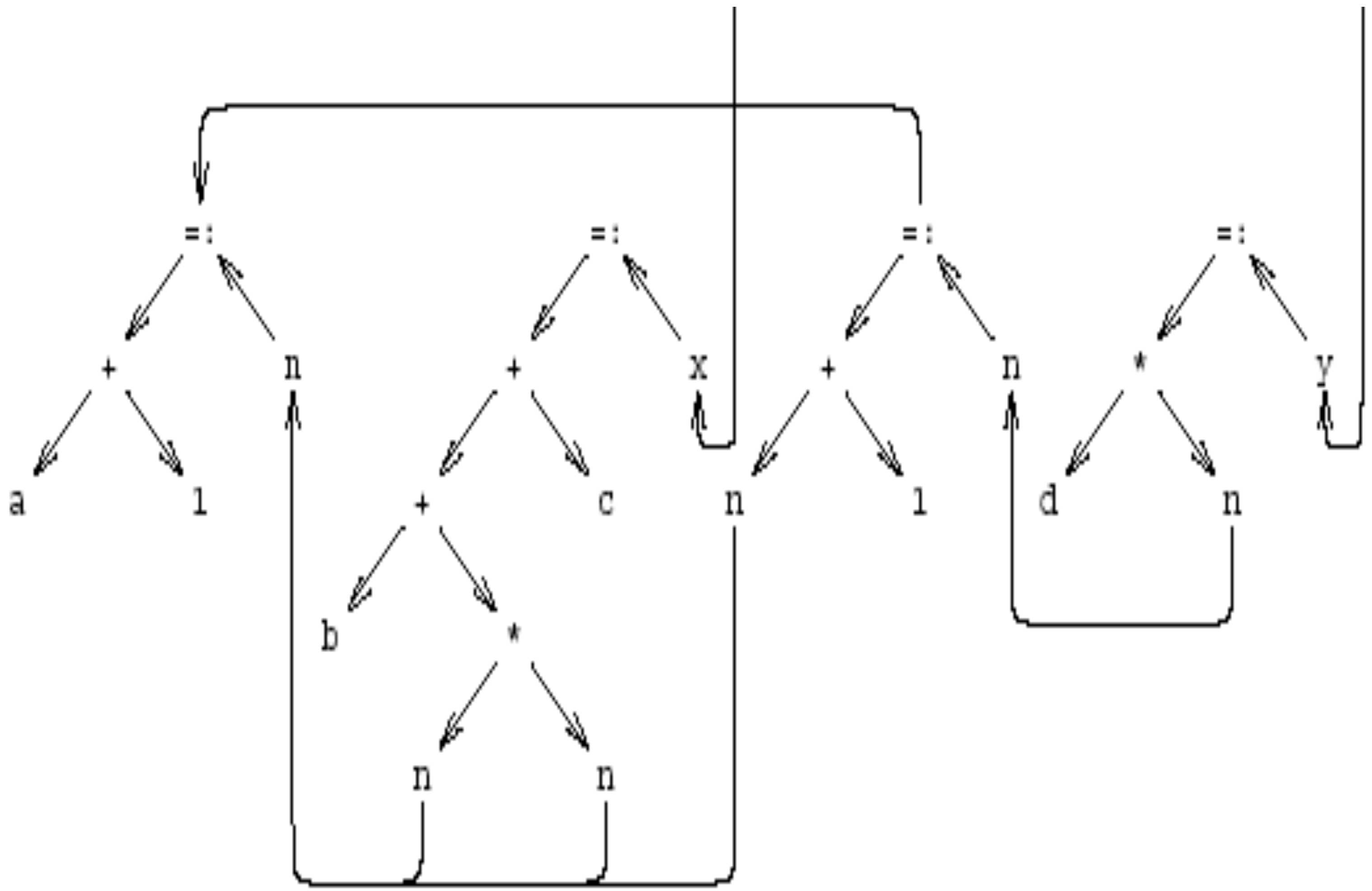
Sources of dependency

- Order of subexpresion evaluation is immaterial
  - As long as inside dependencies are respected
- The order of uses of a variable X are immaterial as long as:
  - X is used between dependent assignments
  - Before next assignment to X

Creating Dependency Graph from AST

- Nodes AST becomes nodes of the graph
- Replaces arcs of AST by dependency arrows
  - Operator $\rightarrow$ Operand
  - Create arcs from assignments to uses
  - Create arcs between assignments of the same variable
- Select output variables (roots)
- Remove ; nodes and their arrows

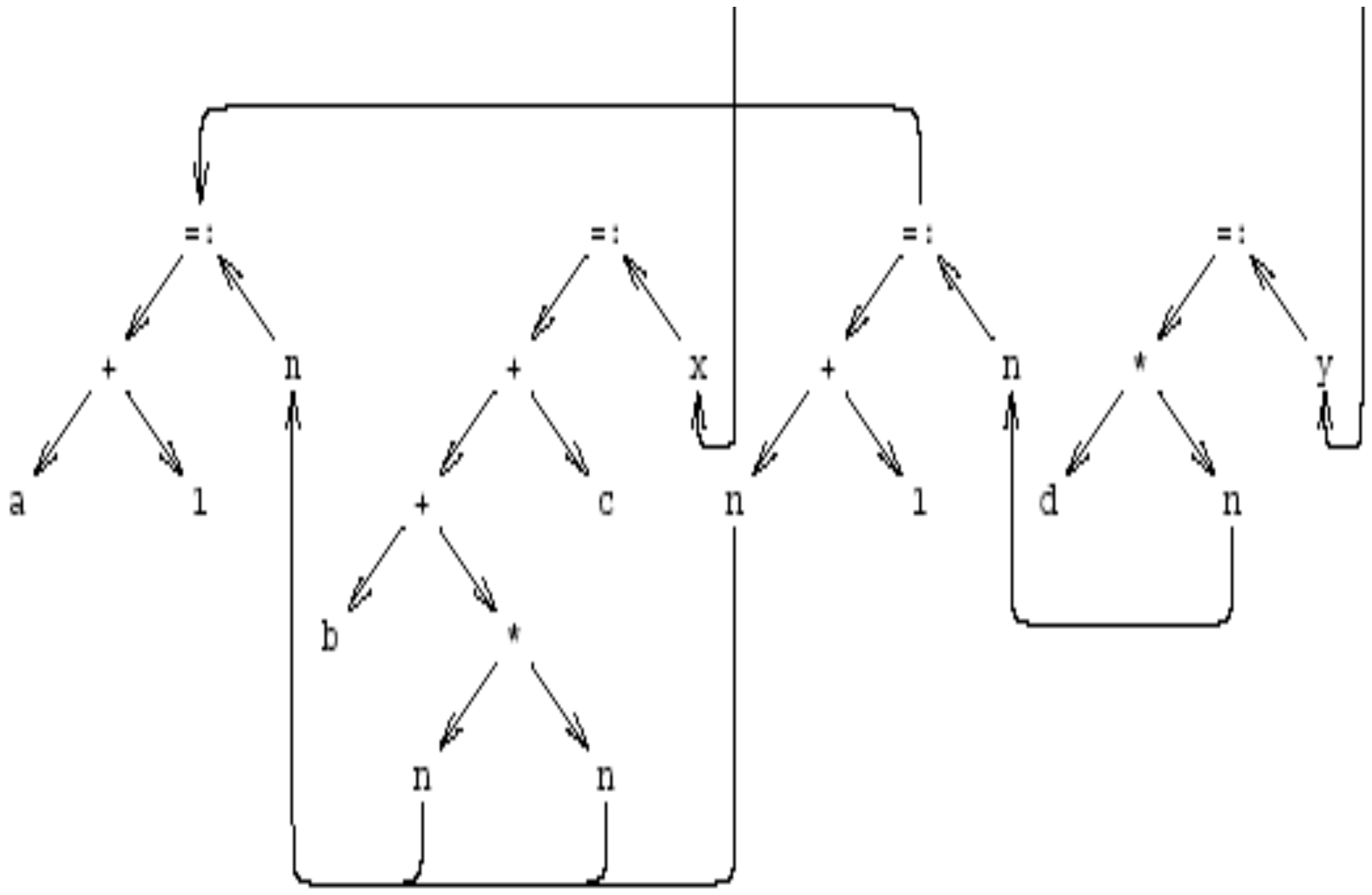
Running Example



Dependency Graph Simplifications

- Short-circuit assignments
  - Connect variables to assigned expressions
  - Connect expression to uses
- Eliminate nodes not reachable from roots

Running Example



Cleaned-Up Data Dependency Graph

$$\begin{tikzcd}
 & & x \arrow[d] & & & & y \arrow[d] & & \\
 & & + \arrow[dl] \arrow[dr] & & & & * \arrow[dl] \arrow[dr] & & \\
 & + \arrow[dl] \arrow[dr] & & c & & d & & + \arrow[dr] \arrow[ddlllll, bend left] & \\
b & & * \arrow[d, bend right] \arrow[d, bend left] & & & & & & 1 \\
 & & + \arrow[dl] \arrow[dr] & & & & & & \\
 & a & & 1 & & & & &
\end{tikzcd}$$

Common Subexpressions

- Repeated subexpressions
- Examples

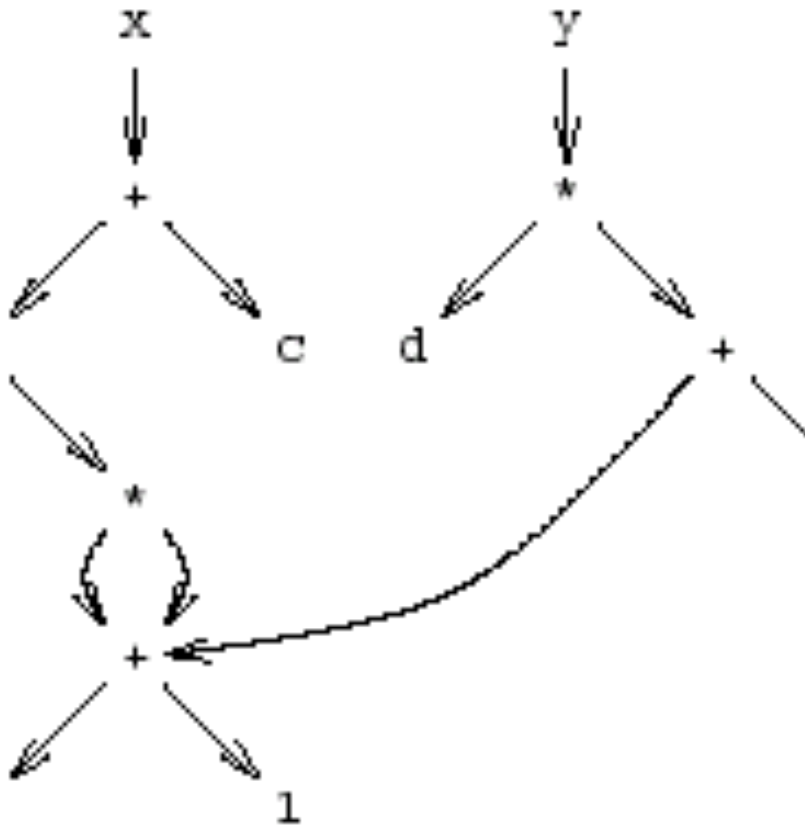
```
x = a * a  +   2 * a * b + b * b;
y = a * a  –   2 * a * b + b * b;
n[i] := n[i] +m[i]
```

- Can be eliminated by the compiler
  - In the case of basic blocks rewrite the DAG

From Dependency Graph into Code

- Linearize the dependency graph
  - Instructions must follow dependency
- Many solutions exist
- Select the one with small runtime cost
- Assume infinite number of registers
  - Symbolic registers
  - Assign registers later
    - May need additional spill
  - Possible Heuristics
    - Late evaluation
    - Ladders

Pseudo Register Target Code



```
Load_Mem     a,R1
Add_Const    1,R1
Load_Reg     R1,X1

Load_Reg     X1,R1
Mult_Reg     X1,R1
Add_Mem      b,R1
Add_Mem      c,R1
Store_Reg    R1,x

Load_Reg     X1,R1
Add_Const    1,R1
Mult_Mem     d,R1
Store_Reg    R1,y
```

Non optimized vs Optimized Code

```
Load_Mem     a,R1
Add_Const    1,R1
Load_Reg     R1,X1

Load_Reg     X1,R1
Mult_Reg     X1,R1
Add_Mem      b,R1
Add_Mem      c,R1
Store_Reg    R1,x

Load_Reg     X1,R1
Add_Const    1,R1
Mult_Mem     d,R1
Store_Reg    R1,y
```

```
Load_Mem     a,R1
Add_Const    1,R1
Load_Reg     R1,R2

Load_Reg     R2,R1
Mult_Reg     R2,R1
Add_Mem      b,R1
Add_Mem      c,R1
Store_Reg    R1,x

Load_Reg     R2,R1
Add_Const    1,R1
Mult_Mem     d,R1
Store_Reg    R1,y
```

```
ıd_Mem      a,R1
l_Const     1,R1
ıd_Reg      R1,R2

.t_Reg      R1,R2
l_Mem       b,R2
l_Mem       c,R2
ɔre_Reg     R2,x

l_Const     1,R1
.t_Mem      d,R1
ɔre_Reg     R1,y
```

```
{
  int n;
  n := a + 1;
  x := b + n * n + c;
  n := n + 1;
  y := d * n;
}
```

Register Allocation

- Maps symbolic registers into physical registers
  - Reuse registers as much as possible
  - Graph coloring (next)
    - Undirected graph
    - Nodes = Registers (Symbolic and real)
    - Edges = Interference
    - May require spilling

Register Allocation for Basic Blocks

- Heuristics for code generation of basic blocks
- Works well in practice
- Fits modern machine architecture
- Can be extended to perform other tasks
  - Common subexpression elimination
- But basic blocks are small
- Can be generalized to a procedure

Problem	Technique	Quality
Expression trees, using register-register or memory-register instructions	Weighted trees; Figure 4.30	
with sufficient registers:		Optimal
with insufficient registers:		Optimal
Dependency graphs, using register-register or memory-register instructions	Ladder sequences; Section 4.2.5.2	Heuristic
Expression trees, using any instructions with cost function	Bottom-up tree rewriting; Section 4.2.6	
with sufficient registers:		Optimal
with insufficient registers:		Heuristic
Register allocation when all interferences are known	Graph coloring; Section 4.2.7	Heuristic

The End