

## Lecture 1: March 2

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## 1.1 Introduction

Several fields in computer science and economics are focused on the analysis of Game theory. Usually they observe Game Theory as a way to solve optimization problems in systems where the participants act independently and their decisions affect the whole system.

Following is a list of research fields that utilize Game Theory:

- Artificial Intelligence (AI) - Multiple Agents settings where the problem is usually a cooperation problem rather than a competition problem.
- Communication Networks - Distribution of work where each agent works independently.
- Computer Science Theory - There are several subfields that use Game Theory:
  - Maximizing profit in bidding
  - Minimum penalty when using distributional environment
  - Complexity
  - Behavior of large systems

## 1.2 Course Syllabus

- Basic definitions in Game Theory, concentrating on Nash Equilibrium
- Coordination Ratio
  - Comparison between global optimum and Nash Equilibrium
  - Load Balancing Models
- Computation of Nash Equilibrium
  - Zero Sum games (Linear Programming)
  - Existence of Nash Equilibrium in general games

- Regret - playing an “unknown” game. Optimizing a player’s moves when the player can only view her own payoff
- Vector Payoff - the Payoff function is a vector and the target is to reach a specific target set
- Congestion and Potential games - games that model a state of load
- Convergence into Equilibrium
- Other...

### 1.3 Strategic Games

A strategic game is a model for decision making where there are  $N$  players, each one choosing an action. A player’s action is chosen just once and cannot be changed afterwards.

Each player  $i$  can choose an action  $a_i$  from a set of actions  $A_i$ . Let  $A$  be the set of all possible action vectors  $\times_{j \in N} A_j$ . Thus, the outcome of the game is an action vector  $\vec{a} \in A$ .

All the possible outcomes of the game are known to all the players and each player  $i$  has a preference relation over the different outcomes of the game:  $\vec{a} \preceq_i \vec{b}$  for every  $\vec{a}, \vec{b} \in A$ . The relation stands if the player prefers  $\vec{b}$  over  $\vec{a}$ , or has equal preference for either.

**Definition** A **Strategic Game** is a triplet  $\langle N, (A_i), (\preceq_i) \rangle$  where  $N$  is the number of players,  $A_i$  is the finite set of actions for player  $i$  and  $\preceq_i$  is the preference relation of player  $i$ .

We will use a slightly different notation for a strategic game, replacing the preference relation with a payoff function  $u_i : A \rightarrow \mathbb{R}$ . The player’s target is to maximize her own payoff. Such strategic game will be defined as:  $\langle N, (A_i), (u_i) \rangle$ .

This model is very abstract. Players can be humans, companies, governments etc. The preference relation can be subjective evolutionary etc. The actions can be simple, such as “go forward” or “go backwards”, or can be complex, such as design instructions for a building.

Several player behaviors are assumed in a strategic game:

- The game is played only once
- Each player “knows” the game (each player knows all the actions and the possible outcomes of the game)
- The players are rational. A rational player is a player that plays selfishly, wanting to maximize her own benefit of the game (the payoff function).
- All the players choose their actions simultaneously

## 1.4 Pareto Optimal

An outcome  $\vec{a} \in A$  of a game  $\langle N, (A_i), (u_i) \rangle$  is Pareto Optimal if there is no other outcome  $\vec{b} \in A$  that makes every player at least as well off and at least one player strictly better off. That is, a Pareto Optimal outcome cannot be improved upon without hurting at least one player.

**Definition** An outcome  $\vec{a}$  is **Pareto Optimal** if there is **no** outcome  $\vec{b}$  such that  $\forall_{j \in N} u_j(\vec{a}) \leq u_j(\vec{b})$  and  $\exists_{j \in N} u_j(\vec{a}) < u_j(\vec{b})$ .

## 1.5 Nash Equilibrium

A Nash Equilibrium is a state of the game where no player prefers a different action if the current actions of the other players are fixed.

**Definition** An outcome  $a^*$  of a game  $\langle N, (A_i), (\preceq_i) \rangle$  is a **Nash Equilibrium** if:

$$\forall_{i \in N} \forall_{b_i \in A_i} (a_{-i}^*, b_i) \preceq_i (a_{-i}^*, a_i^*).$$

$(a_{-i}, x)$  means the replacement of the value  $a_i$  with the value  $x$ .

We can look at a Nash Equilibrium as the best action that each player can play based on the given set of actions of the other players. Each player cannot profit from changing her action, and because the players are rational, this is a “steady state”.

**Definition** Player  $i$  **Best Response** for a given set of other players actions  $a_{-i} \in A_{-i}$  is the set:  $BR(a_{-i}) := \{b \in A_i \mid \forall_{c \in A_i} (a_{-i}, c) \preceq_i (a_{-i}, b)\}$ .

Under this notation, an outcome  $a^*$  is a Nash Equilibrium if  $\forall_{i \in N} a_i^* \in BR(a_{-i}^*)$ .

## 1.6 Matrix Representation

A two player strategic game can be represented by a matrix whose rows are the possible actions of player 1 and the columns are the possible actions of player 2. Every entry in the matrix is a specific outcome and contains a vector of the payoff value of each player for that outcome.

For example, if  $A_1$  is  $\{r1, r2\}$  and  $A_2$  is  $\{c1, c2\}$  the matrix representation is:

	c1	c2
r1	$(w1, w2)$	$(x1, x2)$
r2	$(y1, y2)$	$(z1, z2)$

Where  $u_1(r1, c2) = x1$  and  $u_2(r2, c1) = y2$ .

## 1.7 Strategic Game Examples

The following are examples of two players games with two possible actions per player. The set of deterministic Nash Equilibrium points is described in each example.

### 1.7.1 Battle of the Sexes

	Sports	Opera
Sports	(2, 1)	(0, 0)
Opera	(0, 0)	(1, 2)

There are two Nash Equilibrium points: (Sports, Opera) and (Opera, Sports).

### 1.7.2 A Coordination Game

	Attack	Retreat
Attack	(10, 10)	(-10, -10)
Retreat	(-10, -10)	(0, 0)

There are two Nash Equilibrium outcomes: (Attack, Attack) and (Retreat, Retreat). A question that raises from this game and its equilibria is how the two players can move from one Equilibrium point, (Retreat, Retreat), to the better one (Attack, Attack). Another the way to look at it is how the players can coordinate to choose the preferred equilibrium point.

### 1.7.3 The Prisoner's Dilemma

There is one Nash Equilibrium point: (Confess, Confess). Here, though it looks natural that the two players will cooperate, the cooperation point (Don't Confess, Don't Confess) is not a steady state since once in that state, it is more profitable for each player to move into 'Confess' action, assuming the other player will not change its action.

	Don't Confess	Confess
Don't Confess	$(-1, -1)$	$(-4, 0)$
Confess	$(0, -4)$	$(-3, -3)$

#### 1.7.4 Dove-Hawk

	Dove	Hawk
Dove	$(3, 3)$	$(1, 4)$
Hawk	$(4, 1)$	$(0, 0)$

There are two Nash Equilibrium points: (Dove, Hawk) and (Hawk, Dove).

#### 1.7.5 Matching Pennies

	Head	Tail
Head	$(1, -1)$	$(-1, 1)$
Tail	$(-1, 1)$	$(1, -1)$

In this game there is no Deterministic Nash Equilibrium point. However, there is a Mixed Nash Equilibrium which is  $(\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})$ . This is a zero sum game (the sum of the profits of each player over all possible outcomes is 0).

#### 1.7.6 Auction

There are  $N$  players, each one wants to buy an object.

- Player  $i$ 's valuation of the object is  $v_i$ , and, without loss of generality,  $v_1 > v_2 > \dots > v_n > 0$ .
- The players simultaneously submit bids -  $k_i \in [0, \infty)$ . The player who submit the highest bid -  $k_i$  wins.

- In a *first price auction* the payment of the winner is the price that she bids. Her payoff is  $u_i = \begin{cases} v_i - k_i, & i = \operatorname{argmax} k_i \\ 0, & \text{otherwise} \end{cases}$ .

A Nash equilibrium point is  $k_1 = v_2 + \epsilon, k_2 = v_2, \dots, k_n = v_n$ . In fact one can see that  $k_3, \dots, k_n$  have no influence.

In a *second price auction* the payment of the winner is the highest bid among those submitted by the players who do not win. Player  $i$ 's payoff when she bids  $v_i$  is at least as high as her payoff when she submits any other bid, regardless of the other players' actions. Player 1 payoff is  $v_1 - v_2$ . This strategy causes the player to bid truthfully.

### 1.7.7 A War of Attrition

Two players are involved in a dispute over an object.

- The value of the object to player  $i$  is  $v_i > 0$ . Time  $t \in [0, \infty)$ .
- Each player chooses when to concede the object to the other player
- If the first player to concede does so at time  $t$ , her payoff  $u_i = -t$ , the other player obtains the object at that time and her payoff is  $u_j = v_j - t$ .
- If both players concede simultaneously, the object is split equally, player  $i$  receiving a payoff of  $\frac{v_i}{2} - t$ .

The Nash equilibrium point is when one of the players concede immediately and the other wins.

### 1.7.8 Location Game

- Each of  $n$  people chooses whether or not to become a political candidate, and if so which position to take.
- The distribution of favorite positions is given by the density function  $f$  on  $[0, 1]$ .
- A candidate attracts the votes of the citizens whose favorite positions are closer to her position.
- If  $k$  candidates choose the same position then each receives the fraction  $\frac{1}{k}$  of the votes that the position attracts.
- Each person prefers to be the unique winning candidate than to tie for first place, prefers to tie the first place than to stay out of the competition, and prefers to stay out of the competition than to enter and lose.

When  $n = 3$  there is no Nash equilibrium. No player wants to be in the middle, since the other players will be as close as possible to the middle player, either from the left or the right.

## 1.8 Mixed Strategy

Now we will expand our game and let the players' choices to be nondeterministic. Each player  $i \in N$  will choose a probability distribution  $P_i$  over  $A_i$ :

1.  $P = \langle P_1, \dots, P_N \rangle$
2.  $P(\vec{a}) = \prod P_i(a_i)$
3.  $u_i(P) = E_{\vec{a} \sim P}[u_i(\vec{a})]$

Note that the function  $u_i$  is linear in  $P_i$ :  $U_i(P_i, \lambda\alpha_i + (1 - \lambda)\beta_i) = \lambda U_i(P_i, \alpha_i) + (1 - \lambda)U_i(P_i, \beta_i)$ .

**Definition**  $support(P_i) = \{a | P_i(a) > 0\}$

Note that the set of Nash equilibria of a strategic game is a subset of its set of mixed strategy Nash equilibria.

**Lemma 1.1** *Let  $G = \langle N, (A_i), (u_i) \rangle$ . Then  $\alpha^*$  is Nash equilibria of  $G$  if and only if  $\forall_{i \in N} support(P_i) \subseteq BR_i(\alpha_{-i}^*)$*

**Proof:**

$\Rightarrow$  Let  $\alpha^*$  be a mixed strategy Nash equilibria ( $\alpha^* = (P_1, \dots, P_N)$ ). Suppose  $\exists_{a \in support(P_i)} a \notin BR_i(\alpha_{-i}^*)$ . Then player  $i$  can increase her payoff by transferring probability to  $a' \in BR_i(\alpha_{-i}^*)$ ; hence  $\alpha^*$  is not mixed strategy Nash equilibria - contradiction.

$\Leftarrow$  Let  $q_i$  be a probability distribution s.t.  $u_i(Q) > u_i(P)$  in response to  $\alpha_{-i}^*$ . Then by the linearity of  $u_i$ ,  $\exists_{b \in support(Q_i), c \in support(P_i)} u_i(\alpha_{-i}^*, b) > U_i(\alpha_{-i}^*, c)$ ; hence  $c \notin BR_i(\alpha_{-i}^*)$  - contradiction.  $\square$

### 1.8.1 Battle of the Sexes

As we mentioned above, this game has two deterministic Nash equilibria, (S,S) and (O,O). Suppose  $\alpha^*$  is a stochastic Nash equilibrium:

- $\alpha_1^*(S) = 0$  or  $\alpha_1^*(S) = 1 \Rightarrow$  same as the deterministic case.
- $0 < \alpha_1^*(S) < 1 \Rightarrow$  by the lemma above  $2\alpha_2^*(O) = \alpha_2^*(S)$  ( $\alpha_2^*(O) + \alpha_2^*(S) = 1$ ) and thus  $\alpha_2^*(O) = \frac{1}{3}$ ,  $\alpha_2^*(S) = \frac{2}{3}$ . Since  $0 < \alpha_2^*(S) < 1$  it follows from the same result that  $2\alpha_1^*(S) = \alpha_1^*(O)$  so  $\alpha_1^*(S) = \frac{1}{3}$ ,  $\alpha_1^*(O) = \frac{2}{3}$ .

The mixed strategy Nash Equilibrium is  $((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3}))$ .

## 1.9 Correlated Equilibrium

We can think of a traffic light that correlates, advises the cars what to do. The players observe an object that advises each player of her action. A player can either accept the advice or choose a different action. If the best action is to obey the advisor, the advice is a correlated equilibrium.

**Definition**  $Q$  is probability distribution over  $A$ .  $\vec{a} \in Q$  is a Nash correlated equilibrium if  $\forall z_i \in \text{support}(Q) E_Q[U_i(a_{-i}, z_i) | a_i = z_i] > E_Q[U_i(a_{-i}, x) | a_i = z_i]$

## 1.10 Evolutionary Equilibrium

This type of game describes an "evolution" game between different species. There are  $B$  types of species,  $b, x \in B$ . The payoff function is  $u(b, x)$ . The game is defined as  $\langle \{1, 2\}, B, (u_i) \rangle$ .

The equilibrium  $b^*$  occurs when for each mutation  $b$  the payoff function satisfies  $(1 - \epsilon)u(b^*, b) + \epsilon u(b, b) < (1 - \epsilon)u(b^*, b^*) + \epsilon u(b^*, b)$ .

This kind of equilibrium is defined as an **evolutionarily stable strategy** since it tolerates small changes in each type.