On vertex Ramsey graphs with forbidden subgraphs

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Abstract

A classical vertex Ramsey result due to Nešetřil and Rödl states that given a finite family of graphs $\mathcal{F}$, a graph A and a positive integer r, if every graph $B \in \mathcal{F}$ has a 2-vertex-connected subgraph which is not a subgraph of A, then there exists an $\mathcal{F}$-free graph which is vertex r-Ramsey with respect to A. We prove that this sufficient condition for the existence of an $\mathcal{F}$-free graph which is vertex r-Ramsey with respect to A is also *necessary* for large enough number of colours r.

We further show a generalisation of the result to a family of graphs and the typical existence of such a subgraph in a dense binomial random graph.

1 Introduction

Let A be a graph and let r be a positive integer. We say that a graph G is *(vertex) r-Ramsey* with respect to A if in every colouring of the vertices of G in r colours there exists a monochromatic copy of A. The existence of r-Ramsey graphs is straightforward: the complete graph K_n is r-Ramsey with respect to A for every $n \geq r(|V(A)|-1)+1$. It is thus natural to ask about the existence of *sparse* Ramsey graphs. One of the ways to define sparseness is to avoid copies of a given graph B (or more generally of any graph from a given finite graph family $\mathcal{F}$) in G. Let us call a graph G *$\mathcal{F}$-free* if it does not contain a subgraph isomorphic to B for every $B \in \mathcal{F}$.

Perhaps the most studied case is when both A and B are complete graphs on s and t vertices, respectively, where $t > s \geq 2$. Denote by $f_{s,t}(n)$ the minimum over all K_t-free graphs G on $[n] := \{1, \dots, n\}$ of the maximum number of vertices in an induced K_s-free subgraph of G. Erdős and Rogers [5] proved that, for a certain $\varepsilon = \varepsilon(s) > 0$, $f_{s,s+1}(n) \leq n^{1-\varepsilon}$ (note that this implies that for every $s \geq 2$ and $r \geq 2$, there exists a K_{s+1}-free graph G which is r-Ramsey with respect to K_s). The result of Erdős and Rogers was subsequently refined by Bollobás and Hind [1] and Krivelevich [6]. Let us also mention that subsequent works by Dudek, Retter and Rödl [3] and by Dudek and Rödl [4] determined $f_{s,s+1}(n)$ up to a power of $\log n$ factor,

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strengthened the known bounds for $f_{s,s+2}(n)$, and further improved the bounds for $f_{s,s+k}(n)$ when s, k are large enough.

Considering general graphs A and B (and in fact, a family of graphs B), Nešetřil and Rödl [7] proved the following (see also [2]):

Theorem 1.1 ([7])**.** *Let $\mathcal{F}$ be a finite family of graphs and let A be a graph. Let $r \geq 2$ be an integer. If every graph from $\mathcal{F}$ has a 2-vertex-connected subgraph which is not a subgraph of A, then there exists an $\mathcal{F}$-free graph which is vertex r-Ramsey with respect to A.*

See [9, 10, 11] for additional results on vertex-Ramsey graphs with forbidden subgraphs.

Our main result shows that the above sufficient condition is also necessary for large enough number of colours r. We say that B is an A-forest of size ℓ if $B = \cup_{i=1}^{\ell} B_i$, where for every $1 \leq i \leq \ell$, B_i is isomorphic to a subgraph of A, and for every $i \geq 2$, $\left|V(B_i) \cap V\left(\cup_{j=1}^{i-1} B_j\right)\right| \leq 1$.

Theorem 1. *Let $\ell > 0$ be an integer. Let B be an A-forest of size ℓ. Let $r > 0$ be an integer such that $r \geq \ell\,(2(|V(A)|-1)(|V(B)|-2)+1)$, and let G be an r-Ramsey graph with respect to A. Then G contains a copy of B.*

Let us first note that since $\ell \leq |V(B)|$, it suffices to take $r = O(|V(A)||V(B)|^2)$. Furthermore, observe that the above implies the necessity of the condition in Theorem 1.1, for r large enough. Indeed, let us say that a graph B is *A-degenerate*, if every 2-vertex-connected subgraph of it is a subgraph of A. Note that any A-degenerate graph can be constructed recursively: (1) any subgraph of A is A-degenerate; (2) if B is an A-degenerate graph, then a union of B with a subgraph of A that shares with B at most 1 vertex is A-degenerate as well. Theorems 1.1 and 1 can be formulated in terms of A-degenerate graphs: there exists an $\mathcal{F}$-free graph which is r-Ramsey with respect to A for all large enough r *if and only if* every graph from $\mathcal{F}$ is not A-degenerate.

Note that the case that B consists of ℓ vertex-disjoint components, each isomporphic to a subgraph of A, is easy since if G is r-Ramsey with respect to A then it contains a large enough family of vertex-disjoint copies of A. On the other hand, if the components of B are not disjoint, we can proceed by induction, deleting a component B_i intersecting other components, finding a copy of $B - B_i$ using inductive hypothesis and then adjoining to it a correctly placed copy of B_i, see details in Section 2.

In the next section, we provide a short proof of Theorem 1.1 for the sake of completeness, followed by the proof of Theorem 1. In Section 3, we discuss generalisations of Theorem 1.1 to a family of graphs (instead of A), and the existence of an $\mathcal{F}$-free graph which is r-Ramsey with respect to A in a dense enough binomial random graph.

2 Proofs of Theorems 1.1 and 1

We say that a graph G is *ε-dense* with respect to a graph A if every induced subgraph of G on $\lfloor \varepsilon |V(G)| \rfloor$ vertices contains a copy of A. Clearly, if G is $1/r$-dense with respect to A, then it is also r-Ramsey with respect to A. Theorem 1.1 follows immediately from Theorem 2.1.

Theorem 2.1. *Let $\mathcal{F}$ be a finite family of graphs. If there are no A-degenerate graphs in $\mathcal{F}$, then there exists a $\delta = \delta(A,\mathcal{F}) > 0$ such that for all large enough n, there exists an $\mathcal{F}$-free $n^{-\delta}$-dense graph on $[n]$ with respect to A.*

Proof. Let $a := |V(A)|$. Let $\varepsilon > 0$ be small enough and set $p = n^{1-a+\varepsilon}$. Consider a hypergraph with vertex set $\binom{[n]}{2}$ whose edge set consists of all possible copies of A on $[n]$. Let $\mathcal{H}_A(n,p)$ be its binomial subhypergraph where each copy of A is chosen independently and with probability p, and let $\mathcal{G}_A(n,p)$ be the random graph constructed as follows: an edge belongs to $\mathcal{G}_A(n,p)$ if and only if this edge belongs to a copy of A in $\mathcal{H}_A(n,p)$. We shall prove that it suffices to remove $O(\sqrt{n})$ vertices of $\mathcal{G}_A(n,p)$ to get the desired graph **whp**.

Let $\delta_0 = \frac{\varepsilon}{2(a-1)}$. Let us show that **whp** $\mathcal{G}_A(n,p)$ is $n^{-\delta_0}$-dense with respect to A. Set $N = \lfloor n^{1-\delta_0} \rfloor$. Then the expected number of N-sets containing no copy of A in $\mathcal{G}_A(n,p)$ is at most the expected number of N-subsets $U \subseteq [n]$ such that $\binom{U}{2}$ does not contain any copy of A in $\mathcal{H}_A(n,p)$ that equals to

$$\begin{aligned}\binom{n}{N}(1-p)^{\binom{N}{a}\frac{a!}{\mathrm{aut}(A)}} &\leq \exp\left[N\left(\delta_0 \ln n + 1 - p\frac{N^{a-1}}{\mathrm{aut}(A)}\right)(1+o(1))\right] \\ &\leq \exp\left[N\left(\delta_0 \ln n - \frac{n^{1-a+\varepsilon+(a-1)(1-\delta_0)}}{\mathrm{aut}(A)}\right)(1+o(1))\right] \\ &\leq \exp\left[-Nn^{\varepsilon/2}\left(\frac{1}{\mathrm{aut}(A)} - o(1)\right)\right] \to 0.\end{aligned}$$

By the union bound, **whp** every N-set contains at least one copy of A in $\mathcal{G}_A(n,p)$, that is, **whp** $\mathcal{G}_A(n,p)$ is $n^{-\delta_0}$-dense.

Let $\delta = \delta_0/2$ and let $C > 0$. Note that **whp** the deletion of any $C\sqrt{n}$ vertices from $\mathcal{G}_A(n,p)$ leads to an $\tilde{n}^{-\delta}$-dense graph on $\tilde{n}$ vertices. Indeed, if $\mathcal{G}_A(n,p)$ is $n^{-\delta_0}$-dense, then, since $\tilde{n}^{1-\delta} = (n - C\sqrt{n})^{1-0.5\delta_0} \geq n^{1-\delta_0}$, every set of $\tilde{n}^{1-\delta}$ vertices in the new graph has at least $n^{1-\delta_0}$ vertices and thus contains a copy of A. Therefore, it suffices to prove that **whp** we can remove $O(\sqrt{n})$ vertices from $\mathcal{G}_A(n,p)$ and get an $\mathcal{F}$-free graph.

Given a graph B and graphs $A_1, \ldots, A_m$ isomorphic to A, we say that $A_1 \cup \ldots \cup A_m$ is an *inclusion-minimal cover* of the edges of B if $E(B) \subseteq E(A_1 \cup \ldots \cup A_m)$ but $E(B) \nsubseteq E(A_1 \cup \ldots \cup A_{i-1} \cup A_{i+1} \cup \ldots \cup A_m)$ for every $i \in [m]$. For every $B \in \mathcal{F}$, consider $B' \subset B$ such that every inclusion-minimal cover $A_1 \cup \ldots \cup A_m$ of the edges of B' satisfies $|(A_i \cap \cup_{j \neq i} A_j) \cap B'| \geq 2$ for every $i \in [m]$. By Claim 2.2 (stated below), **whp** the number of copies of B' in $\mathcal{G}_A(n,p)$ is at most $\sqrt{n}$. We can now delete a single vertex from each such copy, and obtain a set of $\tilde{n} \geq n - |\mathcal{F}|\sqrt{n}$ vertices that induces an $\mathcal{F}$-free graph, as required. □

We note that a slight adjustment of the proof of Theorem 2.1 allows one to argue for the existence of $\mathcal{F}$-free ε-dense graph for *induced* copies of A.

Claim 2.2. ***Whp*** *the number of copies of B' in $\mathcal{G}_A(n,p)$ is at most $\sqrt{n}$.*

Proof. Let $b := |V(B')|$ and let $k := |E(B')|$. Let $X_{B'}$ be the number of copies of B' in $\mathcal{G}_A(n,p)$. We shall bound $\mathbb{E}X_{B'}$ from above.

A copy of B' may appear in $\mathcal{G}_A(n,p)$ only through hyperedges $A_1, \ldots, A_\ell \in E(\mathcal{H}_A(n,p))$ such that $B' \subset A_1 \cup \ldots \cup A_\ell$. For any possible inclusion-minimal cover of edges of B' by copies $A_1, \ldots, A_\ell$ of A, let us denote by v_i the number of vertices in the intersection of A_i and B'. Then, each A_i in this cover contributes a factor of $O(n^{a-v_i}p)$ to $\mathbb{E}X_{B'}$. More formally, if $B' = B_1 \cup \ldots \cup B_\ell$, where each B_i is a subgraph of a copy of A, then, for every i,

$$\mathbb{P}(\exists A' \in \mathcal{H}_A(n,p)\colon A' \supset B_i) = O(n^{a-v_i}p).$$

Since there are $O(n^b)$ choices of B' in K_n, we get that

$$\begin{aligned}\mathbb{E}X_{B'} &= O\left(n^b \max_{A_1\cup\ldots\cup A_\ell \supset B'} n^{a\ell - v_1 - \ldots - v_\ell} p^\ell\right) \\ &= O\left(n^{b+\max_{A_1\cup\ldots\cup A_\ell}(\ell(1+\varepsilon) - v_1 - \ldots - v_\ell)}\right), \qquad (1)\end{aligned}$$

where the maximum and minimum are taken over all inclusion-minimal covers $A_1 \cup \ldots \cup A_\ell$ of edges of B' by copies of A.

Let $A_1 \cup \ldots \cup A_\ell$ be an inclusion-minimal cover of the edges of B' by copies of A, and let V_i be the set of vertices in the intersection of A_i with B' (as above, we let $v_i = |V_i|$). Since each V_i has at least two common vertices with $\cup_{j\neq i} V_j$ and $\ell \geq 2$, we get

$$\sum_{i=1}^{\ell} v_i \geq |V_1 \cup \ldots \cup V_\ell| + \ell = b + \ell, \qquad (2)$$

that is $\sum_{i=1}^{\ell} v_i \geq b + \ell$. Indeed, for every i, let $S_i = V_i \cap (\cup_{j\neq i} V_j)$, $s_i = |S_i| \geq 2$. Then $V_1 \cup \ldots \cup V_\ell = S_1 \cup \ldots \cup S_\ell \cup \Sigma$, where Σ is the set of vertices that are covered once. Then $|\Sigma| = \sum_{i=1}^{\ell}(v_i - s_i)$, and $|S_1 \cup \ldots \cup S_\ell| \leq \frac{1}{2}\sum_{i=1}^{\ell} s_i$, since each vertex in this union is covered at least twice. We thus obtain,

$$|V_1 \cup \ldots \cup V_\ell| = |\Sigma| + |S_1 \cup \ldots \cup S_\ell| \leq \sum_{i=1}^{\ell} v_i - \frac{1}{2}\sum_{i=1}^{\ell} s_i \leq \sum_{i=1}^{\ell} v_i - \ell,$$

where the last inequality follows since each s_i is at least 2.

We may assume that $\varepsilon < \frac{1}{2k}$. Due to (1) and (2), we get

$$\mathbb{E}X_{B'} = O(n^{k\varepsilon}) = o(\sqrt{n}).$$

By Markov's inequality, **whp** we have less than $\sqrt{n}$ copies of B' in $\mathcal{G}_A(n,p)$. □

We now turn to the proof of our main theorem.

Proof of Theorem 1. Let $a := |V(A)|$ and $b := |V(B)|$. If $a = 1$, that is, $A = K_1$, then note that B is the empty graph on b vertices and thus every graph on at least b vertices contains a copy of B. If $a = b = 2$, then we have $B \subseteq A$ and thus every graph which is r-Ramsey with respect to A contains a copy of B.

We assume that $a \geq 2, b \geq 3$. We enumerate the vertices of A: $V(A) = \{v_1, \ldots, v_a\}$. Given a graph G, and a copy A' of A in G, we define a mapping $\phi_{A'} : V(A) \to V(G)$ such that for every $v_i \in V(A)$, we set $\phi_{A'}(v_i)$ to be the vertex $v \in V(G)$ which is in the role of v_i in the copy A' of A. Given $v \in V(G)$, we denote by $\mathcal{A}_i(v)$ the set of copies A' of A in G for which

$\phi_{A'}(v_i) = v$. Furthermore, we denote by $s_i(v)$ the maximal size of a subset of $\mathcal{A}_i(v)$, in which every two copies of A in G intersect only at v.

We prove by induction on ℓ, the minimum size of an A-forest of B, where the base case $\ell = 1$ is trivial.

We now consider two cases separately. First, assume that in an A-forest of B of size ℓ all components B_i are disjoint. Let M be the maximum size of a family of vertex disjoint copies of A in G. Then, we can colour each copy in a maximum family of vertex disjoint copies of A in two separate colours, and colour all the other vertices in $(2M+1)$-th colour, without producing a monochromatic copy A. As G is r-Ramsey with respect to A, we conclude that $r < 2M + 1$. Since $r \geq 2\ell$, we find ℓ disjoint copies of A in G, and therefore a copy of B in G.

We now turn to the case where, without loss of generality, B_ℓ intersects $\cup_{i=1}^{\ell-1} B_i$ in an A-forest of B. Let $\tilde{B} \coloneqq \cup_{i=1}^{\ell-1} B_i$, and let $\{x\} \coloneqq V(\tilde{B}) \cap V(B_\ell)$. We may further assume that for $A \supseteq B_\ell$, x corresponds to v_k in A, for some $1 \leq k \leq a$.

Let $U = \{v \in V(G) \colon s_k(v) \leq b-2\}$. We require the following claim.

Claim 2.3. *$G[U]$ can be coloured in $2(a-1)(b-2)+1$ colours, without a monochromatic copy of A.*

Proof. For every $v \in U$, let $\mathcal{S}_k(v)$ be a maximal by inclusion subfamily of $\mathcal{A}_k(v)$ composed of copies of A in $G[U]$, where every two copies of A in the subfamily intersect only at v, and let $S_k(v) = \cup_{A' \in \mathcal{S}_k(v)} V(A')$. By definition of U, $|\mathcal{S}_k(v)| \leq b-2$ and $|S_k(v)| \leq (a-1)(b-2)+1$.

Define an auxiliary directed graph $\overrightarrow{\Gamma}$ on the vertices of U, where for every v and for every $u \in S_k(v) \setminus \{v\}$, $\overrightarrow{\Gamma}$ contains a directed edge from v to u. We thus have that $\Delta^+(\overrightarrow{\Gamma}) \leq (a-1)(b-2)$. Hence, the underlying undirected graph Γ is $2(a-1)(b-2)$-degenerate. Indeed, consider $V' \subseteq V(\Gamma)$. We will show that in the induced subgraph $\Gamma[V']$ there exists a vertex of degree at most $2(a-1)(b-2)$. We have

$$\sum_{v \in V'} d_{\Gamma[V']}(v) = 2|E(\Gamma[V'])| \leq 2 \sum_{v \in V'} d^+_{\overrightarrow{\Gamma}}(v) \leq 2(a-1)(b-2)|V'|,$$

and thus there must be at least one vertex $v \in V'$ with $d_{\Gamma[V']}(v) \leq 2(a-1)(b-2)$. Therefore, Γ is $(2(a-1)(b-2)+1)$-colourable. We colour $G[U]$ according to this colouring.

Suppose towards contradiction that there is a monochromatic copy A' of A in $G[U]$, and let $w = \phi_{A'}(v_k)$. Since A' is monochromatic, it does not have common vertices with $S_k(w)$ other than w — however this contradicts the maximality of $\mathcal{S}_k(w)$. □

Recalling that G is r-Ramsey with respect to A, and that $G[U]$ can be coloured in $2(a-1)(b-2)+1$ colours without containing a monochromatic copy of A, we have that $G[V \setminus U]$ is $(r - (2(a-1)(b-2)+1))$-Ramsey with respect to A. Observing that

$$r - (2(a-1)(b-2)+1) \geq (\ell - 1)\left(2(a-1)(b-2)+1\right),$$

we have by induction that $G[V \setminus U]$ contains a copy of $\tilde{B}$. Let v be the vertex in this copy of $\tilde{B}$ that corresponds to x. Since $v \notin U$ we have that $s_k(v) \geq b-1$, and hence there is a subset of size at least $b-1$ in $\mathcal{A}_k(v)$ such that every two copies A' of A in this subset intersect only at

v. Noting that $|V(\tilde{B})| \leq b-1$, we have that at least one copy A' of A in this subset completes $\tilde{B}$ to B, that is, $\tilde{B} \cup A'$ contains a copy of B (see Figure 1 for an illustration). $\square$

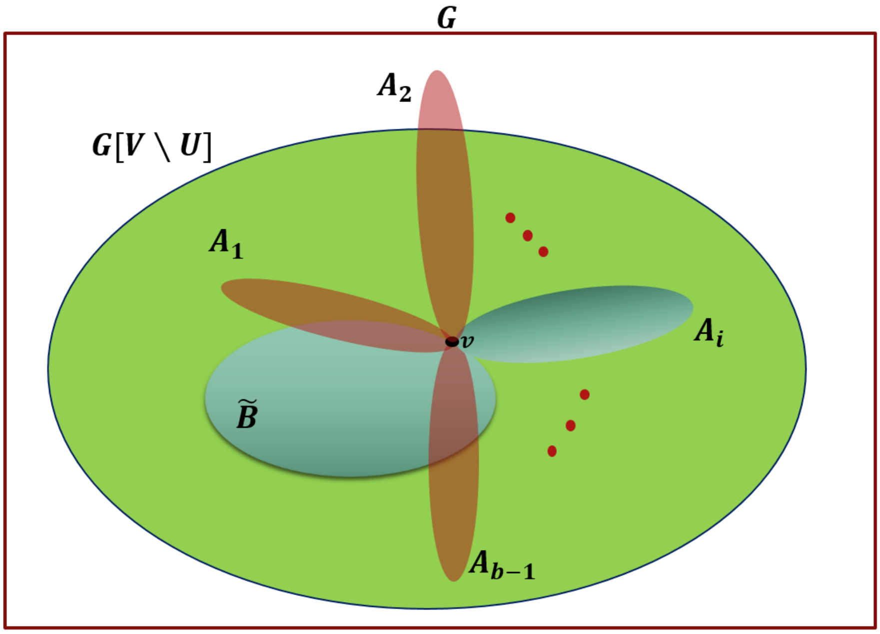

Figure 1: The subgraph $G[V \setminus U]$ and a copy of $\tilde{B}$ in it. A copy A_i of A together with $\tilde{B}$ contain a copy of B. Note that some of the A_j's may have vertices outside $V \setminus U$.

3 Remarks and observations

Let us finish with two remarks.

Remark 1. Theorems 2.1 and 1 can be generalised to families of graphs instead of a single graph A. Let $\mathcal{A}, \mathcal{F}$ be two finite graph families, and $\varepsilon > 0$. The proof of Theorem 1 is quite similar. For the proof of Theorem 2.1, let us say that a graph G is an $\mathcal{F}$-free ε-*dense* with respect to $\mathcal{A}$ if it is B-free for every $B \in \mathcal{F}$, and every induced subgraph of G on exactly $\lfloor \varepsilon |V(G)| \rfloor$ vertices contains a copy of every $A \in \mathcal{A}$. A graph B is $\mathcal{A}$-*degenerate*, if every 2-vertex-connected subgraph of it is isomorphic to a subgraph of some $A \in \mathcal{A}$. If every $B \in \mathcal{F}$ is not $\mathcal{A}$-degenerate, then there exists an $\mathcal{F}$-free ε-dense graph with respect to $\mathcal{A}$ — indeed, let A be the disjoint union of the graphs from $\mathcal{A}$, and apply Theorem 2.1.

Remark 2. For a non-A-degenerate family $\mathcal{F}$ (consisting of graphs that are not A-degenerate) and sufficiently small $\delta > 0$, we claim the likely existence of an $\mathcal{F}$-free $n^{-\delta}$-dense subgraph in the binomial random graph $G(n, n^{-2/a+\delta})$, where a is the number of vertices in A. Indeed, consider the hypergraph with vertex set $\binom{[n]}{2}$, and edge set being all the possible cliques of size a, K_a, on $[n]$. Let $\mathcal{H}_a(n, p')$ be its binomial subgraph. Let us first show that there exists a coupling between $\mathcal{H}_a(n, p')$, and the graph considered in the proof of sufficiency of Theorem 2.1, $\mathcal{H}_A(n, p)$, such that $p = \Theta(p')$ and $\mathcal{H}_A(n, p) \subseteq \mathcal{H}_a(n, p')$. Indeed, let $p = n^{1-a+\delta\binom{a}{2}}$. Consider an a-set, and let p' be the probability that at least one copy of A appears on this a-set. Clearly, $p = \Theta(p')$. Let Q be the conditional distribution of a binomial random hypergraph of copies of A on $[a]$, under the condition that at least one such copy exists. We can now draw $\mathcal{H}_A(n, p)$ as follows. We first choose every a-set with probability p', and then in every set that we chose we construct a random A-hypergraph with distribution Q, independently for different a-sets. We thus have that $\mathcal{H}_A(n,p) \subseteq \mathcal{H}_a(n,p')$, and we can continue the proof in the same manner as in Theorem 2.1. Now, we take q such that $q^{\binom{a}{2}} = p'$. Therefore, by the above coupling between $\mathcal{H}_a(n,p')$ and $\mathcal{H}_A(n,p)$ and by Theorem 3.1 stated below, **whp** $G(n,q) \supset \mathcal{G}_{K_a}(n,p') \supset \mathcal{G}_A(n,p)$.

Theorem 3.1 (Riordan [8])**.** *Let $\varepsilon > 0$ be small enough and $q \leq n^{-\frac{2}{a}+\varepsilon}$, $p \sim q^{\binom{a}{2}}$. Then there exists a coupling between $G(n,q)$ and $\mathcal{H}_a(n,p)$ such that* ***whp*** *for every edge of $\mathcal{H}_a(n,p)$ there exists a copy of K_a in $G(n,q)$ with the same vertex set.*

We note that Riordan in [8, Section 5] discusses a coupling between $G(n,q)$ and $\mathcal{H}_A(n,p)$, and provides sufficient conditions for its existence for some A, however here we settle for higher values of $q(n)$ with respect to $p(n)$, thus making such coupling simpler.

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