Expanders Are Universal for the Class of All Spanning Trees (Extended Abstract)

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Abstract

Given a class of graphs $\mathcal{F}$, we say that a graph G is *universal for* $\mathcal{F}$, or $\mathcal{F}$*-universal*, if every $H \in \mathcal{F}$ is contained in G as a subgraph. The construction of sparse universal graphs for various families $\mathcal{F}$ has received a considerable amount of attention. One is particularly interested in tight $\mathcal{F}$-universal graphs, i. e., graphs whose number of vertices is equal to the largest number of vertices in a graph from $\mathcal{F}$. Arguably, the most studied case is that when $\mathcal{F}$ is some class of trees.

Given integers n and Δ, we denote by $\mathcal{T}(n,\Delta)$ the class of all n-vertex trees with maximum degree at most Δ. In this work, we show that every n-vertex graph satisfying certain natural expansion properties is $\mathcal{T}(n,\Delta)$-universal or, in other words, contains every spanning tree of maximum degree at most Δ. Our methods also apply to the case when Δ is some function of n. The result has a few very interesting implications. Most importantly, since random graphs are known to be good expanders, we obtain that the random graph $G(n,p)$ is asymptotically almost surely (a.a.s.) universal for the class of all bounded degree spanning (that is, n-vertex) trees provided that $p \geq cn^{-1/3}\log^2 n$ where $c > 0$ is a constant. Moreover, a corresponding result holds for the random regular graph of degree pn. In fact, we show that if Δ satisfies $\log n \leq \Delta \leq n^{1/3}$, then the random graph $G(n,p)$ with $p \geq c\Delta n^{-1/3}\log n$ and the random r-regular n-vertex graph with $r \geq c\Delta n^{2/3}\log n$ are a.a.s. universal for $\mathcal{T}(n,\Delta)$. Another interesting consequence is the existence of locally sparse n-vertex graphs that are universal for $\mathcal{T}(n,\Delta)$. For $\Delta \in O(1)$, we show that one can (randomly) construct n-vertex $\mathcal{T}(n,\Delta)$-universal graphs with clique number at most five. This complements the construction of Bhatt, Chung, Leighton, and Rosenberg (1989), whose $\mathcal{T}(n,\Delta)$-universal graphs with merely $O(n)$ edges contain large cliques of size $\Omega(\Delta)$.

We also derive some lower bounds and show that there exist very good expanders which are not universal for $\mathcal{T}(n,\Delta)$. In particular, we see that there are expanders of minimum degree $\Omega(n/\log n)$ which are not $\mathcal{T}(n,c\sqrt{n})$-universal. Finally, we show robustness of random graphs with respect to being universal for $\mathcal{T}(n,\Delta)$ in the context of the Maker-Breaker tree-universality game.

1 Introduction

A graph G is *universal* for a class of graphs $\mathcal{F}$ (equivalently, we say that G is $\mathcal{F}$*-universal*) if a copy of every member of $\mathcal{F}$ is contained in G. Since universality for the class $\mathcal{F}$ implies that the maximum degree of G is at least as large as the maximum degrees of all graphs in $\mathcal{F}$, it is natural to consider only classes with bounded maximum degree. There exists a rich literature on explicit and randomized constructions of universal graphs [2, 20, 6, 12, 17, 18, 19, 27, 42, 30, 3, 15, 16, 4]. One of the classes for which universality has been studied extensively is the class of bounded degree trees. For two positive integers n and Δ, let $\mathcal{T}(n,\Delta)$ be the class of all n-vertex trees with maximum degree at most Δ. Bhatt, Chung, Leighton, and Rosenberg [12] gave an explicit construction of very sparse n-vertex $\mathcal{T}(n,\Delta)$-universal graphs of maximum degree bounded by a function in Δ. For $\Delta \in O(1)$, their universal graphs have only $O(n)$ edges.

In this work, instead of constructing specific universal graphs, we are rather interested in determining for which edge densities almost all n-vertex graphs become $\mathcal{T}(n,\Delta)$-universal. In particular, we want to know

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for which edge probabilities p, the binomial random graph $G(n,p)$ asymptotically almost surely (a.a.s.) becomes universal for the class of all bounded degree spanning trees. Moreover, we want to identify particular pseudo-random properties that guarantee this universality and that are a.a.s. satisfied by $G(n,p)$.

Since every connected component of a graph contains a spanning tree, we know that, for large values of c, the random graph $G(n,c/n)$ a.a.s. contains a copy of some tree that covers a significant proportion of the vertices of G. We may now ask whether this is true for every *specific* tree with size linear in n. For paths, that is, trees with maximum degree two, Ajtai, Komlós and Szemerédi [1] showed that if $c > 1$ the random graph $G(n,c/n)$ indeed contains a.a.s. a path of length linear in n. On the other hand, for fixed c, the random graph $G(n,c/n)$ a.a.s. has maximum degree $(1+o(1))\log n/\log\log n$ and therefore we cannot expect to embed trees of larger maximum degree. Thus, a more reasonable question is to ask whether for every bounded degree tree T with size linear in n, the random graph $G(n,c/n)$ a.a.s. contains a copy of T.

This question was first addressed by Fernandez de la Vega [25], who showed that, for fixed $\Delta \geq 2$ and $7/8 \leq \alpha < 1$, there exists a constant $c = c(\Delta,\alpha)$ with $\Delta - 1 < c \leq 8(\Delta-1)$ such that, for every specific tree $T \in \mathcal{T}((1-\alpha)n,\Delta)$, the random graph $G(n,c/n)$ a.a.s. contains a copy of T. Alon, Krivelevich, and Sudakov [5] showed for all $\varepsilon \in (0,1)$ the existence of a constant $c = c(\Delta,\varepsilon)$ such that $G(n,c/n)$ a.a.s. contains a copy of all trees in $\mathcal{T}((1-\varepsilon)n,\Delta)$, thus extending the result Fernandez de la Vega to *almost spanning trees*, that is, to arbitrary small values of ε. A better bound for $c(\Delta,\varepsilon)$ and a resilience version of this result were obtained by Balogh, Csaba, Pei, and Samotij [7] and by Balogh, Csaba, and Samotij [8], respectively.

Besides being valid for small values of ε, the results of Alon et al. as well as of Balogh et al. exhibit a substantial difference to that of Fernandez de la Vega. Instead of (i) showing that a *particular* tree in $\mathcal{T}((1-\varepsilon)n,\Delta)$ (that is, one tree for every n) is a.a.s. contained in $G(n,p)$, they show instead that (ii) $G(n,p)$ is a.a.s. *universal* for the whole class $\mathcal{T}((1-\varepsilon)n,\Delta)$, that is, contains a copy of every tree in $\mathcal{T}((1-\varepsilon)n,\Delta)$ simultaneously. Note that, for $\Delta \geq 3$, the size of the class $\mathcal{T}((1-\varepsilon)n,\Delta)$ is exponential in n and therefore the union-bound is not sufficient to derive (ii) from (i).

In order to show that $G(n,c/n)$ is a.a.s. universal for $\mathcal{T}((1-\varepsilon)n,\Delta)$, both Alon et al. [5] and Balogh, Csaba, Pei, and Samotij [7] showed that $G(n,c/n)$ exhibits certain pseudo-random properties that imply large expansion of small sets of vertices (after one deletes few vertices with very small degrees). This allows to apply the classical tree-embedding result of Friedman and Pippenger [27] (as was done in [5]) or its somewhat stronger version due to Haxell [30] (as was done in [7]) to embed every bounded degree tree that covers all but an ε-fraction of the vertices of $G(n,c/n)$. Recently, Sudakov and Vondrák [43] gave a randomized algorithm to efficiently embed bounded degree almost spanning trees in graphs with certain expansion properties. We discuss the result of Haxell in Section 3.

For $pn > \log n$, the situation changes drastically. In this regime, $G(n,p)$ is connected and we may ask for the existence of *spanning* trees. The very specific case of embedding a Hamilton path was resolved by Komlós and Szemerédi [35] and, independently, Bollobás [13], who proved that if $pn \geq \log n + \log\log n + \omega(1)$ (where $\omega(1)$ is any function which tends to infinity as $n \to \infty$), then $G(n,p)$ a.a.s. contains a Hamilton cycle. Frieze and Krivelevich [28] and Krivelevich and Sudakov [37] investigated pseudo-random conditions expressed in terms of the spectral gap of the host graph which guarantee the existence of Hamilton paths. Hefetz, Krivelevich, and Szabó [31] showed *Hamilton-connectedness* (that is, the existence of a Hamilton path between any two vertices) of graphs with expansion properties similar to those we introduce in the next section. We discuss their result in Section 3.

Addressing the question of embedding a *particular* tree $T \in \mathcal{T}(n,\Delta)$ (again one tree for every n), Krivelevich [36] showed that if $np \geq \frac{40}{\varepsilon}\Delta\log n + n^{\varepsilon}$ for some $\varepsilon > 0$, then the random graph $G(n,p)$ a.a.s. contains a copy of T. Moreover, it is shown in [36] that this bound on p is asymptotically tight in the order of magnitude if $n^{\varepsilon} \leq \Delta \leq n/\log n$. Extending and improving a result of Alon, Krivelevich, and Sudakov [5], Hefetz, Krivelevich, and Szabó [32] showed that if, in addition, T has a linear number of leaves or contains a bare path (that is, a path in which all vertices have degree two in T) of length linear in n, then $G(n,p)$ a.a.s. contains a copy of T already for $pn = (1+o(1))\log n$.

To the best of our knowledge, until now there exist no results directly addressing the question whether $G(n,p)$ is a.a.s. $\mathcal{T}(n,\Delta)$-universal for $p \in o(1)$. For $\mathcal{G}(n,\Delta)$, the class of all graphs on n vertices with maximum degree at most Δ, Dellamonica, Kohayakawa, Rödl, and Ruciński [21] showed that there exists a constant $c := c(\Delta)$ such that, for $pn \geq cn^{1-1/(2\Delta)}\log^{1/\Delta} n$, the random graph $G(n,p)$ is a.a.s. $\mathcal{G}(n,\Delta)$-universal, improving an earlier result in [4]. As a special case, this result also applies to the subclass $\mathcal{T}(n,\Delta)$ and implies that the random graph $G(n,p)$ is a.a.s. $\mathcal{T}(n,\Delta)$-universal for such values of p. Recently, Dellamonica et al. [22] improved their result to $pn \geq cn^{1-1/\Delta}\log n$. However, in all these bounds $pn = n^{1-o(1)}$ for $\Delta \to \infty$.

1.1 Notation

Let $\mathbb{N}$ and $\mathbb{R}^+$ be the sets of positive integers and positive real numbers, respectively. For two functions $f, g\colon \mathbb{N} \to \mathbb{R}^+$, we write $f \in o(g)$ or, equivalently, $f \ll g$, to denote the fact that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$ and $f \in O(g)$ or, equivalently, $g \in \Omega(f)$, to denote the fact that there exists an absolute positive constant c such that $f(n) \leq cg(n)$ for all positive integers n.

Given a graph G, we denote its vertex set by $V(G)$ and its edge set by $E(G)$. All graphs considered in this work are finite (where typically $|V(G)|$ will be denoted by n), simple, and undirected. For a set $X \subseteq V(G)$, let $G[X]$ be the subgraph of G induced by X and let $N_G(X)$ be the *external* neighborhood of X, that is, $N_G(X) = |\{y \in V(G) \setminus X \mid \exists x \in X\colon \{x,y\} \in E(G)\}|$. For a vertex $v \in V(G)$, let $N_G(v) := N_G(\{v\})$ and let $\deg_G(v) := |N_G(v)|$. For two sets $X, Y \subseteq V(G)$, $e_G(X,Y)$ denotes the number of ordered pairs (x,y) with $x \in X$ and $y \in Y$ such that $\{x,y\} \in E(G)$. Note that if X and Y intersect, then all edges in the intersection are counted twice.

Finally, for two graphs H and G, an *embedding* φ of H in G is an injective graph homomorphism, that is, an injective map $\varphi\colon V(H) \to V(G)$ such that $\{v,w\} \in E(H)$ implies $\{\varphi(v), \varphi(w)\} \in E(G)$. We say φ embeds H *onto* G if φ is bijective and we say G *contains a copy of* H if there exists an embedding of H in G.

2 Our Results

2.1 Tree-Universality of (n,d)-Expanders

The main contribution of this work is establishing tree-universality for the members of a certain class of graphs with good expansion properties, which we term (n,d)-*expanders*. For all positive integers n and all positive real numbers d, let

$$m(n,d) := \left\lceil \frac{n}{2d} \right\rceil.$$

The following notion is an adaptation of the expansion properties investigated in [7] and [31].

Definition 2.1. ((n,d)-expander) *Let n be a positive integer, let d be a positive real number. A graph G is an (n,d)-*expander *if $|V(G)| = n$ and G satisfies the following two conditions:*

E1: $|N_G(X)| \geq d|X|$ holds for all $X \subseteq V(G)$ with $1 \leq |X| < m(n,d)$.

E2: $e_G(X,Y) > 0$ holds for all disjoint $X, Y \subseteq V(G)$ with $|X| = |Y| = m(n,d)$.

A simple calculation (Lemma 3.1 in Section 3) shows that these properties are monotone, that is, if d and d_0 satisfy $3 \leq d_0 \leq d \leq n/6$, then every (n,d)-expander is also an (n,d_0)-expander.

The main result of this work is the following theorem which states that (n,d)-expanders are tree-universal.

Theorem 2.1. (Tree-Universality) *There exists an absolute positive constant c such that the following statement holds. Let n and Δ be two positive integers satisfying $\log n \leq \Delta \leq cn^{1/3}$. Then every $(n, 7\Delta n^{2/3})$-expander is universal for $\mathcal{T}(n,\Delta)$.*

Note that the class $\mathcal{T}(n,\log n)$ also includes all n-vertex trees of maximum degree Δ smaller than $\log n$ (for example, for $\Delta \in O(1)$ and n large). Thus, Theorem 2.1 also applies to the situation where $\Delta < \log n$ by setting Δ to $\log n$. The proof of Theorem 2.1 is given in Section 4.

There are many known constructions of expanders (see, e.g. [38]). Thus, by verifying the conditions in Definition 2.1, one obtains explicit constructions of relatively sparse universal graphs for $\mathcal{T}(n,\Delta)$.

2.2 Random Graphs

Random graphs are well-known to typically exhibit strong expansion properties. For example, the random graph $G(n,p)$ with $pn \geq 7d\log n$ is a.a.s. an (n,d)-expander. We omit the proof of this claim, which can be shown by a simple union-bound argument. Thus, as a direct consequence of Theorem 2.1, such a random graph is a.a.s. universal for the class of all bounded-degree spanning trees.

Theorem 2.2. *There exists an absolute positive constant c such that the following statement holds. Let $\Delta := \Delta(n)$ satisfy $\Delta \geq \log n$. Then the random graph $G(n,p)$ is a.a.s. universal for $\mathcal{T}(n,\Delta)$, provided that $pn \geq c\Delta n^{2/3}\log n$.*

Theorem 2.2 implies that $G(n, cn^{-1/3}\log^2 n)$ is a.a.s. universal for $\mathcal{T}(n,\log n)$ and thus also for $\mathcal{T}(n,O(1))$ if c is a large enough constant.

Likewise, the random r-regular n-vertex graph with $\max\{7d\log n, \sqrt{n}\log n\} \leq r \ll n$ (where rn is even) is a.a.s. an (n,d)-expander. We omit the proof of this claim, which can be shown similarly to Theorem 2.2 in [39] which applies a switching argument originally introduced in [40]. Thus, again as a direct consequence of Theorem 2.1, such a random r-regular graph is a.a.s. tree-universal.

Theorem 2.3. *There exists an absolute positive constant c such that the following statement holds. Let $\Delta := \Delta(n)$ satisfy $\Delta \geq \log n$. Then the r-regular random graph on n vertices is a.a.s. universal for $\mathcal{T}(n,\Delta)$, provided that $c\Delta n^{2/3}\log n \leq r \ll n$, and rn is even.*

Note that the restriction $r \ll n$ in Theorem 2.3 is likely to be an artifact resulting from the use of a switching argument in our proof and we believe the result should extend to linear values of $r = r(n)$. Similarly as before, Theorem 2.3 implies that the random r-regular graph on n vertices with $r \geq cn^{2/3}\log^2$ (where rn is even) is a.a.s. universal for $\mathcal{T}(n, O(1))$ if c is a large enough constant.

2.3 Locally Sparse Expanders

Bhatt, Chung, Leighton, and Rosenberg [12] gave an explicit construction of $\mathcal{T}(n,\Delta)$-universal graphs on n vertices whose maximum degree is bounded by a function of Δ. Thus, for constant Δ, the number of edges in this graph is in $O(n)$. In comparison, the random graph we consider in Theorem 2.2 a.a.s. has $\Theta(n^{5/3}\log^2 n)$ edges. However, the graph constructed in [12] is *locally dense*, that is, it contains a large number of cliques of size $\Omega(\Delta)$ (cf. Lemma 8 in [12]). Here, we show how to construct *locally sparse* graphs that are universal for all bounded degree trees.

We first observe that the expected number of cliques of size k in the random graph $G(n,p)$ is $\binom{n}{k}p^{\binom{k}{2}}$. Therefore, by Markov's inequality, if $n^k p^{\frac{k(k-1)}{2}} \ll 1$, then $G(n,p)$ a.a.s. does not contain any clique of size k. Consequently, for $p(n) = n^{-1/3}\log^2 n$, the random graph $G(n,p)$ is a.a.s. both K_8-free and universal for $\mathcal{T}(n, \log n)$. Thus, for sufficiently large n, there exists a $\mathcal{T}(n, \log n)$-universal graph with clique number at most seven.

We can strengthen this observation by showing that, for an appropriate choice of chosen d, r, and p, the random graph $G(n,p)$ is still a.a.s. an (n,d)-expander even if we make it K_r-free by deleting a carefully chosen set of edges. (We omit the proof of this claim.) Together with Theorem 2.1, this implies the existence of locally sparse tree-universal graphs.

THEOREM 2.4. *There exists an absolute positive constant c such that the following statement holds. Let n and r be two positive integers with $r \geq 5$. Then there exists a graph with clique number at most r that is universal for $\mathcal{T}(n, cn^{1/3-2/(r+2)}/\log n)$.*

In particular, Theorem 2.4 implies that there exists a $\mathcal{T}(n, cn^{1/21}/\log n)$-universal graph with clique number at most five for all positive integers n if c is a small enough constant.

2.4 Lower Bound Constructions

In Theorem 2.1, we gave an *upper* bound of $7\Delta n^{2/3}\log n$ on the minimum value d^* such that, for all $d \geq d^*$, every (n,d)-expanders are $\mathcal{T}(n,\Delta)$-universal. We now discuss *lower* bounds on d^*, that is, constructions of (n,d)-expanders with (relatively) large values of d which are not universal for $\mathcal{T}(n,\Delta)$.

For random graphs, Krivelevich [36] showed the following: for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $n^\varepsilon \leq \Delta \leq \frac{n}{\log n}$, then there exists a tree $\mathcal{T}(n,\Delta)$ of which the random graph $G(n,p)$ with $pn = \delta\Delta\log n$ a.a.s. does not contain a copy. In contrast, Theorem 2.2 shows that there exist an absolute positive constant c such that $pn = c\Delta n^{2/3}\log n$ is sufficient for $G(n,p)$ to become universal for $\mathcal{T}(n,\Delta)$. This huge gap of order $n^{2/3}$ seems to be mainly an artifact of the proof of Theorem 2.1.

In the setting of (n,d)-expanders and for the case $\Delta \in O(1)$, we know of no lower bounds on the smallest value of d necessary for every (n,d)-expander to be $\mathcal{T}(n,\Delta)$-universal (except for the trivial lower bound Δ). For $\Delta \in n^{\Omega(1)}$, however, we can show that (in contrast to the random graph setting) this value grows faster than $\Omega(\Delta\log n)$. To this end, recall that the radius of a connected graph G is defined as

$$r(G) := \min_{u\in V}\max_{v\in V}\operatorname{dist}(u,v)$$

where the *distance* $\operatorname{dist}(u,v)$ between two vertices u and v is the length of a shortest path connecting u and v in G. For example, the radius of the star-graph $K_{1,n}$ is 1 and the radius of a path of even length $2k$ is k. A crucial observation is that we cannot embed a spanning graph onto a host graph with a strictly larger radius: the embedding itself would be a proof that the host graph has small radius, too. We will show that, for certain values of n and Δ, the class $\mathcal{T}(n,\Delta)$ contains trees with a relatively small radius, whereas there are (n,d)-expanders (with quite large d) with a fairly large radius.

First, consider the rooted tree $T_{\Delta,r}$ with $r+1$ levels of vertices, where all vertices except for the leaves have Δ neighbors. Thus, there is one vertex (the root) on the 0-th level and $\Delta(\Delta-1)^{i-1}$ vertices on the i-th level for $i \in \{1,\ldots,r\}$. Then $T_{\Delta,r}$ has $1+\Delta\sum_{i=1}^{r}(\Delta-1)^{i-1} = 1+\frac{\Delta(\Delta-1)^r-1}{\Delta-2}$ vertices in total, has radius r, and is in $\mathcal{T}(n,\Delta)$. On the other hand, we show below that there exist very strong expanders with radius at least $r+1$.

DEFINITION 2.2. *Let n be a positive integer and let H be a graph on n vertices. We define $\mathcal{G}_H$ to be the class of 2n-vertex graphs obtained from H by replacing each vertex v of H by two vertices u_v and u'_v and each edge $\{v,w\}$ of H by either the two edges $\{u_v,u_w\}$ and $\{u'_v,u'_w\}$ or the two edges $\{u_v,u'_w\}$ and $\{u'_v,u_w\}$.*

For the class $\mathcal{G}_H$, the following result holds if we choose H to be the complete graph (in the case $r = 2$) or a suitable pseudo-random graph with girth $r+1$ (in the case $r \geq 3$). The details of the proof are omitted.

LEMMA 2.1. *There exists an absolute positive constant c such that the following statement holds. Let n and r be two positive integers that satisfy $r \geq 2$ and $cn^{1/(r-1)} \geq 3 \log n$. Then there exists an n-vertex graph H such that all graphs in $\mathcal{G}_H$ have radius at least $r+1$ and a graph chosen uniformly at random from $\mathcal{G}_H$ is a.a.s. a $(2n, cn^{1/(r-1)} \log^{-1} n)$-expander.*

Combining Lemma 2.1 with our discussion of the radius of $T_{\Delta,r}$, we get the following theorem.

THEOREM 2.5. *There exists an absolute positive constant c such that the following statement holds. Let Δ and r be positive integers satisfying $\Delta \geq c^{-1} \log n$ with $n := 1 + \frac{\Delta(\Delta-1)^r - 1}{\Delta - 2}$. Then there exists an $(n, c\Delta^{1+1/r} \log^{-1} n)$-expander which is not universal for $\mathcal{T}(n, \Delta)$.*

Note that although all graphs in $\mathcal{G}_H$ have an even number of vertices we do not require this restriction in Theorem 2.5: If we duplicate a vertex of an (n, d)-expander with radius r and connect the duplicate to all of the vertex's neighbors we obtain an $(n+1, d/2)$-expander which still has radius r.

Theorem 2.5 implies that there even exist $(n, cn/\log n)$-expanders which are not universal for $\mathcal{T}(n, (1+o(1))\sqrt{n})$. In comparison, we do not expect the same to be true a.a.s. for $G(n, 7c)$, which (a.a.s.) is a canonical example of an $(n, cn/\log n)$-expander. Moreover, our construction complements a result of Komlós, Sárközy, and Szemerédi [34]. They showed that, for every given δ, there exists a constant $c := c(\delta)$ such that every graph of minimum degree $(1+\delta)n/2$ is $\mathcal{T}(n, cn/\log n)$-universal. It is clear that this bound is sharp, since if we allow the minimum degree to be at most $\lfloor n/2 \rfloor - 1$, then the host graph may be disconnected. However, our $(\lfloor n/2 \rfloor - 1)$-regular construction shows that even host graphs that have an edge between all disjoint pairs of vertex sets of relatively small size $O(\log n)$ may be not $\mathcal{T}(n, (1+o(1))\sqrt{n})$-universal. Finally, we remark that Böttcher, Taraz, and Würfl [14] observed a similar effect in the context of (ε, δ)-regular graphs and independently proposed a construction with similar properties as $\mathcal{G}_H$.

2.5 Universality for Almost All Labeled Trees

For all positive integers n, let $\mathcal{T}_n$ be the family of all labeled trees on the vertex set $\{1, \ldots, n\}$. Bender and Wormald [11] showed that for every constant $p \in (0,1)$ there is a subfamily $\mathcal{T}_n^* \subseteq \mathcal{T}_n^*$ with $|\mathcal{T}_n^*| = (1-o(1))|\mathcal{T}_n|$ such that the random graph $G(n,p)$ is a.a.s. universal for $\mathcal{T}_n^*$. Note that this notion differs substantially from the (weaker) notion of being *almost-universal* for $\mathcal{T}_n$ (see, e.g. [29, 15]), which means that if T is drawn uniformly at random from all trees of $\mathcal{T}_n$, then G a.a.s. contains a copy of T.

It is well known (see, e.g. [41]) that a tree chosen uniformly at random from $\mathcal{T}_n$ has a.a.s. maximum degree at most $(1+o(1)) \log n / \log \log n$. Therefore, the subfamily $\mathcal{T}_n^*$ of all trees in $\mathcal{T}_n$ with maximum degree at most $2 \log n / \log \log n$ satisfies $|\mathcal{T}_n^*| = (1-o(1))|\mathcal{T}_n|$. Thus, Theorem 2.2 strengthens the result of Bender and Wormald as it allows us to replace the constant $p \in (0,1)$ with a function $p \in o(1)$.

THEOREM 2.6. *There exists an absolute positive constant c and a subfamily $\mathcal{T}_n^*$ of $\mathcal{T}_n$ satisfying the condition $|\mathcal{T}_n^*| = (1-o(1))|\mathcal{T}_n|$ such that the random graph $G(n,p)$ is a.a.s. universal for $\mathcal{T}_n^*$, provided that $p \geq cn^{-1/3} \log^2 n$.*

2.6 The Maker-Breaker Game

In recent studies of extremal properties of random graphs (like tree-universality), a central concept is that of *robustness*. This means that we require some property to be not only *typically* present in the respective graph class (that is, to appear a.a.s. in a random graph drawn from this class), but also to persist after a modification of the random instance. There are numerous ways to model robustness, for example by the notion of resilience or via positional games. Here, we study the robustness of tree-universality in expanders in the setting of a Maker-Breaker game.

An $(a\!:\!b)$ Maker-Breaker game is played on a finite hypergraph $(X, \mathcal{F})$ between two players, *Maker* and *Breaker*. The vertex set of the hypergraph is the *board* and the hyperedges are the *winning sets* in the game. The game is played in turns, starting with Maker's turn. In each of their turns, Maker claims a and Breaker claims b previously unclaimed vertices. The numbers a and b are called the *biases* of Maker and Breaker, respectively. Maker's objective is to claim all elements of a winning set by the end of the game. In this case, Maker wins the game. Breaker's objective is to claim at least one element in each winning set by the end of the game. In this case, Breaker wins the game. The game ends when all vertices have been claimed, by which time either Maker or Breaker have won.

We say that an $(a\!:\!b)$ Maker-Breaker game on $(X, \mathcal{F})$ is *Maker's win* if Maker has a strategy that allows him to win the game regardless of Breaker's strategy, otherwise the game is *Breaker's win*. Clearly, every $(a\!:\!b)$ Maker-Breaker game $(X, \mathcal{F})$ is either Maker's or Breaker's win and the decision which of the two holds depends only on the parameters a, b, X, and $\mathcal{F}$. For a more detailed discussion, we refer to [10].

We now formulate a Maker-Breaker game for preserving tree-universality of a graph. Given a graph, Maker tries to claim a set of edges which induces a $\mathcal{T}(n,\Delta)$-universal subgraph.

DEFINITION 2.3. *For two positive integers n and Δ, the* Maker-Breaker $\mathcal{T}(n,\Delta)$-universality game *on a graph G is the Maker-Breaker game on the hypergraph $(E(G),\mathcal{F})$, where $\mathcal{F}$ consists of all edge sets $F \subseteq E(G)$ such that the subgraph $(V(G),F)$ is $\mathcal{T}(n,\Delta)$-universal.*

Our main finding in this section is a condition for Maker's win in the $(1:b)$ Maker-Breaker Tree-Universality game on an (n,d)-expander.

THEOREM 2.7. *There exists an absolute positive constant c such that the following statement holds. Let n, b, and Δ be integers that satisfy $\Delta \geq \log n$. Then the $(1:b)$ Maker-Breaker $\mathcal{T}(n,\Delta)$-universality game is Maker's win on every (n,d)-expander with $d \geq cb\Delta n^{2/3}\log n$.*

Theorem 2.7 implies that, for $\Delta \leq \log n$ and in particular for $\Delta \in O(1)$ and n sufficiently large, the $(1:b)$ Maker-Breaker $\mathcal{T}(n,\Delta)$-universality game is Maker's win on every (n,d)-expander with $d \geq cbn^{2/3}\log^2 n$.

We omit the proof of Theorem 2.7, which can be derived using Beck's generalized Erdős-Selfridge criterion for Breaker's win [24, 9]. Together with the fact that the random graph $G(n,p)$ with $pn \geq 7d\log n$ is a.a.s. an (n,d)-expander (see above), Theorem 2.7 implies the following condition for Maker's win in the tree-universality game on binomial random graphs.

COROLLARY 2.1. *There exists an absolute positive constant c such that the following statement holds. Let $b := b(n)$ and let $\Delta := \Delta(n)$ satisfy $\Delta \geq \log n$. Then the $(1:b)$ Maker-Breaker $\mathcal{T}(n,\Delta)$-universality game is a.a.s. Maker's win on the random graph $G(n,p)$, provided that $pn \geq cb\Delta n^{2/3}\log^2 n$.*

Correspondingly, Theorem 2.7 and the fact that the random r-regular n-vertex graph with $\max\{7d\log n, \sqrt{n}\log n\} \leq r \ll n$ (where rn is even) is a.a.s. an (n,d)-expander (see above) imply the following conditions for Maker's win for the tree-universality game on binomial random graphs.

COROLLARY 2.2. *There exists an absolute positive constant c such that the following statement holds. Let $b := b(n)$ and let $\Delta := \Delta(n)$ satisfy $\Delta \geq \log n$. Then the $(1:b)$ Maker-Breaker $\mathcal{T}(n,\Delta)$-universality game is a.a.s. Maker's win on the random r-regular graph, provided that $r \geq cb\Delta n^{2/3}\log^2 n$, $r \in o(n)$, and rn is even.*

3 Properties of (n,d)-Expanders

We now present all properties of (n,d)-expanders that are needed to prove the universality results in the remainder of this work. First, we observe that the expansion properties given in Definition 2.1 are monotone in d.

LEMMA 3.1. *Let n be a positive integer and let d and d_0 be two positive real numbers such that $3 \leq d_0 \leq d \leq n/6$. Then every (n,d)-expander is also an (n,d_0)-expander.*

Proof. Let $m := m(n,d)$ and $m_0 := m(n,d_0)$. Since $m_0 \geq m$, condition (E2) holds immediately for the parameter m_0, and since $d_0 \leq d$, condition (E1) holds immediately for the parameters m and d_0. Thus, it is sufficient to verify that $|N_G(X)| \geq d_0|X|$ holds for all $X \subseteq V(G)$ with $m \leq |X| < m_0$. For such a set X, we have by condition (E2) that $|N_G(X)| \geq n-|X|-m$ and therefore

$$|N_G(X)| \geq 2d_0(m_0-1) - 2m_0 \geq d_0 m_0 \geq d_0|X|.$$

The lemma follows.

The following fact is an important insight into the structure of sparse expanders and is frequently used in the proof of Theorem 2.1. It allows us to bound the number of vertices with small neighborhood in any sufficiently large vertex set of an expander.

LEMMA 3.2. *Let G be a graph, let m be a positive integer, and let $W \subseteq V(G)$ satisfy $|W| \geq m^2$. Suppose that $e_G(X,Y) > 0$ for all $X \subseteq V(G) \setminus W$ and all $Y \subseteq W$ that satisfy $|X| = |Y| = m$. Then there are at most $m-1$ vertices in $V(G) \setminus W$ that have less than m neighbors in W.*

In the following proof a simple counting argument shows that if there exist m exceptional vertices in $V(G) \setminus W$, then there are at least m vertices in W which are not in their neighborhood — a contradiction to (E2) of Definition 2.1.

Proof. Let $V := V(G)$. Assume for contradiction that there exists a set $X \subseteq V \setminus W$ with $|X| = m$ such that $|N_G(v) \cap W| < m$ for all $v \in X$.

On one hand, since we have $|X| = m$ and therefore $|V \setminus (X \cup N_G(X))| \leq m-1$, we get

$$e_G(X,W) \geq |W \cap N_G(X)| \geq |W| - (m-1) \geq m^2 - m + 1.$$

On the other hand,

$$e_G(X,W) = \sum_{x \in X} |N_G(x) \cap W| \leq m(m-1) = m^2 - m,$$

which is clearly a contradiction. Thus, no such set X exists and there are at most $m-1$ vertices in $V \setminus W$ with fewer than m neighbors in W.

3.1 Partitioning Expanders

We now show that we can partition the vertex set of an (n,d)-expander in such a way that the neighborhoods of small expanding sets distribute between the parts according to the sizes of the parts. In Section 4, this technique plays a major role in the proof of our main result, the tree-universality of sparse expanders (Theorem 2.1).

LEMMA 3.3. (PARTITION LEMMA) *There exists an absolute positive constant n_0 such that the following statement holds. Let k and n be two positive integers and d a positive real number such that $n \geq n_0$ and $k \leq \log n$. Furthermore, let $n_1, \ldots, n_k$ be positive integers satisfying $n = n_1 + \cdots + n_k$ and let $d_i := \frac{n_i}{5n} d$ satisfy $d_i \geq 2\log n$ for all $i \in \{1, \ldots, k\}$.*

Then, for every (n,d)-expander G, the vertex set $V(G)$ can be partitioned into k disjoint sets $U_1, \ldots, U_k$ of sizes $n_1, \ldots, n_k$, respectively, such that

$$|N_G(X) \cap U_i| \geq d_i|X| \tag{3.1}$$

holds for all sets $X \subseteq V$ with $1 \leq |X| < m(n,d)$ and all $i \in \{1, \ldots, k\}$. Moreover, for all $i \in \{1, \ldots, k\}$ and all $W_i \subseteq V(G)$ with $U_i \subseteq W_i$, the induced subgraph $G[W_i]$ is a $(|W_i|, d_i)$-expander.

This statement can be shown using the probabilistic method: Using the union bound and a tail bound on the hypergeometric distribution, we show that a uniformly random partition of an (n,d)-expander into k parts of sizes $n_1, \ldots, n_k$ satisfies (3.1).

Before we prove Lemma 3.3, we first state a well-known result (see, e.g., [33, Theorem 2.10]) for bounding the tail probabilities of the hypergeometric distribution $\textsc{Hyp}(n,m,\ell)$. A random variable X distributed according to $\textsc{Hyp}(n,m,\ell)$ models the number of white balls found among ℓ balls drawn without replacement from an urn containing n balls, m of which are white. Recall that we have $\Pr[X = k] = \binom{m}{k}\binom{n-m}{\ell-k}/\binom{n}{\ell}$ for all $0 \leq k \leq n$ and $\mathbb{E}[X] = m\ell/n$.

THEOREM 3.1. *Let ε be a positive constant satisfying $\varepsilon \leq 3/2$ and let $X \sim \textsc{Hyp}(n,m,\ell)$. Then*

$$\Pr\left[|X - \mathbb{E}[X]| > \varepsilon\right] \leq \mathrm{e}^{-\frac{\varepsilon^2}{3}\mathbb{E}[X]}.$$

Proof. (Proof of Lemma 3.3) Choose n_0 such that $\log n \leq n^{2/15}$. Let $V = V(G)$ and $m := m(n,d)$. We show the existence of a partition $U_1, \ldots, U_k$ which respects (3.1) by a simple probabilistic argument.

Choose a partition $U_1, \ldots, U_k$ of V into disjoint sets of respective sizes $n_1, \ldots, n_k$ uniformly at random. We show that with positive probability, (3.1) holds for all sets $X \subseteq V$ with $1 \leq |X| < m$ and all $i \in \{1, \ldots, k\}$.

Let $X \subseteq V$ be a set with $1 \leq |X| < m$ and let $i \in \{1, \ldots, k\}$. Then the random variable $|N_G(X) \cap U_i|$ is distributed according to the hypergeometric distribution $\textsc{Hyp}(n, n_i, |N_G(X)|)$ with

$$\mathbb{E}\big[|N_G(X) \cap U_i|\big] = \frac{n_i}{n}\,|N_G(X)| \geq \frac{n_i}{n}\,d\,|X| = 5d_i|X|.$$

We apply Theorem 3.1 with $\varepsilon = 4/5$ and obtain

$$\Pr\left[|N_G(X) \cap U_i| \leq d_i|X|\right] \leq \mathrm{e}^{-\frac{16}{15}d_i|X|} \leq n^{-\frac{32}{15}|X|}.$$

Let q be the probability that there exists a set $X \subseteq V$ with $1 \leq |X| < m$ and an $i \in \{1, \ldots, k\}$ which violates property (3.1). Then, by the union bound,

$$q \leq \sum_{i=1}^{k}\sum_{j=1}^{m}\binom{n}{j} n^{-\frac{32}{15}j} < \sum_{i=1}^{k}\sum_{j=1}^{n} n^j n^{-\frac{32}{15}j} \leq kn^{-\frac{2}{15}} \leq 1$$

for sufficiently large n.

We have shown that with positive probability the randomly chosen partition $U_1, \ldots, U_k$ satisfies property (3.1); therefore, such a partition exists and the first statement of Lemma 3.3 holds.

Finally, let $U_1, \ldots, U_k$ be such a partition that satisfies property (3.1). Let $i \in \{1, \ldots, k\}$ and consider a set $W \subseteq V$ with $U_i \subseteq W$ and the induced graph $H = G[W]$. Then, by the choice of d_i, we have

$$m(|W|, d_i) \geq m(n_i, d_i) \geq m(n,d).$$

Thus, since G is an (n,d)-expander, condition (E2) in Definition 2.1 with $m = m(|W|, d_i)$ holds for H. By (3.1), $|N_H(X)| \geq d_i|X|$ holds for all $X \subseteq V(H)$ with $1 \leq |X| < m(n,d)$. Thus, similar to the proof of Lemma 3.1, it is sufficient to verify that the condition $|N_H(X)| \geq d_i|X|$ holds also for all $X \subseteq V(H)$ with $m(n,d) \leq |X| < m(|W|, d_i)$. Since G is an (n,d)-expander, we have $|N_H(X)| \geq |W| - |X| - m(n,d)$ for such a set X and therefore

$$|N_H(X)| \geq 2d_i(m(n_i,d_i) - 1) - 2m(n_i,d_i) \geq d_i|X|$$

and the second statement of Lemma 3.3 holds.

3.2 Almost Spanning Trees, Hamilton Paths, and Star Matchings

We now summarize three known results on embedding almost spanning trees, Hamilton paths, and star matchings in graphs with large expansion. These results are crucial for the proof of Theorem 2.1 (Tree-Universality).

In [30], Haxell extended a result of Friedman and Pippenger [27] and showed that one can embed every almost spanning tree with bounded maximum degree in a graph with sufficiently large expansion. Here, we present a formulation of this result in the flavor of Theorem 3 in [7].

Theorem 3.2. *Let d, m, and k be positive integers and let H be a non-empty graph satisfying the following two conditions:*

(i) $|N_H(X)| \geq d|X| + 1$ *holds for all sets* $X \subseteq V(H)$ *with* $1 \leq |X| \leq m$,

(ii) $|N_H(X)| \geq d|X| + k$ *holds for all sets* $X \subseteq V(H)$ *with* $m < |X| \leq 2m$.

Then the graph H contains a copy of every tree T with $|V(T)| \leq k + 1$ and maximum degree at most d.

In terms of (n, d)-expanders, we may reformulate the previous theorem as follows.

Corollary 3.1. *Let n and Δ be two positive integers and let d be a positive real number with $d \geq 2\Delta$. Then every (n, d)-expander is $\mathcal{T}(n-4\Delta m(n, d), \Delta)$-universal.*

Proof. Without loss of generality we may suppose that $\Delta \geq 2$. Let $m := m(n, d)$ and let $k := |V(T)|$. Furthermore, let H be an (n, d)-expander with $n = k+4\Delta m$ and let $T \in \mathcal{T}(k, \Delta)$.

Then, for all $X \subseteq V(H)$ with $1 \leq |X| \leq m$, we have by (E1) that

$$|N_H(X)| \geq 2\Delta|X| \geq \Delta|X| + 1.$$

Furthermore, we have $|N_H(X)| \geq n - |X| - m$ for all $X \subseteq V(H)$ with $m \leq |X| \leq 2m$, and therefore

$$|N_H(X)| \geq k + 4\Delta m - 3m \geq k + 2\Delta m \geq \Delta|X| + k.$$

The corollary then follows from Theorem 3.2.

Next, we state a result of Hefetz, Krivelevich, and Szabó [31] on the Hamilton-connectedness of expanders with edge-connectivity between large sets. For this, let us briefly revisit the notion of *Hamilton-connectedness*. An *x-y-Hamilton path* in a graph is a path with endvertices x and y that visits each vertex of the graph exactly once. A graph is *Hamilton-connected* if there exists an x-y-Hamilton path for every pair of vertices x and y in the graph. The following theorem is a simplified version of the results in [31].

Theorem 3.3. *Let n and d be two positive integers such that n is sufficiently large and $12 \leq d \leq \sqrt{n}$. Then a graph H on n vertices is Hamilton-connected if it satisfies the following two conditions:*

(i) $|N_H(X)| \geq d|X|$ *holds for all sets* $X \subseteq V(H)$ *with* $0 < |X| \leq \frac{n \log d}{d \log n}$,

(ii) $e_H(X, Y) > 0$ *holds for all disjoint* $X, Y \subseteq V(H)$ *with* $|X| = |Y| \geq \frac{n \log d}{1035 \log n}$.

As before, we give a reformulation of this result in terms of (n, d)-expanders.

Corollary 3.2. *There exists an absolute positive constant n_0 such that the following statement holds. Let n be a positive integer with $n \geq n_0$ and let d be a positive real number with $d \geq \log n$. Then every (n, d)-expander is Hamilton-connected.*

Proof. Let G be an (n, d)-expander. For $d_{\text{T 3.3}} = \mathrm{e}^{1035}$, consider the two conditions of Theorem 3.3. Condition (i) holds by Lemma 3.1 for sufficiently large n. Condition (ii) holds since

$$m \leq \frac{n}{d} \leq \frac{n}{\log n} = \frac{n \log d_{\text{T 3.3}}}{1035 \log n}.$$

Thus, G is Hamilton-connected by Theorem 3.3.

Finally, we state a version of Hall's marriage theorem for expanders which shows that we can embed a star matching in a bipartite graph with large expansion in one direction and large minimum degree in the other direction.

Lemma 3.4. *Let d and m be two positive integers and let G be a graph. Suppose that two disjoint vertex sets U and W satisfy the following three conditions:*

(i) $|N_G(X) \cap W| \geq d|X|$ *holds for all* $X \subseteq U$ *with* $1 \leq |X| \leq m$,

(ii) $e_G(X, Y) > 0$ *holds for all* $X \subseteq U$ *and* $Y \subseteq W$ *with* $|X| = |Y| \geq m$,

(iii) $|N_G(w) \cap U| \geq m$ *for all* $w \in W$.

Then, for every map $k\colon U \to \{0, \ldots, d\}$ that satisfies $\sum_{u \in U} k(u) = |W|$, there exists a partition of W into $|U|$ disjoint subsets $\{W_u\}_{u \in U}$ each satisfying $|W_u| = k(u)$ and $W_u \subseteq N_G(u) \cap W$. We call the set of edges between the vertices of U and their respective parts in W a star matching.

Proof. To prove this lemma, we show for all $X \subseteq U$ the generalized Hall's condition,

$$|N_G(X) \cap W| \geq \sum_{x \in X} k(x). \tag{3.2}$$

We distinguish three cases:

First, if $|X| < m$, then $k(x) \leq d$ for all $x \in X$ and (i) implies (3.2).

Second, if $m \leq |X| \leq |U| - m$, then $k(u) \geq 1$ for all $u \in U \setminus X$ and (ii) implies (3.2).

Third, if $|U| - m < |X|$, then (iii) directly implies (3.2).

Thus, (3.2) holds for all $X \subseteq U$ and the lemma is a direct consequence of the Max-Flow Min-Cut Theorem [23, 26].

4 Tree-Universality of (n,d)-Expanders

This section is devoted to the proof of our main result, Theorem 2.1 (Tree-Universality), which we presented in the introduction. The proof is based on a case distinction on whether the embedded tree contains a long bare path or many leaves. This extends the ideas in [36].

Definition 4.1. *Let T be a tree. A* leaf *of T is a vertex of degree one in T. A* bare path *is a path in T whose vertices have all degree two in T. If we remove all leaves from T, we call the leaves and bare paths in the remaining tree* second level leaves *and* second level bare paths*, respectively. For distinction, we call the leaves and bare paths of the original tree T also* first level leaves *and* first level bare paths*.*

The following observation was already made in [36] and states that a tree with bounded maximum degree contains a long bare path or many leaves.

Lemma 4.1. *Let T be a tree, let P be a bare path of maximum length in T, and let L be the set of leaves in T. Then*

$$2\big(|V(P)|+1\big)\big(|L|-1\big) \geq |V(T)|.$$

Proof. Let $V := V(T)$, let $L := \{v \in V \mid \deg_T(v) = 1\}$ and let $B := \{v \in V \mid \deg_T(v) \geq 3\}$. Then we have $|B| \leq |L| - 2$, since

$$-2 = 2|E(T)| - 2|V(T)| = \sum_{v\in V} \big(\deg_T(v) - 2\big) \geq |B| - |L|.$$

Next, we root T at an arbitrary leaf. This allows us to injectively map the bare paths of T to the set $L \cup B$ by assigning every bare path to the leaf or branching vertex adjacent to it farther away from the root. Therefore, the number of bare paths is at most $|L| + |B|$. Since every vertex in V is either in L, in B, or in a bare path of T, this implies that $|V| \leq |L| + |B| + |V(P)|(|L| + |B|)$ and therefore the lemma.

Consider the setting of Theorem 2.1. Let c be a sufficiently small positive constant and assume that n is sufficiently large. Let Δ satisfy $\log n \leq \Delta \leq cn^{1/2}$, let $T \in \mathcal{T}(n, \Delta)$, and let G be an (n, d)-expander with

$$d := 7\Delta n^{2/3}.$$

Recall that $m := m(n, d) = \lceil \frac{n}{2d} \rceil$. Lemma 4.1 tells us that T contains a bare path on $50\Delta m$ vertices or has at least $25\Delta m^2$ leaves, since

$$2(50\Delta m + 1)(25\Delta m^2 - 1) < n$$

for sufficiently large n and small c. In fact, we even have that

$$2(50\Delta m + 1)(25\Delta m^2 - 1) < n/\Delta$$

for sufficiently large n. Therefore, if L is the set of leaves in T, then the tree $T - L$ still contains a bare path on $50\Delta m$ vertices or has at least $25\Delta m^2$ leaves, since $|T - L| \geq n/\Delta$. Based on this observation, we consider three cases.

Case 1. *T contains a first level bare path on at least $50\Delta m$ vertices.*

In this case, we use Corollary 3.1 to first embed all of T in G except for the bare path (whose removal splits T into two rooted trees). Then we apply Corollary 3.2 to also embed the bare path by connecting the two roots by a path covering all the unused vertices in G. The details of this argument are given in Proposition 4.1.

Case 2. *T has at least $25\Delta m^2$ first level leaves and contains a second level bare path on at least $50\Delta m$ vertices.*

In this case, T has many leaves. We use Corollary 3.1 to first embed all of T in G except for the second level bare path and the leaves. Then we use Corollary 3.2 to embed the bare path. Finally, we use Lemma 3.4 to embed the leaves of T. Note that once T without the leaves is embedded, we know which vertices in G are the images of the parents of the leaves of T. We call these vertices in G the *portals* of the leaves. In order to embed the leaves, we need to find a star matching in G between the set of portals and the set of vertices which remain free after the embedding of T without the leaves. However, if we are not careful, then after embedding T without the leaves (and thus fixing the set of portals), some of the remaining vertices of G may be not connected to any of the portals. As these vertices would prevent us from finding a star matching, we call them *exceptional* vertices. We solve this problem by forcing the second level bare path to cover all exceptional vertices. The details of this argument are given in Proposition 4.2.

Case 3. *T has at least $25\Delta m^2$ first level leaves and at least $25\Delta m^2$ second level leaves.*

In this case, T has many (first level) leaves that are attached to second level leaves. We again use Corollary 3.1 to embed T without these two levels of leaves and then embed the leaves of each level separately using Lemma 3.4. Here, there again may exist a set of exceptional vertices which can spoil the embedding of the second level leaves. We apply a similar argument as in Case 2, only this time the original leaves of T take the role that the second level bare path played before, that is, cover the set of exceptional vertices. The details for this argument are given in Proposition 4.3.

In the remainder of this section we show three results (Proposition 4.1, Proposition 4.2, and Proposition 4.3) which cover the cases discussed above. Theorem 2.1 (Tree-Universality) is a direct consequence of these three propositions.

PROPOSITION 4.1. (CASE 1) *The statement of Theorem 2.1 holds with $\mathcal{T}(n, \Delta)$ restricted to trees that contain a first level bare path on at least $50\Delta m$ vertices.*

Proof. We construct an embedding φ of T onto G as follows. We split T into two parts and embed them consecutively. These parts are a (first level) bare path P on exactly $50\Delta m$ vertices (chosen as a subpath of a longest bare path in T) and, after removing P, the remaining forest $F := T[V(T) \setminus V(P)]$ on $n - 50\Delta m$ vertices which consists of two trees. Note that we have $|V(F)| \geq |V(P)|$ since $50\Delta m \leq n/3$ for sufficiently large n. Let s_P and t_P be the two end-vertices of P and let s_F and t_F, respectively, be their two neighbors in F. We first find an embedding φ_F of the forest F in G and then we will find an embedding φ_P of the path P in $G[V \setminus \varphi_F(F)]$. In this, we make sure that $\{\varphi_P(s_P), \varphi_F(s_F)\}$ and $\{\varphi_P(t_P), \varphi_F(t_F)\}$ are edges in G.

We start by partitioning V into U_F and U_P which (partially) host the embeddings of F and P. For this, we apply Lemma 3.3 (Partition Lemma) to partition V into two sets U_F and U_P with $|U_F| = |V(F)| + 4\Delta m$ and $|U_P| = |V(P)| - 4\Delta m$. Note that U_F and U_P are each of size at least $20\Delta m$. Since

$$\frac{|U_P|}{5n}\, d \geq \frac{|U_P|}{10m} \geq 2\Delta \ (\geq 2 \log n),$$

the prerequisites of Lemma 3.3 are satisfied. Thus, $G[U_F]$ is a $(|U_F|, 2\Delta)$-expander and, for every set W_P with $U_P \subseteq W_P \subseteq V$, $G[W_P]$ is a $(|W_P|, 2\Delta)$-expander.

Now, we turn to the actual constructions of φ_F and φ_P. First, we determine φ_F. By Corollary 3.1, there exists an embedding φ_F of F in $G[U_F]$. Note that since this result only allows us to embed almost spanning trees, U_F was chosen to be somewhat larger than $|V(F)|$.

Next, we move the unused $4\Delta m$ vertices of U_F to U_P and embed P by applying Corollary 3.2. Let $W_P := V \setminus \varphi_F(F)$. Since $U_P \subseteq W_P$, we already know that $G[W_P]$ is a $(50\Delta m, 2\Delta)$-expander. Moreover, because of Lemma 3.3 (Partition Lemma), we are able find two (distinct) vertices $v \in N_G(\varphi_F(s_F)) \cap U_P$ and $w \in N_G(\varphi_F(t_F)) \cap U_P$ to which we embed s_P and t_P, respectively. Since $2\Delta \geq \log n$, $G[W_P]$ contains a Hamilton path connecting v and w by Corollary 3.2. Let φ_P be the embedding of the bare path P to this Hamilton path. This concludes the construction of φ and the proof of the proposition.

PROPOSITION 4.2. (CASE 2) *The statement of Theorem 2.1 holds with $\mathcal{T}(n, \Delta)$ restricted to trees that have at least $25\Delta m^2$ first level leaves and contain a second level bare path on at least $50\Delta m$ vertices.*

Proof. Let $L \subseteq V(T)$ be the set of (first level) leaves of T. By the assumptions of the proposition, we have $|L| \geq 25\Delta m^2$. Let K be the set of neighbors of the leaves in T, that is, $K = N_T(L)$. Since T contains a second level bare path on at least $50\Delta m$ vertices, we can find two vertex-disjoint second level bare paths on exactly $25\Delta m$ vertices in T (e.g., two subpaths of a longest bare path). Of these two second level bare paths, let P be the one that contains *at most* $|K|/2$ vertices of K.

Let F be the forest $T[V(T) \setminus (L \cup V(P))]$; note that F consists of two trees. Like in Proposition 4.1, let s_P and t_P be the end-vertices of P and let s_F and t_F be their respective neighbors in F. We partition K into the two parts $K_F := K \cap V(F)$ and $K_P := K \cap V(P)$. Since each vertex in K is adjacent to at most Δ vertices in L, we have $|K| \geq 25m^2$. Moreover, $|K_F| \geq |K_P|$ by the choice of P and therefore

$$|K_F| \geq m^2. \tag{4.3}$$

Similarly as in Proposition 4.1, we construct an embedding φ of T in G in several steps. First, we construct an embedding φ_F of the forest F, then an embedding φ_P of the path P, and finally an embedding φ_L of the leaves L. In this, we make sure that the images of s_P and t_P are adjacent to those of s_F and t_F, respectively, and that the image of each leaf in L is adjacent to the image of its respective neighbor in K.

Again, we partition the vertices of G into sets which partially host the embeddings of F, P, and L. For this, we apply Lemma 3.3 (Partition Lemma) to partition V into the three parts U_F, U_P, and U_L satisfying $|U_F| = |V(F)| + 4\Delta m$, $|U_L| = |L| + \Delta m$, and $|U_P| = |V(P)| - 5\Delta m$. Then U_F and U_P are each of size at least $20\Delta m$ and U_L is of size at least $20\Delta m^2$. Hence, the subgraph $G[U_F]$ is a $(|U_F|, 2\Delta)$-expander and also $G[W_P]$ is a $(|W_P|, 2\Delta)$-expander for every set W_P with $U_P \subseteq W_P \subseteq V$. Moreover, $|N_G(X) \cap U_L| \geq 2\Delta m|X|$ holds for every set $X \subseteq V$ with $1 \leq |X| < m$. Therefore,

$$|N_G(X) \cap W_L| \geq |N_G(X) \cap U_L| - \Delta m \geq \Delta |X| \tag{4.4}$$

holds for every set W_L of size $|L|$ with $W_L \subseteq U_L$ and for all sets $X \subseteq V \setminus W_L$ with $1 \leq |X| < m$.

In order to construct φ, we first apply Corollary 3.1 to find an embedding φ_F of F in U_F. Let $W_F := \varphi(F)$. Later, we move the remaining $4\Delta m$ vertices in $U_F \setminus W_F$ to U_P.

Next, we embed the second level bare path P. However, before doing so, we identify the exceptional set of vertices $Z \subseteq U_L$ which might later spoil the application of Lemma 3.4 for the embedding of L. This set, denoted by Z, contains all vertices in U_L that have fewer than m neighbors in $\varphi(K)$, the set of portals. At this point of the construction of φ, we only know $\varphi(K_F)$, which is equal to $\varphi_F(K_F)$. However, since $|K_F| \geq |K_P|$ and therefore $|\varphi_F(K_F)| \geq |\varphi(K)|/2$, we may already define Z.

Let $Z := \{u \in U_L \mid |N_G(u) \cap \varphi_F(K_F)| < m\}$. We already know that $|\varphi(K_F)| = |K_F| \geq m^2$. Thus, since G is an (n, d)-expander, we have by Lemma 3.2 that $|Z| \leq m \leq \Delta m$. Let $W_L \subseteq U_L$ be an arbitrary set of size $|L|$ that contains no vertex in Z. In the third step of the embedding, W_L will be the image of L under φ_L. Note that however we embed K_P, this choice of W_L ensures that $|N_G(u) \cap \varphi(K)| \geq m$ holds for every vertex $u \in W_L$. In fact, the only reason why we separated the embedding of the second level bare path P from the embedding of F is to take care of the exceptional set Z.

Now, we return to the embedding φ_P of the second level bare path P. So far, we constructed the embedding φ_F of F to the set $W_F \subseteq U_F$ of size $|V(F)|$ and reserved the set $W_L \subseteq U_L$ of size $|L|$ for the embedding φ_L of L. Let $W_P := V \setminus (W_F \cup W_L)$. Then $|W_P| = |P|$ and we have already seen that $G[W_P]$ is a $(|W_P|, 2\Delta)$-expander. Moreover, as in the proof of the previous proposition, we can choose two distinct vertices $v \in N_G(\varphi_F(s_F)) \cap U_P$ and $w \in N_G(\varphi_F(t_F)) \cap U_P$ as the images of s_P and t_P in φ_P, respectively. Afterwards, we define φ_P by embedding P onto a Hamilton path between v and w in $G[W_P]$ given by Corollary 3.2.

Finally, we construct an embedding φ_L of L by applying Lemma 3.4. At this point, the embedding of K is already given by the embeddings φ_F of K_F and φ_P of K_P. Thus, it suffices to verify that the conditions of Lemma 3.4 are satisfied. Condition (i) holds by (4.4), condition (ii) holds since G is an (n, d)-expander, and condition (iii) holds since we excluded Z from U_L when choosing W_L. Thus, we find an embedding φ_L of L to G that respects the edges between K and L in T. This concludes the construction of φ and the proof of the proposition.

PROPOSITION 4.3. (CASE 3) *The statement of Theorem 2.1 holds with $\mathcal{T}(n, \Delta)$ restricted to trees that have at least $25\Delta m^2$ (first level) leaves and at least $25\Delta m^2$ second level leaves.*

Proof. Let L' be the set of first level leaves of T and M' be the set of second level leaves of T, that is, the set of leaves of $T[V(T) \setminus L]$. By the assumptions of the proposition, $|L'| \geq 25\Delta m^2$ and $|M'| \geq 25\Delta m^2$. Note that every second level leaf $v \in M'$ has at least one first level leaf attached to it since otherwise v would have been in L' to begin with.

As in the proof of Proposition 4.2, we split T into three parts. Let M be an arbitrary subset of M' of size exactly $25\Delta m^2$, let $L := N_T(M) \cap L'$ be the set of first level leaves with neighbors in M, and let F be the induced subtree $F := T[V(T) \setminus (L \cup M)]$.

Let $K := N_{T-L}(M)$ be the neighbors of the second level leaves M in T without the first level leaves. Then $|K| \geq 25m^2$. Note that by definition M is the set of portals of L, that is $M = N_T(L)$. Also note that $|M \cup L| \leq 25\Delta^2 m^2$ and thus $|V(F)| \geq 16\Delta m$ for sufficiently large n.

We construct an embedding φ of T in G by defining three partial embeddings φ_F, φ_M, and φ_L of F, M, and L, respectively. As before, we make sure that these embeddings respect the edges linking K to M and M to L in T. A crucial step in this process will be again to handle the exceptional set Z that might spoil the embedding of M. In the proof of Proposition 4.2, we forced Z to be covered by the image of the second level path, now we will force Z to be covered by the images of L.

We again apply Lemma 3.3 (Partition Lemma) and partition V into three parts U_F, U_M, and U_L satisfying $|U_F| = |V(F)| + 4\Delta m$, $|U_M| = |M| + \Delta m$, and $|U_L| = |L| - 5\Delta m$. This implies that U_F is of size at least $20\Delta m$ and that U_M and U_L are each of size at least $20\Delta m^2$. Hence, $G[U_F]$ is a $(|U_F|, 2\Delta)$-expander. Moreover, for every set W_L of size $|L|$ with $U_L \subseteq W_L$ we have for all sets $X \subseteq V$ with $1 \leq |X| < m$, that

$$|N_G(X) \cap W_L| \geq |N_G(X) \cap U_L| \geq 2\Delta m|X| \tag{4.5}$$

and for every set W_M of size $|M|$ with $W_M \subseteq U_M$ we have for all sets $X \subseteq V$ with $1 \leq |X| < m$, that

$$|N_G(X) \cap W_M| \geq |N_G(X) \cap U_M| - \Delta m \geq \Delta m|X|. \tag{4.6}$$

The construction of the embedding φ closely follows that of the previous proposition. We first apply Corollary 3.1 to find an embedding φ_F of F in U_F. Let $W_F := \varphi_F(F)$. We later move $U_F \setminus W_F$ to U_L.

Next, we give an embedding φ_M of M. For this, let Z be the set of exceptional vertices in M, that is, let $Z := \{u \in U_M \mid |N_G(u) \cap \varphi_F(K)| < m\}$. Then $|Z| < m$ by Lemma 3.2. Let W_M be an arbitrary subset of $U_M \setminus Z$ of size $|M|$ and let W_L be the set given by $V \setminus (W_F \cup W_M)$. Then, by the choice of W_M and by (4.5) and (4.6), we can apply Lemma 3.4 to find an embedding φ_M of M in U_M which respects the edges between K and M in T.

Finally, we give an embedding φ_L of L. Because of (4.5) and (4.6), Lemma 3.4 also hold for L. However, this time there is no exceptional set, since (4.6) guarantees the minimum degree constraint (iii) in Lemma 3.4. Hence, we can embed L such that the edges between M and L in T are respected. This concludes the proof of Proposition 4.3 and hence of Theorem 2.1.

5 Conclusion

We have shown that, for sufficiently large n, every $(n, 7n^{2/3} \max\{\Delta, \log n\})$-expander is universal for $\mathcal{T}(n, \Delta)$, the class of all n-vertex trees with maximum degree at most Δ. This implies that binomial random graphs and random regular graphs with sufficiently large (average) degree are a.a.s. universal for $\mathcal{T}(n, \Delta)$. Furthermore, we obtain constructions of locally sparse $\mathcal{T}(n, \Delta)$-universal graphs. We have also discussed $\mathcal{T}(n, \Delta)$-universality in the setting of the Maker-Breaker game.

One major open problem is to establish the smallest value of p for which $G(n, p)$ becomes a.a.s. universal for $\mathcal{T}(n, \Delta)$. Here, our work leaves a substantial gap of $n^{2/3}$ compared to the lower bound in [36]. Also, it would be interesting to see why the corresponding lower bound for (n, d)-expanders in Theorem 2.5 differs so drastically from that in [36] and to possibly find pseudo-random sufficient conditions which do not yield this discrepancy. In the spirit of Theorem 2.4, it would be nice to see constructions of tree-universal graphs which are triangle-free or even have large girth. Finally, although our embedding results are (for the most part) constructive, they do not give an efficient algorithm to find the embeddings. Here, an algorithmic version would be also desirable.

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