Percolation on high-dimensional product graphs

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Abstract

We consider percolation on high-dimensional product graphs, where the base graphs are regular and of bounded order. In the subcritical regime, we show that typically the largest component is of order logarithmic in the number of vertices. In the supercritical regime, our main result recovers the sharp asymptotic of the order of the largest component, and shows that all the other components are typically of order logarithmic in the number of vertices. In particular, we show that this phase transition is *quantitatively* similar to the one of the binomial random graph.

This generalises the results of Ajtai, Komlós, and Szemerédi [1] and of Bollobás, Kohayakawa, and Łuczak [5] who showed that the d-dimensional hypercube, which is the d-fold Cartesian product of an edge, undergoes a phase transition quantitatively similar to the one of the binomial random graph.

1 Introduction

1.1 Background and motivation

In 1960, Erdős and Rényi [15] discovered the following fundamental phenomenon: the component structure of the binomial random graph $G(d+1,p)$[1] undergoes a remarkable *phase transition* around the probability $p = \frac{1}{d}$. More precisely, if we let $y = y(\epsilon)$ be the unique solution in $(0,1)$ of the equation

$$y = 1 - \exp\left(-(1+\epsilon)y\right), \tag{1}$$

then Erdős and Rényi's work [15] implies the following[2]:

Theorem 1.1 ([15])**.** *Let $\epsilon > 0$ be a small enough constant. Then, with probability tending to one as d tends to infinity,*

(a) if $p = \frac{1-\epsilon}{d}$, then all components of $G(d+1,p)$ are of order $O\left(\frac{\log d}{\epsilon^2}\right)$; and,

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[1] As we mainly consider d-regular graphs, we use the slightly unusual notation of $G(d+1,p)$ instead of $G(n,p)$, to make the comparison of the results simpler.

[2] In fact, Erdős and Rényi worked in the closely related *uniform* random graph model $G(d+1,m)$.

(b) *if* $p = \frac{1+\epsilon}{d}$*, then* $G(d+1,p)$ *contains a unique giant component of order* $(1+o(1))yd$*, where* y *is defined according to (1). Furthermore, all the other components of* $G(d+1,p)$ *are of order* $O\left(\frac{\log d}{\epsilon^2}\right)$*.*

We note that y is the survival probability of a Galton-Watson tree with offspring distribution $Bin\left(d, \frac{1+\epsilon}{d}\right)$, and we have that $y = 2\epsilon - O(\epsilon^2)$. The regime where $p = \frac{1-\epsilon}{d}$ is often referred to as the *subcritical regime*, while if $p = \frac{1+\epsilon}{d}$ it is often called the *supercritical regime*. We refer the reader to [4, 17, 25] for a systematic coverage of random graphs.

We can think of the binomial random graph as perhaps the simplest example of a *percolation* model. Percolation is a mathematical process, initially studied by Broadbent and Hammersley [9] to model the flow of a fluid through a porous medium whose channels may be randomly blocked. The underlying mathematical model is simple: given a fixed *host* graph G and some probability $p \in (0,1)$, in (bond) percolation we consider the random subgraph G_p of G obtained by retaining every edge independently with probability p. The component structure of the percolated subgraph G_p is of particular interest, and in this broader setting the phase transition described in Theorem 1.1 can be viewed as an example of a *percolation threshold*. See [7, 19, 26] for a comprehensive introduction to percolation theory.

Percolation is often studied on lattice-like graphs, where in contrast to $G(d+1,p)$ there is some non-trivial underlying geometry controlling the potential adjacencies in the random subgraph. One particular percolation model that has received considerable interest is that of percolation on the *d-dimensional hypercube* Q^d. Here, Q^d is a graph with the vertex set $V(Q^d) = \{0,1\}^d$, where two vertices are adjacent if they differ in exactly one coordinate. It was conjectured by Erdős and Spencer [16] that Q^d_p undergoes a similar phase transition to $G(d+1,p)$ when p is around $\frac{1}{d}$. Ajtai, Komlós, and Szemerédi [1] confirmed this conjecture, and their work was extended by Bollobás, Kohayakawa, and Łuczak [5].

Theorem 1.2 ([1, 5])**.** *Let* $\epsilon > 0$ *be a small enough constant. Then, with probability tending to one as* d *tends to infinity,*

(a) *if* $p = \frac{1-\epsilon}{d}$*, then all components of* Q^d_p *are of order* $O_\epsilon(d)$*; and,*

(b) *if* $p = \frac{1+\epsilon}{d}$*, then* Q^d_p *contain a unique giant component of order* $(1+o(1))y2^d$*, where* y *is defined according to (1). Furthermore, all the other components of* Q^d_p *are of order* $O_\epsilon(d)$*.*

Observe that in the setting of the hypercube, $|V(Q^d)| = 2^d$, and so $d = \log_2 |V(Q^d)|$. Thus, one can see that there is a striking similarity between the behaviour of $G(d+1,p)$ shown in Theorem 1.1, and the behaviour of Q^d_p shown in Theorem 1.2. Denoting the number of vertices in the host graph by n, the components in the subcritical regime are typically of order at most logarithmic in n, whereas in the supercritical regime there is a unique giant component of asymptotic order yn, and all other components are of order logarithmic in n. In other words, with the right scaling, the phase transitions which occur around the percolation threshold in $G(d+1,p)$ and Q^d_p display quantitatively similar behaviour. It is known that a similar phenomenon occurs in other random graph models, such as graphs with a fixed degree sequence [29] or percolation on pseudo-random graphs [18]. We will informally refer to this as the *Erdős-Rényi component phenomenon.*

Part of the Erdős-Rényi component phenomenon is related to the so-called *discrete duality principle*, which roughly says that if we remove the giant component from a supercritical random graph, then the distribution of what remains in some way resembles the distribution of a subcritical random graph, for an appropriate choice of parameters. See, for example, [21] for a discussion of the discrete duality principle in hypercube and lattice percolation. There is also perhaps some similarity to the concept of *mean-field behaviour*, a phenomenon in which for many models studied in the context of statistical physics there is a critical dimension above which certain quantitative properties of the model when viewed at the critical probability are no longer dependent on the dimension of the model, and so in some sense independent of the host graph. For example, Nachmias [30] showed that percolated transitive expander graphs demonstrate mean-field behaviour in terms of the width of the scaling window around the critical probability and more recently this behaviour has been shown to hold also in the percolated hypercube [8, 23].

It is thus a natural question to ask whether such a phenomenon holds for percolation in a wider family of graphs. The d-dimensional hypercube embeds naturally into d-dimensional space, but we can also view Q^d as a high-dimensional object in terms of its product structure, since Q^d can be obtained as the *Cartesian product* of d-copies of a single edge. It is perhaps not unreasonable to expect that percolation on other *high-dimensional graphs* might display similar behaviour.

Given an integer $t > 0$ and a sequence of graphs $G^{(1)}, G^{(2)}, \ldots, G^{(t)}$, the Cartesian product of $G^{(1)}, G^{(2)}, \ldots, G^{(t)}$, denoted by $G = G^{(1)} \square \cdots \square G^{(t)}$ or $G = \square_{i=1}^{t} G^{(i)}$, is the graph with the vertex set

$$V(G) = \left\{ v = (v_1, v_2, \ldots, v_t) \colon v_i \in V\left(G^{(i)}\right) \text{ for all } i \in [t] \right\},$$

and the edge set

$$E(G) = \left\{ uv \colon \begin{array}{l} \text{there is some } i \in [t] \text{ such that } u_j = v_j \\ \text{for all } i \neq j \text{ and } u_i v_i \in E\left(G^{(i)}\right) \end{array} \right\}.$$

We call $G^{(1)}, G^{(2)}, \ldots, G^{(t)}$ the *base graphs of* G.

Percolation on such product graphs has been quite well studied. Apart from the hypercube, perhaps the two most well-known examples are the d-dimensional torus $T_{n,d}$, which is the Cartesian product of d cycles of length n, and the Hamming graph K_n^d, which is the Cartesian product of d complete graphs of order n (see Chapter 13 of [21] for a survey on many important results on these models). However, in both of these models, most research has focused on the case where the dimension d is fixed, with the asymptotics studied as the number of vertices n in the base graphs tends to infinity. It is interesting to note that it is known that the percolated torus in a fixed dimension does not quite exhibit the Erdős-Rényi component phenomenon. Indeed, in the supercritical regime, the second-largest component is known to be of asymptotic order $\Theta\left(d^{\frac{d-1}{d}} \log^{\frac{d}{d-1}} n\right)$, and not $O(\log |T_{n,d}|) = O(d \log n)$ [24], where the asymptotics here are in terms of n, with d treated as a constant.

In this paper, we will focus on percolation on Cartesian products of *many* graphs, that is, like in the hypercube, where the base graphs are of bounded order and we are interested in the asymptotics as the *dimension* of the product, that is, the number of non-trivial base graphs, tends to infinity.

Recently, Lichev [28] considered percolation on high-dimensional product graphs, under the assumption that the *isoperimetric constants* of the base graphs were not shrinking too quickly. The *isoperimetric constant* $i(H)$, also known as the Cheeger constant, of a graph H is given by

$$i(H) = \inf_{\substack{S \subseteq V(H), \\ |S| \leq |V(H)|/2}} \frac{e(S, S^C)}{|S|}.$$

The isoperimetric constant broadly measures the global connectivity of a graph, by measuring how easy it is to separate the graph into two large parts.

Lichev [28] showed that the component structure of such graphs undergoes a phase transition when p is around $\frac{1}{d}$ where $d \coloneqq d(G)$ is the average degree of the host graph G.

Theorem 1.3 (Theorem 1.1 of [28]). *Let $C, \gamma > 0$ be constants. Let $G^{(1)}, \ldots, G^{(t)}$ be connected graphs such that for all $j \in [t]$, $\Delta\left(G^{(j)}\right) \leq C$ and $i\left(G^{(j)}\right) \geq t^{-\gamma}$. Let $G = \square_{j=1}^{t} G^{(j)}$. Form G_p by retaining every edge of G independently with probability p. Then, with probability tending to one as $d \coloneqq d(G)$ tends to infinity:*

(a) *if* $p = \frac{1-\epsilon}{d}$*, then all components of* G_p *are of order at most* $\exp\left(-\frac{\epsilon^2 t}{9C^2}\right) n$*; and,*

(b) *if* $p = \frac{1+\epsilon}{d}$*, then there exists a positive constant* $c = c(\epsilon, C, \gamma)$ *such that the largest component of* G_p *is of order at least* cn*.*

Observe that, in comparison to Theorems 1.1 and 1.2, Theorem 1.3 only gives a qualitative description of the phase transition, in the sense that the largest component in the supercritical regime is shown to be linear in order, but neither its uniqueness nor the leading constant are determined, and the largest component in the subcritical regime is only shown to have sublinear order.

This is not quite as strong as the Erdős-Rényi component phenomenon, which determines the asymptotic order and uniqueness of the giant component in the supercritical regime, as well as bounds the order of the largest and second-largest component in the subcritical and supercritical regime, respectively, as logarithmic in the order of the host graph.

On the other hand, the assumptions on the base graphs in Theorem 1.3 are quite mild, requiring only that the isoperimetric constants in the base graphs are not tending to 0 too quickly (in fact, as will be elaborated in Section 5, in a forthcoming paper [11] we show that the isoperimetric constants can tend to 0 even faster). It is thus an interesting question as to what additional assumptions, if any, on the structure of the base graphs would be sufficient to ensure that the product graph displays the Erdős-Rényi component phenomenon after percolation.

1.2 Main results

We will see that one natural assumption to make is that our host graph G is regular, which in particular will be the case for a product graph if and only if the base graphs are all regular.

Our first result concerns the component structure in the subcritical regime, and in fact will hold for any d-regular graph, not necessarily a product graph. Our first result concerns the component structure in the subcritical regime, and in fact will hold for *any* graph with maximum degree d.

Theorem 1. *Let* G *be a graph on* n *vertices with maximum degree* d*, let* $\epsilon > 0$ *be a small enough constant and let* $p = \frac{1-\epsilon}{d}$*. Then, with probability tending to one as* n *tends to infinity, all the components of* G_p *are of order at most* $\frac{9 \log n}{\epsilon^2}$*.*

The proof of Theorem 1 is relatively short, utilising the Breadth First Search (BFS) algorithm. In particular, it implies some known results in other more specific models, such as $G(d+1, p)$ [15], Q^d_p [1, 5], and the torus and Hamming graphs [21]. We note that for regular graphs with constant degree d, a more accurate result (with respect to the leading constant) has been obtained in [31].

A second natural assumption will be that the product graphs all have bounded order. Roughly, this will guarantee that the asymptotic behaviour is coming from the product structure.

Our second result then concerns the component structure in the supercritical regime for the product of many regular graphs of bounded order.

Theorem 2. *Let* $C > 1$ *be a constant and let* $\epsilon > 0$ *be sufficiently small. For all* $i \in [t]$*, let* $G^{(i)}$ *be a non-trivial connected regular graph of degree* $d(G^{(i)})$ *such that* $\left|V\left(G^{(i)}\right)\right| \le C$*. Let* $G = \square_{i=1}^t G^{(i)}$ *and let* $p = \frac{1+\epsilon}{d}$*, where* $d \coloneqq d(G) = \sum_{i=1}^t d\left(G^{(i)}\right)$ *is the degree of* G*. Let* $n \coloneqq |V(G)|$*. Then,* ***whp***[3]*, there exists a unique giant component of order* $(1+o(1))yn$ *in* G_p*, where* $y = y(\epsilon)$ *is defined as in (1). Furthermore,* ***whp****, all the remaining components of* G_p *are of order* $O_\epsilon(d)$*.*

[3] With high probability, that is, with probability tending to 1 as t tends to infinity.

We note that, in terms of the asymptotic order of the second-largest component, in our setting $d = \Theta(\log |G|)$, and the dependency of the leading constant on ϵ arising from our proof is inverse polynomial, of order $\frac{1}{\epsilon^3}$ (compare this with Theorem 1, and the well-known result that the second-largest component of $G(d+1,p)$ has order $\Theta(\log d/\epsilon^2)$ [15]).

Theorems 1 and 2 together show that when the underlying asymptotic geometry of the graph arises from a product structure, where the base graphs are bounded and regular, the percolated graph exhibits the Erdős-Rényi component phenomenon. We note that, in particular, these results generalise the known results for Q^d_p [1, 5], while providing shorter and simpler proofs. Furthermore it can be shown that the assumption of regularity is necessary. We will discuss this, and the extent to which the other assumptions, in both Theorems 1.3 and 2, can be weakened in more detail in Section 5.

Finally, we note that unlike the previous proofs in the case of the hypercube [1, 5], our proof does not rely on any strong isoperimetric inequality. Indeed, we only require relatively weak assumptions on the expansion of the host graph G (see Theorem 2.3), and instead use the product structure of G to describe more precisely the structure of the percolated subgraph.

Similar arguments have proven to be useful in other contexts, for example in the setting of site percolation on the hypercube, where the lack of a strong enough vertex isoperimetric inequality complicates the analysis of the component structure. Recently, the first and fourth author [12] use similar ideas to verify a longstanding conjecture of Bollobás, Kohayakawa, and Łuczak [6, Conjecture 11] on the size of the second-largest component in the supercritical regime in this model. We expect these methods to be useful in other contexts where the analysis is constrained by the lack of a strong enough isoperimetric inequality. In particular, the proofs in this paper should be relatively easy to modify to the setting of site percolation on high-dimensional product graphs.

The structure of the paper is as follows. In Section 2, we introduce some notation, terminology and preliminary lemmas which will serve us throughout the rest of the paper. In Sections 3 and 4 we prove Theorems 1 and 2, respectively. Finally, in Section 5 we discuss our results and avenues for future research.

2 Preliminaries

2.1 Notation and terminology

Let us introduce some notation and terminology, which we will use throughout the rest of the paper.

Recall that given a product graph $G = \square_{i=1}^t G^{(i)}$, we call the $G^{(i)}$ the *base graphs* of G. Given a vertex $u = (u_1, u_2, \dots, u_t)$ in $V(G)$ and $i \in [t]$ we call the vertex $u_i \in V(G^{(i)})$ the *i-th coordinate* of G. As is standard, we may still enumerate the vertices of a given set M, such as $M = \{v_1, \dots, v_m\}$ with $v_i \in V(G)$. Whenever confusion may arise, we will clarify whether the subscript stands for enumeration of the vertices of the set, or for their coordinates. When $G^{(i)}$ is a graph on a single vertex, that is, $G^{(i)} = (\{u\}, \varnothing)$, we call it *trivial* (and *non-trivial*, otherwise). We define the *dimension* of $G = \square_{i=1}^t G^{(i)}$ to be the number of base graphs $G^{(i)}$ of G which are non-trivial (we note that the dimension of G is not an invariant of G, and in fact depends on the choice of the base graphs). Note that in Theorem 2, we assumed that the base graphs have more than one vertex, implying that they are non-trivial. Given $H \subseteq G = \square_{i=1}^t G^{(i)}$, we call H a *projection of* G if H can be written as $H = \square_{i=1}^t H^{(i)}$ where for every $1 \le i \le t$, $H^{(i)} = G^{(i)}$ or $H^{(i)} = \{v_i\} \subseteq V(G^{(i)})$; that is, H is a projection of G if it is the Cartesian product graph of base graphs $G^{(i)}$ and their trivial subgraphs. In that case, we further say that H is the projection of G onto the coordinates corresponding to the trivial subgraphs. For example, let $u_i \in V(G^{(i)})$ for $1 \le i \le k$, and let $H = \{u_1\}\square \cdots \square\{u_k\}\square G^{(k+1)}\square \cdots \square G^{(t)}$. In this case we say that H is a projection of G onto the first k coordinates.

We assume throughout the paper that $t \to \infty$, and our asymptotic notation will be with respect to t. Given a graph H and a vertex $v \in V(H)$, we denote by $C_v(H)$ the component of v in H. All logarithms are with the natural base, unless we explicitly state otherwise. We omit rounding signs for the sake of clarity of presentation.

2.2 The BFS algorithm

For the proofs of our main results, we will use the Breadth First Search (BFS) algorithm. This algorithm explores the components of a graph G by building a maximal spanning forest.

The algorithm maintains three sets of vertices:

- S, the set of vertices whose exploration is complete;
- Q, the set of vertices currently being explored, kept in a queue; and
- T, the set of vertices that have not been explored yet.

The algorithm receives as input a graph G and a linear ordering σ on its vertices. It starts with $S = Q = \emptyset$ and $T = V(G)$, and ends when $Q \cup T = \emptyset$. At each step, if Q is non-empty, the algorithm queries the vertices in T, in the order σ, to ascertain if they are neighbours in G of the first vertex v in Q. Each neighbour which is discovered is added to the back of the queue Q. Once all neighbours of v have been discovered, we move v from Q to S. If $Q = \emptyset$, we move the next vertex from T (according to σ) into Q. Note that the set of edges queried during the algorithm forms a maximal spanning forest of G.

In order to analyse the BFS algorithm on a random subgraph G_p of a graph G with n vertices and m edges, we will utilise the *principle of deferred decisions*. That is, we will take a sequence $(X_i \colon 1 \leq i \leq m)$ of i.i.d Bernoulli(p) random variables, which we will think of as representing a positive or negative answer to a query in the algorithm. When the i-th edge of G is queried during the BFS algorithm we will include it in G_p if and only if $X_i = 1$. It is clear that the forest obtained in this way has the same distribution as a forest obtained by running the BFS algorithm on G_p.

2.3 Preliminary Lemmas

We will make use of two standard probabilistic bounds. The first one is a typical Chernoff type tail bound on the binomial distribution (see, for example, Appendix A in [2]).

Lemma 2.1. *Let $n \in \mathbb{N}$, let $p \in [0,1]$, and let $X \sim Bin(n,p)$. Then for any $t \geq 0$,*

$$\mathbb{P}\left[|X - np| \geq t\right] \leq 2\exp\left(-\frac{t^2}{3np}\right).$$

The second one is the well-known Azuma-Hoeffding inequality (see, for example, Chapter 7 in [2]),

Lemma 2.2. *Let $X = (X_1, X_2, \dots, X_m)$ be a random vector with range $\Lambda = \prod_{i\in[m]} \Lambda_i$ and let $f : \Lambda \to \mathbb{R}$ be such that there exists $C \in \mathbb{R}^m$ such that for every $x, x' \in \Lambda$ which differ only in the j-th coordinate,*

$$|f(x) - f(x')| \leq C_j.$$

Then, for every $t \geq 0$,

$$\mathbb{P}\left[\left|f(X) - \mathbb{E}\left[f(X)\right]\right| \geq t\right] \leq 2\exp\left(-\frac{t^2}{2\sum_{i=1}^n C_i^2}\right).$$

We will use the following result of Chung and Tetali [10, Theorem 2], which bounds the isoperimetric constant of product graphs.

Theorem 2.3 ([10])**.** *Let $G^{(1)}, \dots, G^{(t)}$ be graphs and let $G = \square_{i=1}^{t} G^{(i)}$. Then*

$$\min_j \{i(G^{(j)})\} \geq i(G) \geq \frac{1}{2} \min_j \left\{ i\left(G^{(j)}\right) \right\}.$$

Finally, we will use the following bound on the number of m-vertex trees in a bounded-degree graph.

Lemma 2.4 ([3, Lemma 2])**.** *Let G be a graph on n vertices with maximum degree d. Let $t_m(G)$ be the number of m-vertex trees in G. Then,*

$$t_m(G) \leq n(ed)^{m-1}.$$

3 Subcritical regime

The proof of Theorem 1 is inspired by Krivelevich and Sudakov's [27] simple proof of the emergence of a giant component in $G(d+1, p)$. However, here we analyse the BFS algorithm instead of the Depth First Search algorithm.

Proof of Theorem 1. Suppose we run the BFS algorithm on G_p, as described in Section 2.2, and assume to the contrary that G_p contains a component K of order larger than $k = \frac{9 \log n}{\epsilon^2}$.

Let us consider the period of the algorithm from the moment when the first vertex of K enters Q to the moment when the $(k+1)$-st vertex of K enters Q. During this period we only query edges adjacent to the first k vertices of K, and so, since G has maximum degree d, we query at most kd edges. However, by assumption G_p induces a tree on these $k+1$ vertices, and hence in this period we receive k positive answers. In particular, there is some interval $I \subseteq [m]$ of length kd such that at least k of the X_i with $i \in I$ are equal to 1.

However, by Lemma 2.1, the probability that this occurs for any fixed interval I is at most

$$\mathbb{P}\left[Bin\left(kd, \frac{1-\epsilon}{d}\right) \geq k\right] \leq \exp\left(-\frac{\epsilon^2 k}{4}\right) = o\left(\frac{1}{n^2}\right),$$

where the last equality is since $k = \frac{9 \log n}{\epsilon^2}$. Since G has maximum degree d, we have that $m \leq \frac{nd}{2}$. Therefore, there are at most $\frac{nd}{2} \leq n^2$ intervals $I \subseteq \left[\frac{nd}{2}\right]$ of length kd, and so by the union bound **whp** there is no such interval, contradicting our assumption. □

4 Supercritical regime

Let us start by giving a brief sketch of the proof in this section.

We start in Section 4.1 by giving a useful technical lemma, which we call a *projection lemma* (Lemma 4.1), which allows us to cover a small set of points M in the product graph with a set of pairwise disjoint projections of large dimension, each of which contains a unique point in M. This allows us to explore the graph 'locally' around each of these points in an independent manner.

Using this, in Section 4.2, we show that **whp** a fixed proportion of the vertices in G_p will be contained in *big* components, of order at least d^k for some fixed integer $k > 0$ and, in Section 4.3, we show that these big components are 'dense' in the host graph, in the sense that every vertex in G is close to some big component of G_p. Finally, in Section 4.4, we use a sprinkling argument to show that these big components are in fact all contained in a unique giant component and determine its asymptotic order. This approach is relatively standard, and is roughly the method used by Ajtai, Komlós, and Szemerédi [1] and Bollobás, Kohayakawa, and Łuczak [5], but the arguments are simplified substantially by our projection lemma.

However, these methods inherently can only establish a superlinear polynomial bound (in d) on the order of the second-largest component. In order to overcome that, the standard approach requires a strong isoperimetric inequality to demonstrate a gap in the size of the components. Nevertheless, utilising the projection lemma in an inductive manner, together with a carefully chosen multi-round exposure, allows us to keep track more precisely of the distribution of the vertices in big components, and in particular to show that any large enough connected set in G_p, where we only require these sets to be linearly large in d, must be adjacent to many vertices in big components. A further sprinkling argument then allows us to give an improved bound on the size of the second-largest component, completing the proof of Theorem 2, without the need for a strong isoperimetric inequality.

4.1 Projection lemma

We begin by establishing the following projection lemma, which will be useful throughout all of the subsequent sections.

Lemma 4.1. *Let $G = \square_{i=1}^t G^{(i)}$ be a product graph with dimension t. Let $M \subseteq V(G)$ be such that $|M| = m \leq t$. Then, there exist pairwise disjoint projections $H_1, \ldots, H_m$ of G, each having dimension at least $t - m + 1$, such that every $v \in M$ is in exactly one of these projections.*

Proof. We argue by induction on pairs (m, t), where t is the dimension of the graph, with $m \leq t$ under the lexicographical ordering. When $m = 1$, we simply take $H_1 = G$.

Let $M \subseteq V(G)$ have size $m \geq 2$ and let us assume the statement holds for all $(m', t') < (m, t)$. Let us write $M = \{v_1, \ldots, v_m\}$, where we stress that the subscript here is an enumeration of the vertices of M and not the coordinates of a fixed vertex.

There is some coordinate i on which at least two of the vertices of M do not agree. Let us denote by M_i the set of the i-th coordinates of the vertices of M, that is, $V(G^{(i)}) \supseteq M_i = \{v_{1,i}, \ldots, v_{\ell,i}\}$, where we may re-order the vertices so that $v_{j,i}$ is the i-th coordinate of the vertex $v_j \in M$, and $2 \leq \ell \leq m$ (note that it is possible that $\ell < m$, since there could be vertices in M which agree on their i-th coordinates).

Let us now consider the pairwise disjoint projections $H_1, \ldots, H_\ell$ of G defined by

$$H_j = G^{(1)} \square \cdots \square G^{(i-1)} \square \{v_{j,i}\} \square G^{(i+1)} \square \cdots \square G^{(t)}.$$

That is, in the j-th projection, we take the i-th coordinate of H_j to be the trivial graph $\{v_{j,i}\}$ (which is the i-th coordinate of the j-th vertex in M_i).

Note that each of these projections has dimension $t - 1$ and contains at least one vertex of M, and that each of the vertices of M is in exactly one of these projections. Hence, each such projection contains at most $m - 1$ vertices from M.

We can thus apply the induction hypothesis to each of these projections, giving rise to m pairwise disjoint projections of G, each of dimension at least $(t - 1) - (m - 1) + 1 = t - m + 1$ and containing exactly one vertex from M. $\square$

4.2 Vertices in large components

We first estimate from below the probability that a vertex of G lies in a polynomially sized (in d) component of G_p, with the proof inspired by [6].

Lemma 4.2. *Let $C > 1$ and $\epsilon > 0$ be constants, let $G^{(1)}, \ldots, G^{(t)}$ be connected regular graphs such that $1 < \left|V\left(G^{(i)}\right)\right| \leq C$ for all $i \in [t]$, let $G = \square_{i=1}^t G^{(i)}$ and let $p \geq \frac{1+\epsilon}{d}$, where $d =$*

$\sum_{i=1}^{t} d\left(G^{(i)}\right)$ *is the degree of* G. *Let* $r>0$ *be an integer and let* $m_r = d^{\frac{r}{4}}$. *Then, there exists a constant* $c = c(\epsilon, r) > 0$ *such that for any* $v \in V(G)$,

$$\mathbb{P}\left[\left|C_v(G_p)\right| \geq cm_r\right] \geq y - o_t(1),$$

where $y = y(\epsilon)$ *is as defined in* (1).

Proof. We will prove the slightly more explicit statement that the result holds with $c(\epsilon, r) = \left(\frac{y(\epsilon)}{5}\right)^r$ by induction on r, over all possible values of C and ϵ, and all choices of $G^{(1)}, \dots, G^{(t)}$.

For $r=1$, we run the BFS algorithm (as described in Section 2.2) on G_p starting from v with a slight alteration: we terminate the algorithm once $\min\left(\left|C_v(G_p)\right|, d^{\frac{1}{2}}\right)$ vertices are in $S \cup Q$. Note that at every point in the algorithm we have $|S \cup Q| \leq d^{\frac{1}{2}}$, and therefore at each point in the algorithm the first vertex u in the queue has at least $d - d^{\frac{1}{2}}$ neighbours (in G) in T. Hence, we can couple the forest F built by this truncated BFS process with a Galton-Watson tree B rooted at v with offspring distribution $Bin\left(d - d^{\frac{1}{2}}, p\right)$ such that $B \subseteq F$ as long as $|B| \leq d^{\frac{1}{2}}$.

Since $\left(d - d^{\frac{1}{2}}\right) \cdot p \geq 1 + \epsilon - o(1)$, standard results imply that B grows infinitely large with probability $y - o(1)$ (see, for example, [13, Theorem 4.3.12]). Thus, with probability at least $y - o(1)$, we have that

$$\left|C_v(G_p)\right| \geq d^{\frac{1}{2}} \geq \left(\frac{y}{5}\right) d^{\frac{1}{4}} = c(\epsilon, 1) m_1.$$

Let $r \geq 2$ and let us assume that the statement holds with $c(\epsilon', r-1) = \left(\frac{y(\epsilon')}{5}\right)^{r-1}$ for all C, ϵ and $G^{(1)}, \dots, G^{(t)}$. We will argue via a two-round exposure. Set $p_2 = d^{-\frac{5}{4}}$ and $p_1 = \frac{p - p_2}{1 - p_2}$ so that $(1-p_1)(1-p_2) = 1-p$. Note that G_p has the same distribution as $G_{p_1} \cup G_{p_2}$, and that $p_1 = \frac{1+\epsilon'}{d}$ where $\epsilon' = \epsilon - o(1)$. In fact, we will not expose either G_{p_1} or G_{p_2} all at once, but in several stages, each time considering only some subset of the edges.

We begin in a manner similar to the case of $r=1$. We run the BFS algorithm on G_{p_1} starting from v, and we terminate the exploration once $\min\left(\left|C_v(G_{p_1})\right|, d^{\frac{1}{2}}\right)$ vertices are in $S \cup Q$. Once again, by standard arguments, we have that $\left|C_v(G_{p_1})\right| \geq d^{\frac{1}{2}}$ with probability at least $y(\epsilon') - o(1) = y - o(1)$. Let us write $W_0 \subseteq C_v(G_{p_1})$ for the set of vertices explored in this process, and assume in what follows that W_0 is of order $d^{\frac{1}{2}}$. Using Lemma 4.1, we can find pairwise disjoint projections $H_1, \dots, H_{d^{\frac{1}{2}}}$ of G, each having dimension at least $t - d^{\frac{1}{2}}$, such that each $v \in W_0$ is in exactly one of the H_i. We denote the vertex of W_0 in H_i by v_i.

Now, by our assumptions on G, we have that $\left|V\left(G^{(j)}\right)\right| \leq C$ for every $j \in [t]$ and hence clearly $d\left(G^{(j)}\right) \leq C$. Thus, each of the H_i is d_i-regular with $d_i \geq d - Cd^{\frac{1}{2}} = (1-o(1))d$. In particular, each v_i has $(1-o(1))d$ neighbours in H_i. Let us define the following set of vertices:

$$W = \bigcup_{i \in [d^{\frac{1}{2}}]} N_{H_i}(v_i).$$

Then, $W \subseteq N_G(W_0)$ and since the H_i are pairwise disjoint we have $|W| \geq d^{\frac{1}{2}}(1-o(1))d \geq \frac{9d^{\frac{3}{2}}}{10}$. We now look at the edges in G_{p_2} between W_0 and W. Let us denote the vertices in W that are connected with W_0 in G_{p_2} by W'. Since $|W| \geq \frac{9d^{\frac{3}{2}}}{10}$ and $p_2 = d^{-\frac{5}{4}}$, $|W'|$ stochastically dominates

$$Bin\left(\frac{9d^{\frac{3}{2}}}{10}, d^{-\frac{5}{4}}\right).$$

Thus, by Lemma 2.1, we have that $|W'| \geq \frac{d^{\frac{1}{4}}}{3}$ and $|W'| \leq d^{\frac{1}{2}}$ with probability at least $1 - \exp\left(-\frac{d^{\frac{1}{4}}}{15}\right) = 1 - o(1)$. In what follows we will assume that these two inequalities hold.

Let $W_i' = W' \cap V(H_i)$. Now, for each i, we apply Lemma 4.1 to find a family of $\ell_i \le d^{\frac{1}{2}}$ pairwise disjoint projections of H_i, we denote them by $H_{i,1}, \ldots, H_{i,\ell_i}$, such that every vertex of W_i' is in exactly one of the $H_{i,j}$, and each of the $H_{i,j}$ is of dimension at least $t - 2d^{\frac{1}{2}}$. Finally, we denote by $v_{i,j}$ the unique vertex of W_i' that is in $H_{i,j}$. Note that, each $H_{i,j}$ is $d_{i,j}$-regular with $d_{i,j} \ge d - 2Cd^{\frac{1}{2}}$.

Crucially, observe that when we ran the BFS algorithm on G_{p_1}, we did not query any of the edges in any of the $H_{i,j}$ - we only queried edges in W_0 and between W_0 and its neighbourhood, and by construction $E(H_{i,j}) \cap E(W_0 \cup N_G(W_0)) = \varnothing$. Note that, since $y = y(\epsilon)$, defined in (1), is a continuous increasing function of ϵ on $(0, \infty)$ and $p_1 \cdot d_{i,j} = 1 + \epsilon_{i,j} \ge 1 + \epsilon - o(1)$ for all i, j, we may apply the induction hypothesis to $v_{i,j}$ in $G_{p_1} \cap H_{i,j}$ and conclude that

$$|C_{v_{i,j}}(H_{i,j} \cap G_{p_1})| \ge c(\epsilon_{i,j}, r-1) d^{\frac{r-1}{4}} \ge (1 - o(1)) c(\epsilon, r-1) d^{\frac{r-1}{4}}$$

with probability at least $y(\epsilon_{i,j}) - o(1) \ge y - o(1)$. Note that these events are independent for each $H_{i,j}$. Hence, since by assumption $|W'| \ge \frac{d^{\frac{1}{4}}}{3}$, it follows from Lemma 2.1 that **whp** at least $\frac{yd^{\frac{1}{4}}}{4}$ of these $v_{i,j}$ have that $|C_{v_{i,j}}(H_{i,j} \cap G_{p_1})| \ge (1 - o(1)) c(\epsilon, r-1) d^{\frac{r-1}{4}}$.

As such, we may conclude that with probability at least $y - o(1)$, we have that

$$|C_v(G_p)| \ge (1 - o(1)) \frac{yd^{\frac{1}{4}}}{4} c(\epsilon, r-1) d^{\frac{r-1}{4}} \ge \left(\frac{y}{5}\right)^r d^{\frac{r}{4}} = c(\epsilon, r) m_r,$$

completing the induction step. □

From now on let us fix a constant $C > 1$, and connected regular graphs $G^{(1)}, \ldots, G^{(t)}$ such that $1 < \left|V\left(G^{(i)}\right)\right| \le C$ for all $i \in [t]$ and let $G = \square_{i=1}^t G^{(i)}$, where we write $d \coloneqq \sum_{i=1}^t d\left(G^{(i)}\right)$ for the degree of G and $n \coloneqq |V(G)|$.

We can now estimate the number of vertices in components of order at least d^k, where we will choose k later to be some large, fixed, integer.

Lemma 4.3. *Let $\epsilon > 0$ and let $p = \frac{1+\epsilon}{d}$. Let $k > 0$ be an integer and let $W \subseteq V(G)$ be the set of vertices belonging to components of order at least d^k in G_p. Then,* ***whp****,*

$$|W| = (1 + o(1))\, yn,$$

where $y = y(\epsilon)$ is as defined in (1).

Proof. By Lemma 4.2, applied with $r = 4k + 1$, every $v \in V(G)$ is contained in a component of order at least d^k in G_p with probability at least $y - o(1)$. Thus,

$$\mathbb{E}[|W|] \ge (1 - o(1))\, yn.$$

Furthermore, since G is d-regular, for every $v \in V(G)$ an easy coupling implies that $|C_v(G_p)|$ is stochastically dominated by the number of vertices in a Galton-Watson tree with offspring distribution $Bin(d, p)$. Thus, by standard arguments (see, for example, [13, Theorem 4.3.12]), we have that for every $v \in V(G)$,

$$\mathbb{P}\left[|C_v(G_p)| \ge d^k\right] \le y + o_d(1).$$

Since d tends to infinity with t, we obtain that

$$\mathbb{E}[|W|] \le (1 + o(1))\, yn,$$

and we conclude that $\mathbb{E}[|W|] = (1 + o(1))\, yn$.

It remains to show that $|W|$ is concentrated about its mean. To this end, let us consider the edge-exposure martingale on G_p, whose length is $|E(G)| = \frac{nd}{2}$. Adding or deleting an edge can change W by at most $2d^k$ vertices. Thus, a standard application of Lemma 2.2 implies that

$$\mathbb{P}\left[\big||W| - \mathbb{E}\left[|W|\right]\big| \geq n^{\frac{2}{3}}\right] \leq 2\exp\left(-\frac{n^{\frac{4}{3}}}{2\sum_{i=1}^{\frac{nd}{2}} 4d^{2k}}\right)$$
$$\leq 2\exp\left(-\frac{n^{\frac{1}{3}}}{5d^{2k+1}}\right) = o(1),$$

since $d = \Theta(\log n)$. $\square$

4.3 Big components are everywhere-dense

In this section, we establish several lemmas showing that, typically, big components in G_p are *everywhere-dense* in G. While the statements seem similar, the subtle differences in the type of density will be of importance in the proof of Theorem 2.

We begin with the first type of density lemma. Note that, in particular, the following implies that for any fixed vertex $v \in G$, **whp** v is adjacent to many vertices which lie in big components in G_p.

Lemma 4.4. *Let $\epsilon > 0$ be a small enough constant, let $p = \frac{1+\epsilon}{d}$, let $k > 0$ be an integer and let $M \subseteq V(G)$ be such that $|M| = m \leq \frac{\epsilon d}{10C}$. Then, the probability that every $v \in M$ has less than $\frac{\epsilon^2 d}{40C}$ neighbours (in G) which lie in components of G_p whose order is at least d^k is at most $\exp\left(-\frac{\epsilon^2 dm}{40C}\right)$.*

Proof. Let $M = \{u_1, \ldots, u_m\}$, where we stress here that the subscript denotes an enumeration of the vertices of M, that is, $u_i \in V(G)$. By Lemma 4.1, we can find pairwise disjoint projections $H_1, \ldots, H_m$ of G such that each H_i is of dimension at least $t - m + 1 \geq t - \frac{\epsilon d}{10C}$ and $u_i \in V(H_i)$ for all $i \in [m]$. Note that since $d \leq Ct$, $t - \frac{\epsilon d}{10C} > 0$.

Let us fix some i. Without loss of generality, we may assume that H_i is the projection of G onto the last $m_i \leq m - 1$ coordinates, that is,

$$H_i = G^{(1)}\square\cdots\square G^{(t-m)}\square\{u_{i,t-m_i+1}\}\square\cdots\square\{u_{i,t}\},$$

where $u_{i,\ell} \in V(G^{(\ell)})$ is the ℓ-th coordinate of u_i, that is, the ℓ-th coordinate of the i-th vertex of M. By our assumption, each $G^{(\ell)}$ has at least 2 vertices and is connected. Thus, for each ℓ, we can choose arbitrarily one of the neighbours of $u_{i,\ell}$ in $G^{(\ell)}$ and denote it by $v_{i,\ell}$, where the subscript in $v_{i,\ell}$ is to stress that it is a neighbour of $u_{i,\ell}$ in $G^{(\ell)}$. We now define the following $\frac{\epsilon d}{10C}$ pairwise disjoint projections of H_i, denoted by $H_i(1), \ldots, H_i\left(\frac{\epsilon d}{10C}\right)$, each having dimension at least $t - \frac{\epsilon d}{5C}$. We set $H_i(j)$ to be the projection of H_i on the first $\frac{\epsilon d}{10C}$ coordinates, such that the j-th coordinate (where $1 \leq j \leq \frac{\epsilon d}{10C}$) is the trivial subgraph $\{v_{i,j}\} \subseteq G^{(j)}$, and the coordinates $1 \leq \ell \leq \frac{\epsilon d}{10C}$ (where $\ell \neq j$) are the trivial subgraphs $\{u_{i,\ell}\} \subseteq G^{(\ell)}$. Note that each $H_i(j)$ is at distance 1 from u_i, since it contains the vertex

$$v_i(j) \coloneqq (u_{i,1}, \cdots, u_{i,j-1}, v_{i,j}, u_{i,j+1}, \cdots, u_{i,t}).$$

Observe that $p = \frac{1+\epsilon}{d}$ is supercritical for every $H_i(j)$, since $d(H_i(j)) \geq d - C \cdot \frac{\epsilon d}{5C} = \left(1 - \frac{\epsilon}{5}\right)d$, and so $p \cdot d(H_i(j)) \geq 1 + \frac{3\epsilon}{5}$, for small enough ϵ. Then, by Lemma 4.2, with probability at least $y\left(\frac{3\epsilon}{5}\right) - o(1)$ we have that $v_i(j)$ belongs to a component of order at least d^k in $G_p \cap H_i(j)$, and we note that by (1), we have that $y\left(\frac{3\epsilon}{5}\right) - o(1) > \epsilon$ for small enough ϵ. These events are independent for different j, and thus by Lemma 2.1, with probability at least $1-\exp\left(-\frac{\epsilon^2 d}{40C}\right)$, at least $\frac{\epsilon^2 d}{40C}$ of the $v_i(j)$ belong to a component of order at least d^k.

These events are independent for different i, and thus the probability that none of the u_i have at least $\frac{\epsilon^2 d}{40C}$ neighbours (in G) in components of G_p whose order is at least d^k is at most $\exp\left(-\frac{\epsilon^2 dm}{40C}\right)$, as required. □

Hence, we expect almost all vertices in G to be adjacent to a vertex in a big component in G_p. However, even the vertices not adjacent to big components, whose proportion is likely to be small, will typically be not too far from a big component.

Lemma 4.5. *Let $\epsilon > 0$ be a small enough constant, let $p = \frac{1+\epsilon}{d}$ and let $k > 0$ be an integer. Then,* ***whp****, every $v \in V(G)$ is at distance (in G) at most 2 from a component of order at least d^k in G_p.*

Proof. Fix $u \in V(G)$, and let u_i be the i-th coordinate of u for each $i \in [t]$. For each $i \in [t]$ let us choose a neighbour v_i of u_i in $G^{(i)}$. For each $1 \le \ell \ne m \le \frac{\epsilon d}{10C}$ let $H_{\ell,m}$ be the projection of G onto the first $\frac{\epsilon d}{10C}$ coordinates, such that the ℓ-th and m-th coordinates of $H_{\ell,m}$ are $\{v_\ell\}$ and $\{v_m\}$, respectively, and the j-th coordinate is $\{u_j\}$ for all other $j \in \left[\frac{\epsilon d}{10C}\right]$.

We note that the $H_{\ell,m}$ are then a set of $\binom{\frac{\epsilon d}{10C}}{2} \ge \frac{\epsilon^2 d^2}{300C^2}$ pairwise disjoint projections of G. Furthermore, every $H_{\ell,m}$ contains a vertex at distance 2 (in G) from u, which we denote by $v_{\ell,m}$. Moreover, since every $H_{\ell,m}$ has dimension $\left(1-\frac{\epsilon}{10C}\right)d$, it follows that every $H_{\ell,m}$ is regular with $d\left(H_{\ell,m}\right) \ge d - C\frac{\epsilon d}{10C} = \left(1-\frac{\epsilon}{10}\right)d$. In particular, $p \cdot d\left(H_{\ell,m}\right) \ge 1 + \frac{4}{5}\epsilon$, for small enough ϵ, and so p is supercritical for every $H_{\ell,m}$.

Hence, by Lemma 4.2 and (1), $v_{\ell,m}$ belongs to a component of $G_p \cap H_{\ell,m}$ of order at least d^k with probability at least $y\left(\frac{4\epsilon}{5}\right) - o(1) \ge \epsilon$, for small enough ϵ. Since the $H_{\ell,m}$ are pairwise disjoint, these events are independent for different $v_{\ell,m}$. Thus, with probability at least $1 - (1-\epsilon)^{\frac{\epsilon^2 d^2}{300C^2}} \ge 1 - \exp\left(-\frac{\epsilon^3 d^2}{10^3 C^2}\right)$ we have that at least one of the $v_{\ell,m}$ belongs to a component whose order is at least d^k. Since $d = \Theta(\log n)$, we have $\exp\left(-\frac{\epsilon^3 d^2}{10^3 C^2}\right) = o\left(\frac{1}{n}\right)$ and thus we can complete the proof with a union bound over the n choices for u. □

Finally, we can use Lemma 4.4 and Lemma 2.4 to show that, typically, big components in G_p are dense with respect to connected sets, in the following sense.

Lemma 4.6. *Let $\epsilon > 0$ be a small enough constant, let $p = \frac{1+\epsilon}{d}$ and let $k > 0$ be an integer. Let W be the set of vertices in G_p belonging to components of order at least d^k and let $C_1 > 0$ be a constant. Then,* ***whp*** *every connected subset $M \subseteq V(G)$ of size $C_1 d$ contains at most $\frac{\epsilon d}{10C}$ vertices $v \in M$ such that $|N_G(v) \cap W| < \frac{\epsilon^2 d}{40C}$.*

Proof. By Lemma 2.4, there are at most $n(ed)^{C_1 d}$ connected subsets $M \subseteq V(G)$, and there are at most $\binom{C_1 d}{\frac{\epsilon d}{10C}} \le 2^{C_1 d}$ ways to choose a subset $X \subset M$ of size $\frac{\epsilon d}{10C}$. By Lemma 4.4, the probability that no vertex in X has at least $\frac{\epsilon^2 d}{40C}$ neighbours in W is at most $\exp\left(-\frac{\epsilon^3 d^2}{400C^2}\right)$.

Hence, by a union bound, the probability that the conclusion of the lemma does not hold is at most

$$n(ed)^{C_1 d} 2^{C_1 d} \exp\left(-\frac{\epsilon^3 d^2}{400C^2}\right) = o(1),$$

where we used that $d = \Theta(t) = \Theta(\log n)$. □

4.4 Proof of Theorem 2

Throughout this section, we take C, $G^{(1)}, \dots, G^{(t)}$, G, ϵ, d and p to be as in the statement of Theorem 2, and let $y = y(\epsilon)$ be as defined in (1). In order to prove Theorem 2, we will use a multi-round exposure argument.

More precisely, let $p_2 = \frac{\epsilon}{2d}$, $p_3 = \frac{1}{d^2}$, and let p_1 be such that $(1-p_1)(1-p_2)(1-p_3) = 1-p$. Note that $p_1 = \frac{1+\frac{\epsilon}{2}-o(1)}{d}$. Let G_{p_1}, G_{p_2} and G_{p_3} be independent random subgraphs of G and let us write $G_1 = G_{p_1}$, $G_2 = G_1 \cup G_{p_2}$ and $G_3 = G_2 \cup G_{p_3}$, where we note that G_3 has the same distribution as G_p. Let $k > 0$ be an integer, which we will choose later. We define $W_i = W_i(k)$ (for $i = 1, 2, 3$) to be the set of vertices in components of order at least d^k in G_i. We note that $W_1 \subseteq W_2 \subseteq W_3$.

We begin by showing that **whp**, there are no components of order larger than $O_\epsilon(d)$ in G_p, which do not meet W_1.

Lemma 4.7. *There exists a constant $C_1 = C_1(\epsilon)$ such that **whp** every component M of G_p with $|M| \geq C_1 d$ intersects with W_1.*

Proof. We start by exposing $G_{p_1} = G_1$. Let $V_1 \subseteq V(G) \setminus W_1$ be the set of all vertices of G that have at least $\frac{\epsilon^2 d}{161C}$ neighbours (in G) in W_1, and let $V_0 = V(G) \setminus (V_1 \cup W_1)$. We first note that, by Lemma 4.6 applied to G_{p_1}, **whp** every connected subset of G with at least $C_1 d$ vertices contains fewer than $\frac{\epsilon d}{21C}$ vertices in V_0. We assume henceforth that this property holds.

We then expose the rest of G_p on $V_0 \cup V_1$. There are at most n components of $G_p[V_0 \cup V_1]$. Given a connected component M of $G_p[V_0 \cup V_1]$ of order at least $C_1 d$, since M spans a connected subset of G, by the above

$$|M \cap V_1| \geq C_1 d - \frac{\epsilon d}{21C} \geq \frac{C_1 d}{2},$$

if C_1 is sufficiently large.

Each vertex in $M \cap V_1$ has at least $\frac{\epsilon^2 d}{161C}$ neighbours in W_1 and so there is a set X of at least $\frac{\epsilon^2 C_1 d^2}{330C}$ edges between M and W_1. We now expose the edges in $X \cap G_{p_2}$. By Lemma 2.1, $X \cap G_{p_2} \neq \emptyset$ with probability at least $1 - \exp\left(-\frac{\epsilon^3 C_1 d}{400C}\right)$. Recalling that $d = \Theta(\log n)$, we let α be the constant such that $d = \alpha \log n$.

Then, for $C_1 \geq \frac{800 \cdot C \cdot \alpha}{\epsilon^3}$, with probability $1 - o\left(\frac{1}{n}\right)$, M is adjacent to a vertex in W_1 in $G_{p_2} \subseteq G_p$. Hence, by a union bound, **whp** every component of $G_p[V_0 \cup V_1]$ of order at least $C_1 d$ is adjacent in G_p to a vertex in W_1, from which the statement follows. □

We now fix $k = 16$.

Lemma 4.8. ***Whp**, all the components of $G_2[W_2]$ are contained in a single component of G_p.*

Proof. We first note that every edge of G belongs to G_2 independently with probability $1 - (1-p_1)(1-p_2) = \frac{1+\epsilon-o(1)}{d}$. Thus, by Lemma 4.5, **whp** every $v \in V(G)$ is at distance at most 2 from a vertex of W_2. We assume henceforth that this property holds.

Suppose the statement of the lemma does not hold, then we can partition the components of $G_2[W_2]$ into two families, denote them by $\mathcal{A}$ and $\mathcal{B}$, such that there are no paths in G_p (and hence in G_{p_3}) between $\mathcal{A}$ and $\mathcal{B}$. Denote by s the number of components in $\mathcal{A} \cup \mathcal{B}$, and let $\ell \leq \frac{s}{2}$ be the number of components in the smaller family. Observe that both $A = \cup_{C \in \mathcal{A}} V(C)$ and $B = \cup_{C \in \mathcal{B}} V(C)$ have at least ℓd^{16} vertices in them.

Now, since every vertex in G is at distance at most 2 from either A or B, we can partition $V(G)$ into two sets A' and B', such that A' contains A and all $v \in V(G) \setminus B$ at distance at most 2 from A, and similarly B' contains B and all $v \in V(G) \setminus A'$ at distance at most 2 from B.

Now, by Lemma 2.3 the isoperimetric constant $i(G)$ of G is at least $\frac{1}{2} \min_{j \in [t]} i\left(G^{(j)}\right)$. Since for all $j \in [t]$, $1 < |V\left(G^{(j)}\right)| \leq C$ and $G^{(j)}$ is connected, we have that $i\left(G^{(j)}\right) \geq \frac{1}{C}$ and thus $i(G) \geq \frac{1}{2C}$. It follows that there are at least $\frac{\ell d^{16}}{2C}$ edges between A' and B'. Now, since every vertex in A' is at distance at most 2 from A, and similarly every vertex in B' is at distance at most 2 from B, we can extend these edges to a family of $\frac{\ell d^{16}}{2C}$ paths of length at most 5 between A and B. Since G is d-regular, every edge participates in at most $5d^4$ paths of length at most 5.

Thus, we can greedily thin this family to a set of $\frac{\ell d^{16}}{2C \cdot 5 \cdot 5d^4} = \frac{\ell d^{12}}{50C}$ edge-disjoint paths of length at most 5 between A and B. The probability none of these paths are in G_{p_3} is thus at most

$$(1-p_3^5)^{\frac{\ell d^{12}}{50C}} \leq \exp\left(-\frac{\ell d^2}{50C}\right).$$

On the other hand, there are at most n components in $G_2[W_2]$ and hence the number of ways to partition these components into two families with one containing ℓ components is at most $\binom{n}{\ell}$. Thus, by the union bound, the probability that the statement does not hold is at most

$$\sum_{\ell=1}^{\frac{s}{2}} \binom{n}{\ell} \exp\left(-\frac{\ell d^2}{50C}\right) \leq \sum_{\ell=1}^{\frac{s}{2}} \left[n \exp\left(-\frac{d^2}{50C}\right)\right]^{\ell} = o(1),$$

since $d = \Theta(\log n)$, completing the proof. □

We are now ready to prove Theorem 2:

Proof of Theorem 2. By Lemma 4.3, we have that **whp** $|W_3| = (1+o(1))\, yn$.

By Lemma 4.7, **whp** there are no components in G_p of order larger than $O_\epsilon(d)$ which do not intersect with W_1. Thus, since every component of $G_p[W_3]$ has size at least d^{16}, **whp** every component of $G_p[W_3]$ meets a component of $G_1[W_1]$, and so (by inclusion) meets a component of $G_2[W_2]$. Hence, by Lemma 4.8, **whp** all these components are contained in a single component of G_p, which we denote by L_1, with $V(L_1) = W_3$.

Finally, once again by Lemma 4.7, **whp** there are no components larger than $O_\epsilon(d)$ which do not intersect with $W_1 \subset W_3$, and thus by the above **whp** there are no components besides L_1 of order larger than $O_\epsilon(d)$.

All in all, **whp** there are $(1+o(1))\, yn$ vertices in the largest component L_1, and all the other vertices are in components whose order is at most $O_\epsilon(d)$, as required. □

5 Discussion and possible future research directions

Already in [28], Lichev asked whether all of the assumptions in Theorem 1.3 were necessary to ensure a threshold for the appearance of a linear sized component. In particular, he asked whether the bound on the maximum degree of the host graphs could be removed, and whether his (relatively mild) isoperimetric requirement on the base graphs could be furthered relaxed. In a forthcoming paper [11] we provide a construction answering the former question in the negative, showing that with irregular base graphs, even ones satisfying a relatively strong isoperimetric inequality, the largest component in the supercritical regime can be sublinear in size. However, using some of the tools developed in this paper we are also able to significantly relax the isoperimetric requirements on the base graphs from a polynomial dependence in the dimension to a superexponential one.

In terms of our Theorem 2, it is also interesting to consider which assumptions are necessary. As mentioned in the introduction, there are examples of products graphs of bounded dimension, for example the d-dimensional torus [21], which do not go through a quantitatively similar phase transition to $G(d+1, p)$.

In terms of our other two assumptions, that the base graphs are regular and of bounded size, in [11] we also give an example to show that the regularity assumption is necessary to a certain extent — if we take our base graphs to be stars, and so quite irregular, then the component behaviour can be quite different, with polynomially sized components appearing already in the subcritical regime. It remains an interesting open question as to whether we can relax the assumption that the base graphs have bounded size to that of just bounded regularity, or even weaker to some assumption of *almost-regularity* of the product graph, under some mild isoperimetric assumptions.

Question 5.1. *Let G be a high-dimensional product graph all of whose base graphs are connected and let d be the average degree of G. Let $\epsilon > 0$ and let $p = \frac{1+\epsilon}{d}$. What assumptions on degree distributions and the isoperimetric constants of the base graphs are sufficient to guarantee that G_p exhibits the Erős-Rényi component phenomenon?*

More generally, it would be interesting to investigate further which other classes of graph exhibit the Erdős-Rényi component phenomenon under percolation and to find the limits of the universality of this phenomenon.

We note that the isoperimetric inequality in Theorem 2.3 is far from optimal in the case of the hypercube. Indeed, whilst Theorem 2.3 is asymptotically tight for linear sized sets, which can have constant edge-expansion, a well-known result of Harper [20] implies that smaller sets in Q^d have almost optimal edge-expansion, expanding by a factor of almost d. In the presence of such a strong isoperimetric inequality, it is relatively simple to show the existence of a gap in the sizes of the components in Q^d_p using a first moment argument, which implies any component of polynomial (in d) order must in fact have order $O(d)$ (see [5]).

We believe that a qualitatively similar isoperimetric inequality should hold for high-dimensional product graphs (with regular base graphs of bounded order), which would lead to a shorter proof of Theorem 2, but one that is less adaptable to other models, such as site percolation.

Conjecture 5.2. *Let G be a high-dimensional product graph all of whose base graphs are connected, regular and of bounded size and let d be the degree of G. If $X \subseteq V(G)$ is such that $\log|X| \ll d$, then $e(X, X^c) = (1 - o(1))d|X|$.*

Since we have seen that percolated high-dimensional product graphs exhibit the Erdős-Rényi phenomenon, a natural question is to ask what other typical properties of the supercritical binomial random graph are also present in these percolated product graphs, and in particular in their giant components. For many natural properties, tight bounds are not yet known even in the case of the hypercube, see [14] for a recent work by the last three authors on the expansion properties of the giant component in the percolated hypercube, and its consequences for the structure of the giant component. We note that a strong isoperimetric inequality for high-dimensional product graphs as discussed in the previous paragraph would likely be a key tool in future research into the combinatorial structure of the giant component in these models. It would also be interesting to study the behaviour of percolation on such graphs close to the critical point, in particular whether they exhibit mean-field behaviour in terms of the width of the critical window, a result which was only recently shown to hold in the hypercube in a breakthrough result of van der Hofstad and Nachmias [22].

Finally, whilst the Cartesian product is perhaps a natural product in the context of percolation, there are many other types of graphs products, such as strong products and tensor products, and it would be interesting to know if 'high-dimensional' graphs with respect to these types of products also exhibit similar behaviour under percolation. As a concrete example, since the d-fold tensor product of a single edge is disconnected and the d-fold strong product of a single edge is complete, it is perhaps interesting to consider percolation in the tensor and strong products of many small cycles. Let us write $T(k, d)$ for the d-fold tensor product of the cycle C_k and similarly $S(k, d)$ for the d-fold strong product. Note that $T(k, d)$ is 2^d-regular and $S(k, d)$ is $2^d + 2d$ regular.

Question 5.3. *Do the percolated subgraphs $T(3, d)_p$ and $S(3, d)_p$ exhibit the Erdős-Rényi component phenomenon at the critical point $p = 2^{-d}$?*

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