Sparse pancyclic subgraphs of random graphs

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Abstract

It is known that the complete graph K_n contains a pancyclic subgraph with $n + (1 + o(1)) \cdot \log_2 n$ edges, and that there is no pancyclic graph on n vertices with fewer than $n + \log_2(n-1) - 1$ edges. We show that, with high probability, $G(n,p)$ contains a pancyclic subgraph with $n + (1+o(1)) \log_2 n$ edges for $p \geq p^*$, where $p^* = (1+o(1)) \ln n/n$, right above the threshold for pancyclicity.

1 Introduction

Say that a graph G is pancyclic if G contains a cycle of every length between 3 and $|V(G)|$. See monograph [6] for generic information on pancyclic graphs. In his influential paper on pancyclic graphs, Bondy [2] asked what is the minimum number of edges in a pancyclic n-vertex graph. This can be rephrased as the minimum number of edges in a pancyclic subgraph of K_n, which motivates the following definition.

Definition 1. *Say that a pancyclic graph G on n vertices has pancyclicity excess k, and denote $Pex(G) = k$, if the minimum number of edges in a pancyclic subgraph of G is $n + k$.*

In other words, a pancyclic subgraph of G achieving the minimum number of edges is formed by a Hamilton cycle and $\text{Pex}(G)$ additional chords. In his paper, Bondy stated that, for every n,

$$\log_2(n-1) - 1 \leq \text{Pex}(K_n) \leq \log_2 n + \log^* n + O(1),$$

and did not provide a proof. Shi [10] later asserted the lower bound, by showing that an n-vertex graph with $n + k$ edges contains at most $2^{k+1} - 1$ distinct cycles, so every subgraph of K_n with fewer than $n + \log_2(n-1) - 1$ edges must have fewer than $2^{\log_2(n-1)} - 1 = n - 2$ cycles in total, regardless of their lengths. On the other hand, there are constructions for every n of an n-vertex pancyclic graph with $\log_2 n + \log^* n + O(1)$ chords (see e.g. [6], Chapter 4.5), so $\text{Pex}(K_n) \leq \log_2 n + \log^* n + O(1)$. What is the exact value of $\text{Pex}(K_n)$ within this range is still an open question.

In this paper, we study the typical behaviour of $\text{Pex}(G)$, for $G \sim G(n,p)$. Cooper and Frieze [4] showed that, for $p \in [0,1]$, the limiting probability of $G \sim G(n,p)$ being pancyclic is

$$\lim_{n\to\infty} \mathbb{P}(G(n,p) \text{ is pancyclic}) = \begin{cases} 1 & \text{if } np - \log n - \log\log n \to \infty; \\ e^{-e^{-c}} & \text{if } np - \log n - \log\log n \to c; \\ 0 & \text{if } np - \log n - \log\log n \to -\infty. \end{cases}$$

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Here and later, if the base of the logarithm is not stated then it is the natural base. The above expression is also the limiting probability of G being Hamiltonian, and the limiting probability of $\delta(G) \geq 2$. In particular, the three properties have the same threshold.

Clearly, $\text{Pex}(G) \geq \log_2(n-1) - 1$ for every pancyclic graph G on n vertices. On the other hand, Cooper [3] showed that if p is above the pancyclicity threshold, then with high probability $G \sim G(n,p)$ is a so called *1-pancyclic* graph, that is, it contains a Hamilton cycle H with the property that, for every $\ell \in [3, n-1]$, there is an edge $e \in E(G)$ such that $H \cup \{e\}$ contains a cycle of length ℓ and a cycle of length $n - \ell + 2$. Observe that if G is a 1-pancyclic n-vertex graph then $\text{Pex}(G) \leq \lceil \frac{n-3}{2} \rceil$. So Cooper's result implies that $\text{Pex}(G(n,p)) \leq \lceil \frac{n-3}{2} \rceil$ with high probability, for all p above the pancyclicity threshold.

Our result in this paper shows that, for $G \sim G(n,p)$, the pancyclicity excess of G is, typically, very close to the above stated general lower bound.

Theorem 1. *There is $p^* = p^*(n) = (1 + \varepsilon(n)) \cdot \frac{\log n}{n}$, where $\varepsilon(n) = O\left(\frac{1}{\log \log}\right)$, such that, if $p \geq p^*$ and $G \sim G(n,p)$, then with high probability g is pancyclic with $Pex(G) = (1 + o(1)) \cdot \log_2 n$.*

It is worth noting that we did not attempt to optimize the error term $\varepsilon(n)$, opting rather for a more simple proof. We therefore leave the question of whether $\text{Pex}(G(n,p))$ also typically satisfies $\text{Pex}(G(n,p)) = (1 + o(1)) \cdot \log_2 n$ for all p above the pancyclicity threshold as an open question.

Paper structure In Section 2 we introduce definitions and notation required for the rest of the paper, as well as auxiliary results to be used in our proof. In Section 3 we introduce a construction of a subgraph of a given n-vertex graph, which, if successful, produces a subgraph with $n + (1 + o(1)) \cdot \log_2 n$ edges. In Section 4 we show that, with high probability, the construction is possible in $G(n,p)$ for $p \geq p^*$, and in Section 5 we complete the proof of Theorem 1 by showing that the constructed subgraph is pancyclic.

2 Preliminaries

2.1 Definitions and notation

The following graph theoretic notation is used throughout the paper.

Let G be a graph and $U, W \subseteq V(G)$ vertex subsets. We denote by $E_G(U,W)$ the set of edges of G with vertex in U and one vertex in W, and $e_G(U,W) = |E_G(U,W)|$. We let $G[U]$ denote the subgraph induced by G on the vertex subset U, by $E_G(U)$ the set of edges in $G[U]$, and by $e_G(U)$ its size. We denote by $N_G(U)$ the (external) neighbourhood of U, that is, the set of vertices in $V(G) \setminus U$ adjacent to a vertex of U. The degree of a vertex $v \in V(G)$, denoted by $d_G(v)$, is the number of edges of G incident to v.

We let $\mathcal{L}(G)$ denote the set of cycle lengths found in G, that is, $\mathcal{L}(G)$ is the set of integers ℓ such that G contains a cycle of length ℓ.

While using the above notation we occasionally omit G if the identity of the specific graph is clear from the context.

We occasionally suppress the rounding signs to simplify the presentation.

Finally, we require the following definition.

Definition 2. *A graph G is called a (k, α)-expander if every subset $U \subseteq V(G)$ with $|U| \leq k$ satisfies $|N_G(U)| \geq \alpha \cdot |U|$.*

2.2 Auxiliary results

Theorem 2.1 (Cycle lengths in $G(n,p)$, a corollary of Łuczak [9]). *Let $p = p(n)$ be such that $np \to \infty$ and let $G \sim G(n,p)$. Then, with high probability, $[3, 0.99n] \subseteq \mathcal{L}(G)$.*

Theorem 2.2 (Tree embeddings in expanders, a corollary of [7] as given in [1]). *Let N, Δ be integers, and let G be a graph. Assume that there exists an integer k such that*

1. *For every $U \subseteq V(G)$ with $|U| \leq k$ we have $|N_G(U)| \geq \Delta \cdot |U| + 1$;*
2. *For every $U \subseteq V(G)$ with $k < |U| \leq 2k$ we have $|N_G(U)| \geq \Delta \cdot |U| + N$.*

Then, for every $v \in V(G)$ and every rooted tree T with at most N vertices and maximum degree at most Δ, the graph G contains a copy of T rooted in v.

Lemma 2.1 (Hamiltonicity and expansion of $G(n,p)$, see e.g. [8], Section 4). *Let $p = p(n)$ be such that $np - \log n - \log\log n \to \infty$, and let $G \sim G(n,p)$. Then, with high probability, there is a subset $S \subseteq V(G)$ of $\frac{n}{4}$ vertices, such that for every $s \in S$ there is a subset $T_s \subseteq V(G)$ of $\frac{n}{4}$ vertices, and for every $t \in T_s$ there is a Hamilton path between s and t.*

3 The constructed pancyclic subgraph

We emulate (an approximation of) the construction in [6].

Definition 3. *Let G be a graph and $H \subseteq G$ be a Hamilton cycle, and let $2 \leq \ell \leq n-2$. We say that an edge $e \in E(G)$ is an* ℓ-shortcut *with respect to H if (at least) one of the two intervals on H that connects the two endpoints of e has length $\ell + 1$.*

The motivation behind this definition is that by using H and an ℓ-shortcut we can find a cycle of length $n - \ell$ in G, by replacing an interval of length $\ell + 1$ with a single edge (the ℓ-shortcut). In the construction described in [6], one creates a sparse pancyclic graph by taking an n-cycle H and K shortcuts $e_0, e_1, ..., e_K$, where K is such that $\frac{1}{2}n \leq 2^{K+1} + K - 1 \leq n$ and e_i is a 2^i-shortcut. Additionally, these shortcuts are consecutive on the cycle, so that e_i, e_{i+1} and their corresponding intervals intersect in a vertex v_i. By taking intervals from the cycle H and a subset of shortcuts, one can now encode a cycle of every length between n and $n - 2^{K+1} + 1$. Next, by adding the edge between the first vertex of e_0 and the second vertex of e_K, all cycle lengths between $K + 2$ and $2^{K+1} + K$ can be encoded. This leaves out only a subset of cycle lengths contained in $[5, K+1]$, and adding these lengths to the set of cycle lengths in the graph can be done by inserting $O(\log^* n)$ additional edges. For the full details of the construction, we refer the reader to [6] Chapter 4.5.

We approximate this construction by finding a Hamilton cycle and shortcuts to encode an interval of $L = \Omega\left(\frac{n}{\sqrt{\log n}}\right)$ consecutive cycle lengths. Like in the deterministic version, we will utilize binary encoding of the cycle lengths, so that the number of required shortcuts is $(1 + o(1))\log_2 n$. Additionally, we will require the shortcuts to reside on a short interval of the cycle (where in the deterministic version they intersected each other in a vertex). Next, by adding certain edges to the subgraph we can add an interval of L cycle lengths with each such added edge. If the said additional edges are chosen well (which we will show is possible to do with high probability), one can get a union of $O(\sqrt{\log n})$ of these intervals that covers all the lengths between some initial length $\ell^* = (1 + o(1))\log_2 n$ and n.

To handle cycle lengths shorter than ℓ^* we will show that, with high probability, almost all of them (that is, all but $o(\log n)$ cycle lengths in $[3,\ell^*]$) can be encoded by $o(\log n)$ carefully chosen shortcuts, this time utilizing an encoding in base $b = \lceil \log\log n \rceil$. The remaining unencoded cycle lengths, which constitute a subset of $[3,\ell^*]$ of size $o(\log n)$, can now be added one-by-one by using at most $o(\log n)$ additional edges, with high probability.

Let
$$p_1 = p_5 = \frac{2\log\log n}{n},\ p_2 = p_3 = \frac{50\log n}{n\cdot\log\log n},\ p_4 = \frac{\log n + 10\sqrt{\log n}}{n},$$
and let
$$p^* = p^*(n) = 1 - \prod_{i=1}^{5}(1-p_i).$$
Letting $\varepsilon(n) := \frac{n}{\log n}\cdot p^* - 1$ we get that $\varepsilon(n) = O(\frac{1}{\log\log n})$, and since the property $\mathrm{Pex}(G) \leq k$ is monotone increasing, it suffices to prove that $\mathrm{Pex}(G) \leq (1+o(1))\log_2 n$ holds with high probability for $G(n,p^*) \sim \bigcup_{i=1}^{5} G(n,p_i)$. We note that we did not attempt to optimize the value of $\varepsilon(n)$ determined by $p_1,...,p_5$, aiming rather for simplicity.

Denote
$$\ell_i := 2^i + 1,$$
and
$$\beta = \beta(n) := \frac{2(\log\log n)^2}{\log n},\ d = d(n) = \lfloor \log_{(5\beta)^{-1}}(n/200) \rfloor.$$
Note that
$$d = \lfloor \log_{(5\beta)^{-1}}(n/200) \rfloor = (1+o(1))\cdot\frac{\log(n/200)}{-\log(5\beta)} = (1+o(1))\cdot\frac{\log n}{\log\log n}.$$
For $1 \leq i \leq 5$ let $G_i \sim G(n,p_i)$. We divide the construction into five steps, where in the i'th step we sample G_i to try and produce a subgraph $H_i \subseteq \bigcup_{j=1}^{i} G_j$. If the construction is successful, the produced subgraph H_5 will be pancyclic with $|E(H_5)| = n + (1+o(1))\cdot\log_2 n$. The steps of our construction are as follows.

1. Let
$$K_0 := \lfloor \log_2\left(\frac{\log n}{6\log\log\log n}\right) \rfloor,$$
and
$$b := \lceil \log\log n \rceil,\ t := \lceil \log_b \log n \rceil.$$
Find a set of vertex disjoint cycles $C_0,...,C_{K_0},C_{\mathrm{short}}$ in G_1 of respective lengths $\ell_0+1,\ell_1+1,...,\ell_{K_0}+1, t\cdot b+1$. The first K_0+1 cycles will later become the first K_0+1 shortcuts, and their corresponding intervals, where the edges of C_{short} will become the shortcuts required to handle short cycles. For every $0 \leq i \leq K_0$, choose an arbitrary edge $e_i \in C_i$ to serve as the shortcut. Denote $H_1 = C_{\mathrm{short}} \cup \bigcup_{i=0}^{K_0} C_i$.

2. For every $0 \leq i \leq K_0$, find a path of length $d+2$ in G_2 between the second vertex of e_i and the first vertex of e_{i+1} (where for $i = K_0$ the path is between e_{K_0} and e_0), so that the K_0+1 paths are pairwise vertex disjoint from each other, and internally vertex disjoint from $V(H_1)$. Call the cycle formed by the union of the paths and the shortcuts C^* and denote $\ell^* := e(C^*), H_2 := H_1 \cup C^*$. We have
$$\ell^* = (1+o(1))\cdot K_0\cdot d = (1+o(1))\cdot\log_2 n.$$

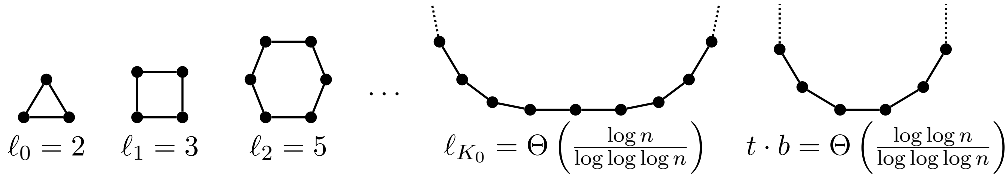

Figure 1: Step 1, with resulting graph H_1 depicted.

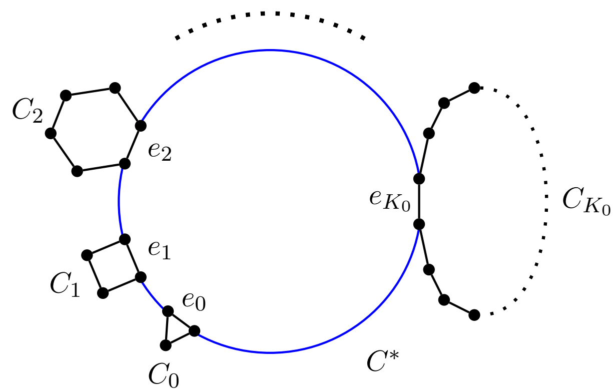

Figure 2: Step 2, with resulting graph H_2 depicted.

3. Let

$$K := \lfloor \log_2 \left(\frac{n}{\sqrt{\log n}} \right) \rfloor,$$
$$L := 2^{K+1} - 1,$$

so that $L+1 \in \left[\frac{n}{\sqrt{\log n}}, \frac{2n}{\sqrt{\log n}}\right]$. For $K_0 < i \leq K$, construct the i'th shortcut by choosing an arbitrary (non-shortcut) edge e_i on C^*, and finding a path of length ℓ_i between its two vertices in G_3, such that these paths are internally vertex disjoint from each other and from $V(H_2)$. Letting C_i denote the cycle comprised of e_i and the ℓ_i-path in G_3 between its vertices, we get that the subgraph $C^* \cup \bigcup_{i=0}^{K} C_i$ contains all cycle lengths in the interval $[\ell^*, \ell^* + L]$. Choose an arbitrary edge $e^* \in C^* \setminus \{e_0, ..., e_K\}$.

Next, denote $E(C_{\text{short}}) = \{e_{\text{short}}\} \cup \{e_{i,j} \mid 0 \leq i \leq t-1, 0 \leq j \leq b-1\}$, with an arbitrary order. Find paths $\{P_{i,j} \mid 0 \leq i \leq t-1, 0 \leq j \leq b-1\}$ in G_3, where $P_{i,j}$ connects the endpoints of $e_{i,j}$, such that the paths are all internally vertex disjoint from each other, and from $V(H_2 \cup \bigcup_{i=K_0}^{K} C_i)$, and $P_{i,j}$ has length $d + 2 + j \cdot b^i$. Now the subgraph $C_{\text{short}} \cup \{P_{i,j} \mid 0 \leq i \leq t-1, 0 \leq j \leq b-1\}$ contains all cycle lengths in $\left[(d+b+1) \cdot t + 1, (d+b+1) \cdot t + b^t\right]$. Note that $b^t \geq \log n > \ell^*$, and that $(d+b+1) \cdot t = O\left(\frac{\log n}{\log\log\log n}\right)$.

Finally for this step, connect one vertex of e^* to one vertex of e_{short} by a path P^* of length $d+2$, internally disjoint from all previous construction, and denote $H_3 := H_2 \cup P^* \cup \{P_{i,j}\}_{i,j} \cup \bigcup_{i=K_0}^{K} C_i$.

4. Construct a Hamilton cycle by connecting the vertex of e^* and the vertex of e_{short} that are not connected by P^* by a path P in G_4, whose internal vertices are exactly $V(G) \setminus V(H_3)$. Denote

$$C_H := H_3 \cup P \setminus \left(\{e_0, ..., e_K, e^*, e_{\text{short}}\} \cup \{e_{i,j}\}_{i,j}\right).$$

Then the constructed $H_4 := H_3 \cup P$ contains the Hamilton cycle C_H and $K + b \cdot t + 3$ additional edges, and all cycle lengths in $[(d+b+1) \cdot t + 1, \ell^* + L] \cup [n - L, n]$.

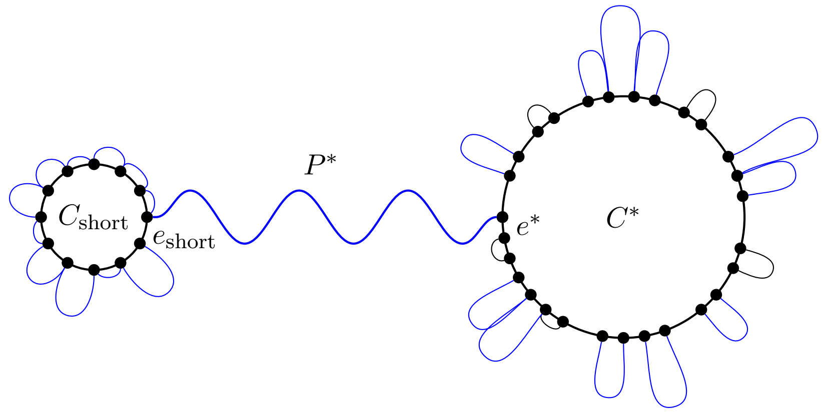

Figure 3: Step 3, with resulting graph H_3 depicted.

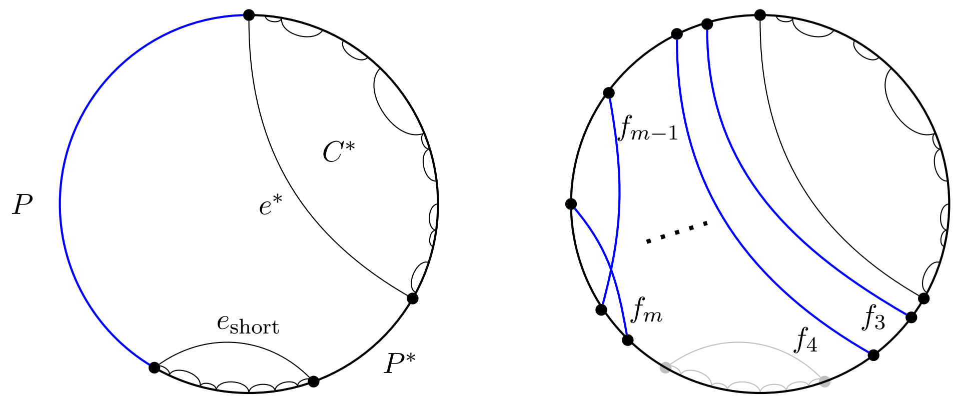

Figure 4: Steps 4 and 5, with resulting graph H_4 (left) and H_5 (right) depicted.

5. Let $m \coloneqq \lfloor n \cdot 2^{-K} \rfloor = o(\log n)$. For $3 \le i \le m$ find an ℓ_i^*-shortcut f_i in G_5, where ℓ_i^* is an integer such that $\left|\ell_i^* - i \cdot 2^K\right| \le n^{0.9}$, and such that the (ℓ_i^*+1)-path accompanying f_i contains $V\left(C^* \cup \bigcup_{i=0}^K C_i\right)$. We now have that $H_4 \cup \{f_i\}$ contains all cycle lengths in $[\ell_i^* + 2 - L, \ell_i^* + 2]$. Since $\ell_i^* \ge \ell_{i+1}^* - L$ for all i, and $\ell^* + L \ge \ell_3^* + 2 - L,\ \ell_m^* \ge n - L$, we get that $H_4 \cup \{f_3, ..., f_m\}$ contains all cycle lengths in $[(d+b+1) \cdot t + 1, n]$.

   Finally, add the remaining at most $(d+b+1) \cdot t = o(\log n)$ cycle lengths by finding in G_5 an edge g_ℓ that constitutes an $(\ell-2)$-shortcut with respect to C_H, for every $\ell \in [3, (d+b+1) \cdot t]$.

   This step adds at most $m + (d+b+1) \cdot t$ edges to the constructed subgraph $H_5 \coloneqq H_4 \cup \{f_3, ..., f_m\} \cup \{g_3, ..., g_{(d+b+1)t}\}$.

Observe that the resulting subgraph H_5 is a union of the Hamilton cycle C_H and an additional set of edges

$$\{e_0, ..., e_K, e^*, e_{\text{short}}\} \cup \{e_{i,j} \mid 0 \le i \le t-1, 0 \le j \le b-1\} \cup \{f_3, ..., f_m\} \cup \{g_3, ..., g_{(d+b+1)t}\}.$$

Therefore H_5 contains at most

$$n + K + b \cdot t + m + (d+b+1) \cdot t = n + (1 + o(1)) \cdot \log_2 n$$

edges. In Section 4 we prove that the construction of H_5 we described is possible with high probability in $G(n, p^*)$. In Section 5 we prove that H_5, if it exists as a subgraph of G, is indeed pancyclic.

4 Finding the subgraph in $G(n,p)$

We follow the steps described in Section 3, and show that, in each step, the desired substructure of the respective random graph G_i, $i=1,2,3,4,5$, exists with high probability.

We will denote the subgraph output by the i'th step of the construction (if successful) by H_i.

Step 1

Recall the notation

$$K_0 := \lfloor \log_2 \left(\frac{\log n}{6 \log\log\log n} \right) \rfloor, \ \ell_i := 2^i + 1$$

and

$$b := \lceil \log\log n \rceil, \ t := \lceil \log_b \log n \rceil.$$

By Theorem 2.1, with high probability, $G_1 \sim G(n, p_1)$ (where $p_1 = \frac{2\log\log n}{n}$) contains a sequence of cycles $C_0, C_1, ..., C_{K_0}, C_{\text{short}}$ of respective lengths $\ell_0 + 1, ..., \ell_{K_0} + 1, b \cdot t + 1$.

The following lemma implies that these cycles are also typically vertex disjoint.

Lemma 4.1. *With high probability, no two cycles of length at most $\ell_{K_0} + 1$ in G_1 intersect each other.*

Proof. Using the union bound we can show that, with high probability, G_1 does not contain a subgraph with at most $2\ell_{K_0} + 1 \leq \frac{\log n}{2\log\log\log n}$ vertices and more edges than vertices, which implies the lemma. Indeed, the probability that such a subgraph exists is at most

$$\sum_{k=4}^{2\ell_{K_0}+1} \binom{n}{k} \cdot \binom{\binom{k}{2}}{k+1} \cdot {p_1}^{k+1} \leq \sum_{k=4}^{2\ell_{K_0}+1} (e^2 n p_1)^k \cdot k \cdot p_1 \leq \log^2 n \cdot (2e^2 \log\log n)^{\frac{\log n}{2\log\log\log n}} \cdot p_1 = o(1).$$

□

Step 2

Recall that $G_2 \sim G(n, p_2)$, where $p_2 = \frac{50 \log n}{n \cdot \log\log n}$. For each $0 \leq i \leq K_0$ let $\{s_i, t_i\} := e_i \in E(C_i)$ be an arbitrary edge of C_i.

Recall that $\beta := \frac{2(\log\log n)^2}{\log n}$ and $d = \lfloor \log_{(5\beta)^{-1}}(n/200) \rfloor = (1+o(1)) \cdot \frac{\log n}{\log\log n}$.

Lemma 4.2. *With high probability G_2 contains paths $Q_0, ..., Q_{K_0}$ such that*

1. *Q_i is a path between t_i and s_{i+1} for $0 \leq i \leq K_0 - 1$, and Q_{K_0} is between t_{K_0} and s_0;*
2. *$Q_0, ..., Q_{K_0}$ all have length $d+2$;*
3. *$Q_0, ..., Q_{K_0}$ are vertex disjoint, and are internally vertex disjoint from $V(H_1)$.*

Recall that $K = \lfloor \log_2 \left(\frac{n}{\sqrt{\log n}} \right) \rfloor$ and $L = 2^{K+1} - 1$. Before getting to the proof of Lemma 4.2, we show the following claim.

Claim 4.3. *With high probability, for every vertex subset $U \subseteq V(G)$ with $|U| \geq n - 2L$, there is a vertex subset $U^* \subseteq U$, with $|U^*| \geq (1-\beta) \cdot n$, such that the induced subgraph $G_2[U^*]$ is a $(\beta n, 1/3\beta)$-expander.*

Proof. First, observe that, with high probability, for every $U, W \subseteq V(G)$ disjoint subsets with $|U| = |W| = \beta n$ there is an edge in G_2 between U and W. Indeed, the probability that there are such subsets with no edge between them is at most

$$\binom{n}{\beta n}^2 \cdot (1-p_2)^{\beta^2 n^2} \leq \left(\frac{e^2}{\beta^2} \cdot \exp(-\beta n p_2)\right)^{\beta n} \leq \left(\frac{\log^2 n}{\log\log n} \cdot \exp\left(-100 \log\log n\right)\right)^{\omega(1)} = o(1).$$

Now, assume that G_2 has the aforementioned property. We reiterate an argument from [5] and show that, in this case, for every such U there is $U^* \subseteq U$ with the desired properties.

For a given U, construct U^* as follows. Set $U_0 = U$. For $i \geq 0$, if $|U_i| \geq (1-\beta)n$ and there is $W_i \subseteq U_i$ with $|W_i| \leq \beta n$ and $|N_{G_2[U_i]}(W_i)| \leq \frac{1}{3\beta}|W_i|$, set $U_{i+1} = U_i \setminus W_i$. Otherwise, terminate the process with $U^* = U_i$.

Clearly, either the resulting $G_2[U^*]$ is a $(\beta n, 1/3\beta)$-expander, or $|U^*| < (1-\beta) \cdot n$. In fact, in the latter case, it must be that $(1-2\beta) \cdot n \leq |U^*| < (1-\beta) \cdot n$, since at most βn vertices are removed in every step of the process. Suppose that this is the case, and denote $W := U \setminus U^*$. Then $|N_{G_2}(W)| \leq \frac{1}{3\beta} \cdot |W| + |V(G) \setminus U| \leq \frac{2}{3}n$. We therefore have that W and $V(G) \setminus N_{G_2}(W)$ are subsets of size at least βn with no edges between them, a contradiction to our assumption. □

With Claim 4.3 at hand we are now able to prove Lemma 4.2 by appealing to Theorem 2.2.

Proof of Lemma 4.2. Assume that G_2 has the property in the assertion of Claim 4.3, and suppose that $Q_0, ..., Q_{i-1}$ have already been constructed. We attempt to construct Q_i.

Let $U := V(G) \setminus \left(V(H_1) \cup \bigcup_{j=0}^{i-1} V(Q_j)\right)$, so that

$$|U| \geq n - 2b \cdot t - 2^{K_0+1} - K_0 \cdot (d+2) \geq n - 2L,$$

and let $U^* \subseteq U$ be a subset of size at least $(1-\beta) \cdot n$ such that $G_2[U^*]$ is a $(\beta n, 1/3\beta)$-expander. Observe that $G_2[U^*]$ satisfies the conditions of Theorem 2.2 for $\Delta = \frac{1}{4\beta}, N = \frac{1}{50}n, k = \frac{1}{2}\beta n$.

Observe that, at this point, the edges of G_2 between $\{s_{i+1}, t_i\}$ (s_0 in the case $i = K_0$) and U^* have not been sampled yet. The probability that s_{i+1} does not have a neighbour in U^* is at most $(1-p_2)^{(1-\beta)n} = o({K_0}^{-1})$. Assume that there is such a neighbour, say u. By Theorem 2.2, $G_2[U^*]$ contains a complete $\frac{1}{5\beta}$-ary tree of depth d rooted in u. This tree has at least $\frac{\beta}{40} \cdot n$ leaves. The probability that none of these leaves is a neighbour of t_i in G_2 is at most

$$(1-p_2)^{\beta n/40} \leq \exp\left(-\frac{1}{40} \cdot \frac{50 \log n}{n \cdot \log\log n} \cdot \frac{2(\log\log n)^2}{\log n} \cdot n\right) = o({K_0}^{-1}).$$

Now, if indeed t_i has a neighbour among the tree's leaves, say w, the path from s_{i+1} to u, down the the tree to w, and from w to t_i is a path of length $d+2$ that intersects $V(C_{\text{short}}) \cup \bigcup_{j=0}^{K_0} V(C_j) \cup \bigcup_{j=0}^{i-1} V(Q_j)$ only in $\{s_{i+1}, t_i\}$.

Finally, for every i we showed that the probability that such a path Q_i does not exist is at most $o({K_0}^{-1})$, and therefore, by the union bound, a sequence $Q_0, ..., Q_{K_0}$ as required exists with high probability. □

Now $\left(\bigcup_{i=0}^{K_0} Q_i\right) \cup \{e_0, ..., e_{K_0}\}$ is a cycle, denote it by C^*. We have

$$\ell^* := |C^*| = (K_0+1) \cdot (d+3) = (1+o(1)) \cdot \log_2 n.$$

Step 3

Recall that $G_3 \sim G(n, p_3)$, with $p_3 = \frac{50 \log n}{n \cdot \log \log n}$. Let $e_{K_0+1}, ..., e_K, e^*$ be distinct edges of $C^* \setminus \{e_0, ..., e_{K_0}\}$, such that e^* is vertex disjoint from $e_{K_0+1}, ..., e_K$, and denote $e_i = \{s_i, t_i\}$ such that s_i is the predecessor of t_i on C^* for all i, according to an arbitrary orientation of C^*.

As a preparation for a proof that the construction in Step 3 is possible with high probability, observe that G_3 and G_2 are drawn from the same distribution, and therefore Claim 4.3 also holds for G_3. That is, we have that, with high probability, for every $U \subseteq V(G)$ with $|U| \geq n - 2L$, there is $U^* \subseteq U$, with $|U^*| \geq (1-\beta) \cdot n$, such that $G_3[U^*]$ is a $(\beta n, 1/3\beta)$-expander. In the proofs of the following two lemmas, we assume that indeed G_3 has this property.

Lemma 4.4. *With high probability G_3 contains paths $Q_{K_0+1}, ..., Q_K$ such that*

1. *Q_i is a path between s_i and t_i for $K_0 + 1 \leq i \leq K$;*
2. *Q_i has length $\ell_i + 1$ for $K_0 + 1 \leq i \leq K$;*
3. *$Q_{K_0+1}, ..., Q_K$ are internally vertex disjoint from each other and from $V(H_2)$.*

Proof. Suppose that $Q_{K_0+1}, ..., Q_{i-1}$ were found, and attempt to construct Q_i.

Here, as in Lemma 4.2, we will appeal to Theorem 2.2.

Let $U \coloneqq V(G) \setminus \left(V(H_2) \cup \bigcup_{j=0}^{i-1} V(Q_j)\right)$ and observe that

$$|U| \geq n - 2^{K+1} - K_0 \cdot (d+2) \geq n - 2L.$$

Let $U^* \subseteq U$ be a subset with at least $(1-\beta) \cdot n$ vertices such that $G_3[U^*]$ is a $(\beta n, 1/3\beta)$-expander.

As in the proof of Lemma 4.2, $G_3[U^*]$ satisfies the conditions of Theorem 2.2 for the same parameters $\Delta = \frac{1}{4\beta}, N = \frac{1}{50}n, k = \frac{1}{2}\beta n$. Recall that $d = \lfloor \log_{(5\beta)^{-1}}(n/200) \rfloor$, and let T be the tree consisting of two complete $\frac{1}{5\beta}$-ary trees of depth d, whose roots are connected by a path of length $\ell_i - 2d - 1$ (which is positive for $i > K_0$). By Theorem 2.2, U^* contains a copy of T (rooted at an arbitrary vertex).

Let $L_{s_i}, L_{t_i} \subseteq U^*$ be the sets of leaves of the embedding of T that correspond to the first and the second subtrees of T that are connected by a path. By the definition of T we have that $|L_{s_i}| = |L_{t_i}| \geq \frac{\beta}{40} n$. Observe that s_i and t_i each belong to at most one other path among $Q_{K_0+1}, ..., Q_{i-1}$. For $v \in \{s_i, t_i\}$ do the following. If $v \notin V(Q_j)$ for $K_0 < j \leq i-1$, then choose an arbitrary subset of L_v of size $\frac{1}{2}|L_v|$ and connect v to one of the vertices in the subset by an edge from $E(G_3)$, if there is a neighbour of v in the subset. If $v \in V(Q_j)$ for some $K_0 < j \leq i-1$, connect v to a vertex of L_v by a previously unexposed edge from $E(G_3)$, if such an edge exists. In both cases, at least $\frac{1}{2}|L_v| \geq \frac{\beta}{80} n$ edges are considered. Therefore, the probability that there is no edge between v and (the subset of) L_v is at most

$$(1-p_3)^{\beta n/80} \leq \exp\left(-\frac{1}{80} \cdot \frac{50 \log n}{n \cdot \log \log n} \cdot \frac{2(\log \log n)^2}{\log n} \cdot n\right) = \exp\left(-\frac{5}{4} \log \log n\right) = o(K^{-1}).$$

In the case that an edge is found, denote it by e_v.

Now, e_{s_i}, e_{t_i} along with the path of length $\ell_i - 1$ in T between the two leaves connected to s_i and t_i constitute a path between s_i and t_i of length $\ell_i + 1$, which is internally contained in U, and therefore, by the definition of U, is internally vertex disjoint from $V(H_2), Q_{K_0}, ..., Q_{i-1}$. Call this path Q_i.

The probability that there is $K_0 + 1 \leq i \leq K$ for which we did not manage to find a path Q_i in this way is at most the probability that G_3 does not have the property in the assertion of Claim 4.3, or s_i or t_i did not have a leaf neighbour in the embedding of T for some i, both of which are of order $o(1)$. □

For $K_0+1 \le i \le K$, denote by C_i the cycle $Q_i \cup \{e_i\}$.

Let $v_1, ..., v_{bt+1}$ be the vertices of C_{short} according to their order on the cycle, let $\sigma : \{0,1,...,t-1\} \times \{0,1,...,b-1\} \to [tb]$ be a bijection and denote $e_{i,j} = \{v_{\sigma(i,j)}, v_{\sigma(i,j)+1}\}$ and $e_{\text{short}} = \{v_1, v_{bt+1}\}$.

Lemma 4.5. *With high probability G_3 contains paths $\{P_{i,j} \mid 0 \le i \le t-1, 0 \le j \le b-1\}$ such that*

1. *$P_{i,j}$ is a path between $v_{\sigma(i,j)}$ and $v_{\sigma(i,j)+1}$, for all i and j;*
2. *$P_{i,j}$ has length $d+2+j \cdot b^i$, for all i and j;*
3. *$\{P_{i,j} \mid 0 \le i \le t-1, 0 \le j \le b-1\}$ are internally vertex disjoint from each other and from $V(H_2) \cup \bigcup_{i=K_0+1}^{K} V(C_i)$.*

Proof. The proof follows similar steps to the proofs of Lemma 4.2 and Lemma 4.4 by appealing to Theorem 2.2. Assume that $P_{\sigma^{-1}(1)}, ..., P_{\sigma^{-1}(k-1)}$ have already been found, and let $(i,j) = \sigma^{-1}(k)$. Let $U^* \subseteq U := V(G) \setminus \left(V(H_2) \cup \bigcup_{r=K_0+1}^{K} V(C_r) \cup \bigcup_{r=1}^{k-1} V(P_{\sigma^{-1}(r)})\right)$ be such that $|U^*| \ge (1-\beta) \cdot n$ and $G_3[U^*]$ is a $(\beta n, 1/3\beta)$-expander.

Let s_{k+1} be a neighbour of v_{k+1} from among the first $\frac{n}{\log\log n}$ vertices of U^*. The edges between v_{k+1} and U^* in G_3 have not been sampled yet, and the probability that no such neighbour exists is at most $(1-p_3)^{\frac{n}{\log\log n}} = o(1/tb)$.

As in the previous proofs, by Claim 4.3 and Theorem 2.2 we have that $G_3[U^*]$ contains a tree which consists of a complete $\frac{1}{5\beta}$-ary tree of depth d with a path of length $j \cdot b^i$ (this can possibly be 0, in which case the path is just a vertex) attached to its root, and such that the other end of the path is s_{k+1}. Let L_k be the set of leaves in the tree, so that $|L_k| \ge \frac{\beta}{40} n$. At most $\frac{n}{\log\log n}$ of the leaves were considered as neighbours of v_k in previous steps, and therefore the probability that v_k does not have a neighbour t_k from among the remaining leaves is at most $(1-p_3)^{\frac{\beta}{50}n} = o(1/tb)$. Now the path from v_{k+1}, through s_{k+1}, along the (jb^i)-path, down the tree to t_k and then to v_k, satisfies all the requirements to be $P_{i,j}$.

The probability that for some k one of the vertices s_{k+1}, t_k was not found is of order $o(1)$, and therefore, with high probability, this construction ends successfully. □

We remain with finding a path between e_{short} and e^*, which is done in the following lemma.

Lemma 4.6. *With high probability G_3 contains a path P^* of length $d+2$ between a vertex of e_{short} and a vertex of e^*, which is internally disjoint from $V(H_2) \cup \bigcup_{i=K_0+1}^{K} V(C_i) \cup \bigcup_{i,j} V(P_{i,j})$.*

Proof. As in previous constructions in this step, let $U = V(G) \setminus \left(V(H_2) \cup \bigcup_{i=K_0+1}^{K} V(C_i) \cup \bigcup_{i,j} V(P_{i,j})\right)$ and let U^* be a large subset spanning an expander. At least $\frac{1}{2}n$ vertices of U^* have not yet been considered as neighbours of v_{bt+1}, and with high probability at least one of them is, denote it by s. By Claim 4.3 and Theorem 2.2, $G_3[U^*]$ contains a complete $\frac{1}{5\beta}$-ary tree of depth d rooted in s. As none of the edges in G_3 of the vertices of e^* have been sampled yet, with high probability there is an edge between one of them and the tree's at least $\frac{\beta}{40}n$ leaves, which together with the path from the leaf to s and with $\{s, v_{bt+1}\}$ forms a path P^* satisfying the conditions. □

Step 4

Denote $X = V(G) \setminus V(H_3)$, and let $s_H \in e^*, t_H \in e_{\text{short}}$ be the vertices of e^*, e_{short} not already connected by P^*.

Claim 4.7. *With high probability there is a path P in G_4 between s_H and t_H, whose vertex set is $V(P) = X \cup \{s_H, t_H\}$.*

Proof. Consider the induced subgraph $G_4[X] \sim G(|X|, p_4)$. We have

$$\begin{aligned} |X| \cdot p_4 &\geq (n - 2L) \cdot \left(\frac{\log n + 10\sqrt{\log n}}{n} \right) \\ &\geq \log n \cdot \left(1 - \frac{4}{\sqrt{\log n}}\right) \cdot \left(1 + \frac{10}{\sqrt{\log n}}\right) \\ &= \log n + \omega(\log \log n) \\ &= \log |X| + \omega(\log \log |X|) \end{aligned}$$

Therefore, by Lemma 2.1, there is a set $S \subseteq X$ with $|S| = \frac{1}{4}|X| \geq \frac{1}{5}n$, and for every $s \in S$ there is a subset $T_s \subseteq X$ with $|T_s| \geq \frac{1}{4}|X| \geq \frac{1}{5}n$, such that there is a Hamilton path in $G_4[X]$ between s and t for every $t \in T_s$.

The set $E_{G_4}(s_H, X)$ has not yet been sampled. The probability that s_H has no neighbour in S is at most $(1 - p_4)^{n/5} = o(1)$. Assume that there is one, and denote it by s. Similarly, the probability that t_H has no neighbour in T_s is at most $(1 - p_4)^{n/5} = o(1)$, denote such a neighbour by t. Now the Hamilton path in $G_4[X]$ between s and t, along with the edges $\{s_H, s\}, \{t_H, t\}$, constitute a path P with $V(P) = X \cup \{s_H, t_H\}$, as desired. □

Denote the obtained Hamilton cycle $H_3 \cup P \setminus (\{e_0, ..., e_K, e^*, e_{\text{short}}\} \cup \{e_{i,j}\}_{i,j})$ by C_H.

Step 5

Let $m \coloneqq \lfloor n \cdot 2^{-K} \rfloor$.

Lemma 4.8. *With high probability G_5 contains edges $f_3, ..., f_m$, such that the following hold for every i.*

1. *There is $\ell_i^* \in \left[i \cdot 2^K - n^{0.9}, i \cdot 2^K + n^{0.9}\right]$ such that f_i is an ℓ_i*-shortcut with respect to C_H;*
2. *The $(\ell_i^* + 1)$-path on C_H that connects the vertices of f_i contains $V\left(C^* \cup \bigcup_{i=0}^{K} C_i\right)$.*

Proof. For every $3 \leq i \leq m$ there is a set of at least $\frac{1}{5} \cdot n^{1.8}$ potential edges that satisfy the conditions. The probability that there is $3 \leq i \leq m$ for which none of these edges appears in G_5 is at most

$$m \cdot (1 - p_5)^{n^{1.8}/5} = o(1).$$

□

Lemma 4.9. *With high probability G_5 contains an $(\ell - 2)$-shortcut with respect to C_H for every $\ell \in [3, (d + b) \cdot t]$.*

Proof. For a given $\ell \in [3, (d + b) \cdot t]$, the probability that such an $(\ell - 2)$-shortcut does not exist is $(1 - p_5)^n = o(1/\log n)$, and by the union bound we obtain the lemma, as $(d + b)t = o(\log n)$. □

5 Proof of Theorem 1

The following lemma, referring to the subgraph $H_5 \subseteq G$ described in Section 3 and whose construction is shown to be possible with high probability in Section 4, completes the proof of Theorem 1.

Lemma 5.1. *The subgraph H_5 is pancyclic.*

Proof. Let $\ell \in [3, n]$. We show that H_5 contains a cycle of length ℓ. We divide the proof into cases based on a subinterval of $[3, n]$ that ℓ resides in. The subintervals are covering $[3, n]$ but not necessarily disjoint, so ℓ may be covered by more than one subinterval.

- If $\ell \in [3, (d+b+1) \cdot t]$, then g_ℓ is an $(\ell-2)$-shortcut with respect to C_H, so that g_ℓ and its accompanying $(\ell-1)$-path form an ℓ-cycle.

- If $\ell \in [(d+b+1) \cdot t + 1, (d+b+1) \cdot t + b^t]$, let $k = \ell - (d+b+1) \cdot t - 1$, so $0 \leq k \leq b^t - 1$ can be encoded in base b using t digits. Let $(k_{t-1}, k_{t-2}, ..., k_1, k_0)$ be its encoding, that is, $0 \leq k_i \leq b - 1$ for all i, and $k = \sum_{i=0}^{t-1} k_i b^i$. Then
$$\{e_{\text{short}}\} \cup \bigcup_{j=k_i} P_{i,j} \cup \bigcup_{j \neq k_i} \{e_{i,j}\}$$
is a cycle of length ℓ in H_5. Indeed, it is a cycle, since it is the result of replacing a subset of the edges of C_{short} with internally disjoint paths, and its length is
$$1 + (b-1) \cdot t + \sum_{i=0}^{t-1} e(P_{i,k_1}) = 1 + (b-1) \cdot t + \sum_{i=0}^{t-1} \left(d + 2 + k_i b^i\right) = 1 + (d+b+1) \cdot t + k = \ell.$$

- If $\ell \in [\ell^*, \ell^* + L]$, let $k = \ell - \ell^*$, so $0 \leq k \leq 2^{K+1} - 1$ can be encoded by $K+1$ binary digits, say $k = \sum_{i=0}^{K} k_i 2^i$, where $k_i \in \{0, 1\}$. Then
$$\left(C^* \cup \bigcup_{i:k_i=1} C_i\right) \setminus \{e_i \mid k_i = 1\}$$
is a cycle in H_5 with length ℓ. It is a cycle because it is the result of replacing edges of C^* with internally disjoint paths. The length is indeed
$$\ell^* + \sum_{i=0}^{K} k_i \cdot (e(C_i) - 1) = \ell^* + \sum_{i=0}^{K} k_i 2^i = \ell^* + k = \ell.$$

- If $\ell \in \left[\left(\frac{1}{2}i - \frac{4}{5}\right) \cdot L, \left(\frac{1}{2}i - \frac{1}{5}\right) \cdot L\right]$, where $3 \leq i \leq m$, then in particular $\ell \in [\ell_i^* + 2 - L, \ell_i^* + 2]$. Similarly to the previous case, let $\ell_i^* + 2 - \ell = k = \sum_{i=0}^{K} k_i 2^i$, where $k_i \in \{0, 1\}$. Denote by C_i^* the cycle of length $\ell_i^* + 2$ comprised of f_i and its accompanying $(\ell_i^* + 1)$-path. Then
$$\left(C_i^* \setminus \bigcup_{i:k_i=1} C_i\right) \cup \{e_i \mid k_i = 1\}$$
is a cycle of length $\ell^* + 2 - k = \ell$.

- If $\ell \in [n-L, n]$ then for $n - \ell = k = \sum_{i=0}^{K} k_i 2^i,\ k_i \in \{0,1\}$, we get a cycle

$$\left(C_H \setminus \bigcup_{i:k_i=1} C_i \right) \cup \{e_i \mid k_i = 1\}$$

  of length $n - k = \ell$.

Observe that

$$(d+b+1) \cdot t + b^t \geq b^{\log_b \log n} = \log n \geq \ell^* \ ;$$
$$\ell^* + L \geq \left(\frac{1}{2} \cdot 3 - \frac{4}{5}\right) \cdot L \ ;$$
$$\left(\frac{1}{2} \cdot i - \frac{1}{5}\right) \cdot L \geq \left(\frac{1}{2} \cdot (i+1) - \frac{4}{5}\right) \cdot L;$$
$$\left(\frac{1}{2} \cdot m - \frac{1}{5}\right) \cdot L \geq \frac{1}{2}(n \cdot 2^{-K} - 1) \cdot (2^{K+1} - 1) - \frac{1}{5}L \geq n - \frac{1}{2}L - \frac{1}{5}L - O(\sqrt{\log n}) \geq n - L \ ;$$

and therefore the subintervals indeed cover $[3, n]$, and so H_5 is pancyclic. □

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