Every graph contains a linearly sized induced subgraph with all degrees odd

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May 26, 2022

Abstract

We prove that every graph G on n vertices with no isolated vertices contains an induced subgraph of size at least $n/10000$ with all degrees odd. This solves an old and well-known conjecture in graph theory.

1 Introduction

We start with recalling a classical theorem of Gallai (see [2], Problem 5.17 for a proof):

Theorem 1 (Gallai's Theorem)**.** *Let G be any graph.*

1. *There exists a partition $V(G) = V_1 \cup V_2$ such that both graphs $G[V_1]$ and $G[V_2]$ have all their degrees even.*
2. *There exists a partition $V(G) = V_o \cup V_e$ such that the graph $G[V_e]$ has all its degrees even, and the graph $G[V_o]$ has all its degrees odd.*

It follows immediately from 1. that every graph G has an induced subgraph of size at least $|V(G)|/2$ with all its degrees even. This is easily seen to be tight by taking G to be a path.

It is natural to ask whether we can derive analogous results for induced subgraphs with all degrees odd. Some caution is required here — an isolated vertex can never be a part of a subgraph with all degrees odd. Thus we restrict our attention to graphs of positive minimum degree.

Let us introduce a relevant notation: given a graph $G = (V, E)$, we define

$$f_o(G) = \max\{|V_0| : G[V_0] \text{ has all degrees odd.}\},$$

and set

$$f_o(n) = \min\{f_o(G) \mid G \text{ is a graph on } n \text{ vertices with } \delta(G) \geq 1\}.$$

The following is a very well known conjecture, aptly described by Caro already more than a quarter century ago [1] as "part of the graph theory folklore":

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Conjecture 2. *There exists a constant $c > 0$ such that for every $n \in \mathbb{N}$ we have $f_o(n) \geq cn$.*

Caro himself proved [1] that $f_o(n) = \Omega(\sqrt{n})$, resolving a question of Alon who asked whether $f_o(n)$ is polynomial in n. The current best bound, due to Scott [3], is $f_o(n) = \Omega(n/\log n)$. There have been numerous variants and partial results about the conjecture, we will not cover them here.

Our main result establishes Conjecture 2 with $c = 0.0001$.

Theorem 3. *Every graph G on n vertices with $\delta(G) \geq 1$ satisfies: $f_o(G) \geq cn$ for $c = \frac{1}{10000}$.*

With some effort/more accurate calculations the constant can be improved but probably to a value which is still quite far from the optimal one; we decided not to invest a substantial effort in its optimization and just chose some constants that work.

A relevant parameter was studied by Scott [4]: given a graph G with no isolated vertices, let $t(G)$ be the minimal k for which there exists a vertex cover of G with k sets, each spanning an induced graph with all degrees odd. Letting

$$t(n) = \min\{t(G) \mid G \text{ is a graph on } n \text{ vertices with } \delta(G) \geq 1\},$$

Scott proved (Theorem 4 in [4]) that

$$\Omega(\log n) = t(n) = O(\log^2 n).$$

As indicated by Scott already, showing that $f_o(n)$ is linear in n proves the following:

Corollary 4. $t(n) = \Theta(\log n)$.

For completeness, we outline its proof here.

Proof. Let G be a graph on n vertices with $\delta(G) \geq 1$. By a repeated use of Theorem 3, we can find disjoint sets $V_1, \dots, V_t$ such that:

1. $V_i \subseteq V(G) \setminus \left(\bigcup_{j=1}^{i-1} V_j\right)$, and
2. all the degrees in $G[V_i]$ are odd, and
3. letting n_i be the number of non-isolated vertices in $G\left[V(G) \setminus \left(\bigcup_{j=1}^{i-1} V_j\right)\right]$, we have that $|V_i| \geq n_i/10000$.

We continue the above process as long as $n_i > 0$. Clearly, the process terminates after $t = O(\log n)$ steps. Moreover, letting $U = V(G) \setminus \left(\bigcup_{i=1}^{t} V_i\right)$, we have that U is an independent set in G. Finally, as shown in the proof of Theorem 4 in [4], every independent set in such G can be covered by $O(\log n)$ odd graphs. This proves that $t(n) = O(\log n)$.

To show a lower bound, we can use the following example due to Scott [4]: assume n is of the form $n = s + \binom{s}{2}$. Let the vertex set of G be composed of two disjoint sets: A of size s associated with $[s]$, and B of size $\binom{s}{2}$ associated with $\binom{[s]}{2}$. The graph G is bipartite with the edges defined as follows: a pair $(i, j) \in B$ is connected to both $i, j \in A$. Observe that if $U \subset V(G)$ spans a subgraph of G with all degrees odd and containing $(i, j) \in B$, then U contains exactly one of $i, j \in A$. Hence if $\mathcal{U} = (U_1, \dots, U_t)$ forms a cover of $V(G)$ with subsets spanning odd subgraphs, then $\mathcal{U}$ separates the set A, and the minimum size of such a separating family is easily shown to be asymptotic to $\log_2 s = \Omega(\log_2 n)$. □

2 Auxiliary results

The following lemma appears as Theorem 2.1 in [1]. For the convenience of the reader we provide its simple proof.

Lemma 2.1. *For every graph G we have that $f_o(G) \geq \frac{\Delta(G)}{2}$.*

Proof. Let $v \in V(G)$ be a vertex with $d_G(v) = \Delta(G)$, and let $U \subseteq N_G(v)$ be an odd subset of size $|U| \geq \Delta(G) - 1$. Apply Gallai's Theorem to $G[U]$ to obtain a partition $U = V_e \cup V_o$, and observe that V_o must be of an even size (so in particular, $|V_e|$ is odd). If $|V_o| \geq \Delta(G)/2$, then we are done. Otherwise, define $V^* = \{v\} \cup V_e$, and observe that $G[V^*]$ has all its degrees odd and is of size at least $\Delta(G)/2$ as required. □

The next lemma appears as Theorem 1 in [3], and again, for the sake of completeness, we give its proof here.

Lemma 2.2. *For every graph G with $\delta(G) \geq 1$ we have that $f_o(G) \geq \frac{\alpha(G)}{2}$.*

Proof. Let $I \subseteq V(G)$ be a largest independent set in G. Since $\delta(G) \geq 1$, every $u \in I$ has at least one neighbor in $V(G) \setminus I$.

Let $D \subseteq V(G) \setminus I$ be a smallest subset dominating all vertices in I. Observe that by the minimality of D for every $w \in D$ there exists some $u_w \in I$ such that $N_G(u_w) \cap D = \{w\}$; let $I_D := \{u_w \mid w \in D\}$.

Let $D' \subseteq D$ be a subset of D chosen uniformly at random, and let $I_0 \subseteq I \setminus I_D$ be a subset consisting of all elements $u \in I \setminus I_D$ that have an odd degree into D'.

Let
$$I_1 = \{u_w \in I_D \mid w \in D' \text{ and } w \text{ has even degree in } D' \cup I_0\},$$
and observe that $G[I_0 \cup I_1 \cup D']$ is an induced subgraph of G with all its degrees odd.

Finally, since $\Pr[u \in I_0] = \frac{1}{2}$, by linearity of expectation we have that
$$\mathbb{E}[|I_0 \cup I_1 \cup D'|] = \mathbb{E}[|I_0|] + \mathbb{E}[|I_1|] + \mathbb{E}[|D'|] \geq \frac{|I| - |D|}{2} + \frac{|D|}{2} = \frac{\alpha(G)}{2}.$$
Hence there exists a set D' for which
$$|I_0| + |I_1| + |D'| \geq \frac{\alpha(G)}{2},$$
as desired. □

Next we argue that if G contains a semi-induced matching with "nice" expansion properties, then it also has a large induced subgraph with all degrees odd.

Lemma 2.3. *Let G be a graph and let M be a matching in G with parts U and W, where every vertex $w \in W$ has only one neighbor in G between the vertices covered by M. Assume that $|N_G(U) \setminus (W \cup N_G(W))| \geq k$. Then $f_o(G) \geq \frac{k}{4}$.*

Proof. Let $X = N_G(U) \setminus (W \cup N_G(W))$ and recall that $|X| \geq k$. Let U_0 be a random subset of U chosen according to the uniform distribution, and let
$$X_0 = \{x \in X : d_G(x, U_0) \text{ is odd}\}.$$

Since $\mathbb{E}[|X_0|] = |X|/2$, it follows that there exists an outcome $U_0 \subseteq U$ for which $|X_0| \geq |X|/2 \geq k/2$. Fix such U_0.

Next, apply Gallai's theorem to $G[X_0]$ to find a subset $X_1 \subseteq X_0$ with $|X_1| \geq |X_0|/2 \geq k/4$ and all degrees in $G[X_1]$ even. Finally, for every $u \in U_0$ with $d_G(u, X_1 \cup U_0)$ even, add an edge of M containing u. Clearly, the obtained graph G_1 has size at least $|X_1| \geq |X|/4 \geq k/4$, and all its degrees are odd. This completes the proof. □

The following simple lemma will be used several times below.

Lemma 2.4. *Let G be a bipartite graph with parts A, B such that $d(b) > 0$ for every $b \in B$. Assume that $|A| \leq \alpha|B|$ for some $0 < \alpha \leq 1$. Then there is an edge $ab \in E(G)$ with $d(a) \geq \frac{d(b)}{\alpha}$.*

Proof. We have:

$$\sum_{ab \in E(G)} \left(\frac{1}{d(b)} - \frac{1}{d(a)} \right) = \sum_{b \in B} d(b) \cdot \frac{1}{d(b)} - \sum_{a \in A, d(a) > 0} d(a) \cdot \frac{1}{d(a)} \geq |B| - |A| \geq (1 - \alpha)|B| \,.$$

Hence there is $b \in B$ with

$$\sum_{a \in N_G(b)} \left(\frac{1}{d(b)} - \frac{1}{d(a)} \right) \geq 1 - \alpha \,.$$

It follows that there is a neighbor a of b for which $\frac{1}{d(b)} - \frac{1}{d(a)} \geq (1 - \alpha)\frac{1}{d(b)}$, implying $d(a) \geq \frac{d(b)}{\alpha}$ as desired. □

For a graph $G = (V, E)$ and $\beta > 0$, define

$$L = L(G; \beta) = \{v \in V : \exists u \in V, uv \in E(G), |N(u) \setminus N(v)| \geq \beta |N(u) \cup N(v)|\} \,.$$

We say that for $v \in L$, an edge uv as above *witnesses* $v \in L$.

Set

$$\begin{aligned} \beta &= \frac{1}{20}\,, \\ \gamma &= \frac{1}{14}\,, \\ \epsilon &= \frac{1}{10}\,. \end{aligned}$$

The next lemma is a key part in the proof of our main theorem. Roughly speaking, the lemma asserts that if $L(G; \beta)$ is small, then G contains a large induced subgraph $G[U]$ with a vertex of large degree in every connected component, allowing to find inside U a large odd subgraph, using previously presented tools. We did not really pursue the goal of optimizing the constants in its statement.

Lemma 2.5. *Let $G = (V, E)$ be a graph on $|V| = n$ vertices with $\delta(G) > 0$ and $|L(G; \beta)| \leq \gamma n$. Then $f_o(G) \geq n/61$.*

Proof. Define

$$\begin{aligned} V_1 &= \{v \in V \setminus L : d(v, L) \geq \epsilon d(v)\}\,, \\ V_2 &= V \setminus (V_1 \cup L). \end{aligned}$$

Suppose first that $|V_1| \geq 12|L|$. Observe that for all $v \in V_1$ we have that $d(v, L) \geq \epsilon d(v)$ and this quantity is positive by the assumption $\delta(G) > 0$. By Lemma 2.4 there exists $uv \in E(G)$ with $v \in V_1$ and $u \in L$ such that $d(u, V_1) \geq 12d(v, L) \geq 12\epsilon d(v) = 1.2d(v)$ (in particular, we have $d(v) \leq \frac{5}{6}d(u)$). Therefore we have that

$$|N(u) \setminus N(v)| \geq d(u) - d(v) = (d(u) - \frac{12}{11}d(v)) + (d(v) - \frac{10}{11}d(v)) \geq \frac{1}{11}(d(u) + d(v)) > \beta|N(u) \cup N(v)|,$$

so in particular v should also be in L with uv witnessing it — a contradiction. We conclude that

$$|V_1| < 12|L| \leq 12\gamma n\,,$$

and therefore $|V_2| \geq (1 - \gamma - 12\gamma)n = \frac{n}{14}$.

Let $v \notin L$. Take an edge $uv \in E(G)$. Then

$$\max\{1, d(u) - d(v)\} \leq |N(u) \setminus N(v)| \leq \beta|N(u) \cup N(v)| \leq \beta(d(u) + d(v))\,,$$

yielding:

$$d(u) \leq \frac{1+\beta}{1-\beta}d(v)\,,$$

and

$$\beta\left(\frac{1+\beta}{1-\beta} + 1\right) d(v) \geq 1\,.$$

This shows that every vertex $v \in V \setminus L$ has degree $d(v) \geq \lceil \frac{1-\beta}{2\beta} \rceil = 10$.

Let now $uv \in E(G)$ with $u, v \notin L$. Then

$$|N(u) \setminus N(v)|, |N(v) \setminus N(u)| \leq \beta|N(u) \cup N(v)|\,,$$

and hence

$$|N(u) \cap N(v)| \geq (1 - 2\beta)|N(u) \cup N(v)|. \tag{1}$$

Since

$$|N(u) \cap N(v)| \leq \min\{d(u), d(v)\} \text{ and } |N(u) \cup N(v)| \geq \max\{d(u), d(v)\},$$

it follows that

$$(1 - 2\beta)d(u) \leq d(v) \leq \frac{d(u)}{1 - 2\beta} < (1 + 3\beta)d(u). \tag{2}$$

Now, for all $v \notin L$ define $R(v) = (\{v\} \cup N(v)) \setminus L$. Notice that as $d(v) \geq 10$ we have $|R(v)| \geq (1 - \epsilon)d(v) + 1 \geq 10$ for $v \in V_2$. Suppose that $R(u) \cap R(v) \neq \emptyset$ for some $u \neq v$ where $v \in V_2$ (note that it might be that $u \in V_1$). Then for $w \in R(v) \cap R(u)$, by (1) we have

$$|N(u) \cap N(w)| \geq (1 - 2\beta)|N(u) \cup N(w)| \text{ and } |N(v) \cap N(w)| \geq (1 - 2\beta)|N(v) \cup N(w)|,$$

which implies, by the identity $|A \triangle B| = |A \cup B| - |A \cap B|$, that

$$|N(u) \triangle N(w)| \leq \frac{2\beta}{1 - 2\beta}|N(u) \cap N(w)| < 3\beta|N(u) \cap N(w)|$$

and

$$|N(v) \triangle N(w)| \leq \frac{2\beta}{1 - 2\beta}|N(v) \cap N(w)| < 3\beta|N(v) \cap N(w)|\,.$$

Therefore, we have

$$\begin{aligned}|N(u)\cap N(v)| &\geq |N(u)\cap N(v)\cap N(w)|\\ &\geq |N(u)\cup N(v)| - |N(u)\triangle N(w)| - |N(v)\triangle N(w)|\\ &> |N(u)\cup N(v)| - \frac{4\beta}{1-2\beta}\max\{d(u),d(v)\}\\ &\geq \left(1-\frac{4\beta}{1-2\beta}\right)|N(u)\cup N(v)| \geq (1-6\beta)|N(u)\cup N(v)|. \end{aligned} \tag{3}$$

Since $v \in V_2$ we conclude that

$$\begin{aligned}|R(u)\cap R(v)| &\geq |N(u)\cap N(v)| - \epsilon d(v)\\ &> \left(1-\frac{4\beta}{1-2\beta}-\epsilon\right)|N(u)\cup N(v)|\\ &\geq \left(1-\frac{4\beta}{1-2\beta}-\epsilon\right)(|N(u)\cup R(v)|-1)\\ &\geq \frac{9}{10}\left(1-\frac{4\beta}{1-2\beta}-\epsilon\right)|N(u)\cup R(v)|\\ &\geq (1-8\beta)\,|N(u)\cup R(v)|\,,\end{aligned}$$

where the second to last inequality follows since $|N(u)| \geq 10$.

Next, let $R_1,\dots,R_k$ be a maximal by inclusion collection of non-intersecting sets $R(v_i), v_i \in V_2$. Due to maximality, every $v \in V_2$ has its set $R(v)$ intersecting with at least one of the R_i's; moreover, the above argument shows that it can intersect only one such set. Define now

$$U_i = \{v \notin L : R(v)\cap R_i \neq \emptyset\}\,.$$

Trivially we have $R_i \subseteq U_i$. Also, $V_2 \subseteq \bigcup_{i=1}^{k} U_i$ due to the maximality of the family $R_1,\dots,R_k$.

We wish to show that all U_i are disjoint and that there are no edges in between different U_i's. (This will add to the above stated fact that the family of U_i's forms a cover of V_2.)

To prove the latter claim, suppose that there exists an edge $w_1w_2 \in E(G)$ for some $w_1 \in U_i, w_2 \in U_j$, $1 \leq i \neq j \leq k$. We will obtain a contradiction by showing that $R_i \cap R_j \neq \emptyset$. Since both $w_1, w_2 \notin L$, by (1) and (2) we conclude that

$$|N(w_1)\cap N(w_2)| \geq (1-2\beta)|N(w_1)\cup N(w_2)| \text{ and } |N(w_1)| \in (1\pm 3\beta)|N(w_2)|\,.$$

Moreover, by (3) we have

$$|N(w_1)\cap N(v_i)| > (1-6\beta)|N(w_1)\cup N(v_i)|, \text{ and } |N(w_2)\cap N(v_j)| \geq (1-6\beta)|N(w_2)\cup N(v_j)|.$$

Since $v_i, v_j \in V_2$, the above inequalities imply that

$$|N(w_1)\cap R_i| > (1-6\beta-\epsilon)|N(w_1)\cup R_i|, \text{ and } |N(w_2)\cap R_j| > (1-6\beta-\epsilon)|N(w_2)\cup R_j|.$$

It follows that

$$|N(w_1)\cap R_i| > (1-6\beta-\epsilon)|N(w_1)|$$

and

$$|N(w_2)\cap R_j| > (1-6\beta-\epsilon)|N(w_2)|\,,$$

and recalling that

$$|N(w_1) \cap N(w_2)| \geq (1-2\beta)|N(w_1) \cup N(w_2)|\,,$$

we conclude that $R_i \cap R_j \neq \emptyset$ — a contradiction. In a similar way we can show that $U_i \cap U_j = \emptyset$.

Next, suppose that $|U_i| \geq (1+19\beta)|R_i|$. Then by looking at the auxiliary bipartite graph between R_i and U_i ($v \in R_i$, $u \in U_i$ are connected by an edge if $uv \in E(G)$) and by applying Lemma 2.4 to this graph we derive that there are $v \in R_i$, $u \in U_i$ with $d(v) \geq (1+19\beta)d(u, R_i)$. Since $uv \in E(G)$ and both $u, v \notin L$, it follows that

$$d(v) < (1+3\beta)d(u)\,.$$

Moreover, since $u \in U_i$ we have:

$$d(u, R_i) \geq (1-8\beta)d(u)\,.$$

All in all, since $d(v) \geq (1+19\beta)d(u, R_i)$ we conclude that

$$(1+3\beta)d(u) > d(v) \geq (1+19\beta)d(v, R_i) > (1+19\beta)(1-8\beta)d(u) > (1+3\beta)d(u)\,,$$

a contradiction.

Therefore, we can assume that $|U_i| \leq (1+19\beta)|R_i|$ for all $1 \leq i \leq k$. Looking at the induced subgraph $G[U_i]$, we note that it has vertex v_i of degree $|R_i| - 1 \geq \frac{9|R_i|}{10} \geq \frac{9}{10(1+19\beta)}|U_i|$. By applying Lemma 2.1 to $G[U_i]$ we find an induced odd subgraph O_i of $G[U_i]$ of size at least $\frac{9}{20(1+19\beta)}|U_i| = \frac{9|U_i|}{39}$.

Finally, since all U_i's are disjoint, there are no edges between any two such U_i's and since $V_2 \subseteq \bigcup U_i$, we conclude that $O = \bigcup_{i=1}^{k} O_i$ is an induced odd subgraph of size at least $\frac{9|V_2|}{39} > n/61$. This completes the proof. □

3 Proof of Theorem 3

The main plan is as follows. We will grow edge by edge a matching M with sides U, W so that every $w \in W$ has exactly one neighbor between the vertices covered by M (which is of course its mate u in the matching). Moreover, the set U has "many" neighbors outside of M not connected to W. If the set of such neighbors is substantially large, then we will be able to apply Lemma 2.3 to get a large induced subgraph with all degrees odd. Otherwise we will show that either there exists a large subset of vertices V' such that $\delta(G[V']) \geq 1$ with small $L(G[V']; 1/20)$ (and then we are done by Lemma 2.5), or that we can extend the matching while enlarging substantially the set of neighbors outside M not connected to W. The details are given below.

We start with $M_0 = \emptyset$, and given M_i, $i \geq 0$, we define

$$\begin{aligned} X_i &= N(U_i) \setminus (W_i \cup N(W_i))\,, \\ V_i &= V \setminus N(U_i \cup W_i)\,. \end{aligned}$$

In particular, we initially have $X_0 = \emptyset$ and $V_0 = V$. We will run our process until the first time we have $|V_i| < n/2$ (in particular, we may assume throughout the process that $|V_i| \geq n/2$). Now, fix $\beta = 1/20$ and $\gamma = 1/14$ (same parameters as set before Lemma 2.5). Our goal is to show that $f_o(G) \geq \frac{n}{T}$, where $T = 10000$. We will maintain $|X_i| \geq \frac{|V \setminus V_i|}{40}$. If at some point we reach $|X_i| \geq \frac{4n}{T}$ then we are done by Lemma 2.3. Hence we assume $|X_i| \leq \frac{4n}{T} = \frac{n}{2500}$. Moreover, if $G[V_i]$ has at least $2n/T$ isolated vertices, then since this set induces an independent set in G, by Lemma 2.2 we are done as well. Therefore, letting $V_i' \subseteq V_i$ be the set of all non-isolated vertices in $G[V_i]$, since $|V_i| \geq n/2$ we obtain that $|V_i'| \geq (1-4/T)|V_i| \geq |V_i|/2$. We can further assume $|L(G[V_i'];\beta)| \geq \gamma|V_i'| \geq \gamma n/4$, as otherwise by Lemma 2.5 we obtain an odd subgraph of size at least $|V_i'|/61 \geq n/244$. Our goal now is to show that under these assumptions we can add an edge to M_i while maintaining $|X_{i+1}| \geq \frac{|V \setminus V_{i+1}|}{40}$.

Consider first the case where every $v \in L := L(G[V_i'];\beta)$ satisfies $d(v,X_i) \geq d(v,V_i)/40$. By Lemma 2.4 applied to the bipartite graph between X_i and L, using the fact that

$$|X_i| \leq \frac{4n}{T} = \frac{n}{2500} \leq \frac{|L|}{44},$$

we derive that there is an edge xv with $x \in X_i$ and $v \in L$ and $d(x,L) \geq 44d(v,X_i) \geq 1.1d(v,V_i) > 0$. Then we can define M_{i+1} by adding xv to M_i and setting $U_{i+1} := U_i \cup \{x\}$ and $W_{i+1} := W_i \cup \{v\}$. By doing so we obtain that

$$\begin{aligned}
|X_{i+1}| &= |N(U_{i+1}) \setminus (W_{i+1} \cup N(W_{i+1})| \\
&\geq |N(U_i) \setminus (W_i \cup N(W_i))| + |N(x,V_i)| - |N(v,X_i)| - |N(v,V_i)| \\
&= |X_i| + d(x,V_i) - d(v,X_i) - d(v,V_i) \\
&\geq |X_i| + d(x,V_i)\left(1 - \frac{1}{44} - \frac{10}{11}\right) \\
&> |X_i| + \frac{3d(x,V_i)}{44}.
\end{aligned}$$

Moreover, since we clearly have that

$$|V_{i+1}| \geq |V_i| - d(x,V_i) - d(v,V_i) \geq |V_i| - \frac{21d(x,V_i)}{11},$$

it follows that at least $\frac{3/44}{21/11} > \frac{1}{40}$ proportion of the vertices deleted from V_i go to X_{i+1}.

In the complementary case there exists a vertex $v \in L$ with $d(v,X_i) \leq d(v,V_i)/40$. Let uv be an edge in $G[V_i']$ witnessing $v \in L$ (that is, $|N(u,V_i) \setminus N(v,V_i)| \geq \beta|N(u,V_i) \cup N(v,V_i)|$). Then we can define M_{i+1} by adding uv to M_i, and set $U_{i+1} := U_i \cup \{u\}$ and $W_{i+1} := W_i \cup \{v\}$. In this case we have:

$$\begin{aligned}
|X_{i+1}| &= |N(U_{i+1}) \setminus (W_{i+1} \cup N(W_{i+1}))| \\
&\geq |N(U_i) \setminus (W_i \cup N(W_i))| + |N(u,V_i) \setminus N(v,V_i)| - |N(v,X_i)| \\
&= |X_i| + |N(u,V_i) \setminus N(v,V_i)| - |N(v,X_i)| \\
&\geq |X_i| + \beta|N(u,V_i) \cup N(v,V_i)| - |N(v,X_i)| \\
&\geq |X_i| + (\beta - \frac{1}{40})|N(u,V_i) \cup N(v,V_i)|.
\end{aligned}$$

Moreover, since we have $|V_{i+1}| \geq |V_i| - |N(u,V_i) \cup N(v,V_i)|$, at least $\beta - \frac{1}{40} = \frac{1}{40}$ proportion of the vertices deleted from V_i go to X_{i+1}.

All in all, in each step, either we find an odd subgraph of size at least $\frac{n}{T}$ (in case that we have "many" isolated vertices, or that $|X_i| \geq \frac{4n}{T}$, or that $L(G[V_i'];\beta)$ is "large"), or we can keep X_i of size at least $\frac{|V \setminus V_i|}{40}$. In particular, if the latter case holds until $|V_i| < n/2$, we obtain that $|X_i| \geq \frac{n}{80}$ and we are done by Lemma 2.3. This completes the proof. □

Acknowledgement. We would like to thank Alex Scott for his remarks, and for pointing out a serious flaw in the previous version.

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