

The number of F -matchings in almost every tree is a zero residue

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Abstract

For graphs F and G an F -matching in G is a subgraph of G consisting of pairwise vertex disjoint copies of F . The number of F -matchings in G is denoted by $s(F, G)$. We show that for every fixed positive integer m and every fixed tree F , the probability that $s(F, \mathcal{T}_n) \equiv 0 \pmod{m}$, where \mathcal{T}_n is a random labeled tree with n vertices, tends to one exponentially fast as n grows to infinity. A similar result is proven for induced F -matchings. As a very special special case this implies that the number of independent sets in a random labeled tree is almost surely a zero residue. A recent result of Wagner shows that this is the case for random unlabeled trees as well.

1 Introduction

The number of independent sets in graphs is an important counting parameter. It is particularly well-studied for trees and tree-like structures. Proding and Tichy showed

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in [12] that the star and the path maximize and minimize, respectively, the number of independent sets among all trees of a given size. Part of the interest in this graph invariant stems from the fact that the number of independent sets plays a role in statistical physics as well as in mathematical chemistry, where it is known as the *Merrifield-Simmons index* [11]. A problem that arises in this context is the inverse problem: determine a graph within a given class of graphs (such as the class of all trees) with a given number of independent sets. It is an open conjecture [9] (see also [8]) that all but finitely many positive integers can be represented as the number of independent sets of some tree. Recently Wagner [14] published a surprising result that may partially explain why the inverse problem for independent sets in trees is difficult. He showed that for every positive integer m , the number of independent sets in a random tree with n vertices is zero modulo m with probability exponentially close to one. Wagner's proof does not give an intuitive explanation of the aforementioned fact. In this paper we give a probabilistic proof for the analogous result in labeled trees. Our proof is intuitive and simple, thus allowing us to generalize the result significantly. We refer the reader to [14] or [7] for further motivation and for a recent survey of previous results regarding the number of independent sets in trees.

Another graph parameter popular in statistical physics and in mathematical chemistry is the *Hosoya index* which is the number of matchings in the graph. While the inverse problem for the number of matchings in trees is easy, as the star with n vertices has exactly n matchings, finding the distribution of this number is still open, as is the case with the number of independent sets. Wagner mentions in [14] that his method could be applied to the number of matchings as well, showing that asymptotically this number is typically divisible by any constant m . This may serve as an explanation for the hardness of obtaining distribution results.

Both independent sets and matchings are special cases of F -matchings. Let F and G be graphs. An F -matching in G is a subgraph of G consisting of pairwise vertex disjoint copies of F . We say that the F -matching is *induced* in G if no additional edge of G is spanned by the vertices of G covered by the matching. These two closely related notions generalize naturally matchings and independent sets. Indeed, if F is the graph with two vertices and one edge then an F -matching is simply a matching. If F is a single vertex then an induced F -matching is an independent set.

The notion of F -matching, which is sometimes called F -packing (and in the spanning case F -factor) has been considered in several papers, but most of them do not deal with the number of F -packings and mainly consider necessary or sufficient conditions for their existence. See, for example, [6] and its references.

Given graphs F and G we denote the set of F -matchings in G by $S(F, G)$ and its size by $s(F, G)$. The set of all induced F -matchings in G is denoted by $S'(F, G)$ with $s'(F, G) = |S'(F, G)|$ being its size.

In this paper G will be drawn at random from a probability space of graphs. We define the *random tree* \mathcal{T}_n to be the set of all n^{n-2} labeled trees on n vertices endowed with the uniform distribution.

Our main results are the following:

Theorem 1. *Let F be a tree that is not a single vertex and let m be a positive integer. Then there is a constant $c = c(F, m) > 0$ such that the number of F -matchings in the random tree \mathcal{T}_n is zero modulo m with probability at least $1 - e^{-cn}$.*

Note that when F is a single vertex, the number of F -matchings in any graph with n vertices is 2^n .

Theorem 2. *Let F be a tree and let m be a positive integer. Then there is a constant $c' = c'(F, m) > 0$ such that the number of induced F -matchings in the random tree \mathcal{T}_n is zero modulo m with probability at least $1 - e^{-c'n}$.*

The fact that the number of independent sets is almost surely a zero residue is an immediate consequence of Theorem 2 — simply take F to be a single vertex.

In the next section we prove Theorem 1, in Section 3 we describe a similar proof of the induced case and in the last section we state some extensions and conclude with a few remarks and open questions. Our extensions include the fact that the assertions of both theorems hold when the random tree \mathcal{T}_n is replaced by a random planar graph on n vertices.

2 The non-induced case

In this section we prove Theorem 1. The proof is probabilistic and has two parts, a probabilistic claim (Lemma 3) and a deterministic claim (Lemma 5). Theorem 1 is an immediate consequence of these claims.

We shall use the following notation. Let T be a tree and assume that $\{u, v\}$ is an edge in T . We define a rooted tree $T^{(u,v)}$ by first setting v as the root — this defines a direction of parenthood in T — and then removing u along with its descendants. Note that $T^{(u,v)}$ is a rooted (undirected) tree. If R is a rooted tree isomorphic to $T^{(u,v)}$ (a fact we denote by $R \cong T^{(u,v)}$) for some edge $\{u, v\} \in T$, we say that T has an R -leaf. The next Lemma states that for every fixed rooted tree R , a random tree has an R -leaf with probability exponentially close to 1.

Lemma 3. *Let R be a rooted tree. There exists a constant $c = c(R) > 0$ such that*

$$\Pr[\exists \{u, v\} \in \mathcal{T}_n \text{ s.t. } R \cong \mathcal{T}_n^{(u,v)}] > 1 - e^{-cn}.$$

Proof. While our object of interest are trees, it is easier to work with functions on $[n] = \{1, 2, \dots, n\}$ via the Joyal mapping ([5], also presented in English in [1]).

We shall briefly describe the Joyal mapping and some of its properties that we need. The Joyal mapping maps f , a function from $[n]$ to itself, to an undirected tree T_f over the set of vertices $[n]$. There are n^n functions in $[n]^{[n]}$, but only n^{n-2} labeled trees over $[n]$. In order to make the mapping into a bijection we distinguish two vertices of a labeled tree by marking them *left* and *right* (we may mark one vertex with both). Now the target set is the set of all labeled trees over $[n]$ together with the markings, and is of size n^n .

The mapping is defined as follows. Let $f: [n] \rightarrow [n]$. Define \vec{G}_f as the functional digraph¹ with vertex set $[n]$ and edge set $\{(i, f(i)) \mid i \in [n]\}$. Every vertex in \vec{G}_f has outdegree one, so every connected component has one directed cycle, and all edges that are not in a cycle are pointing towards the cycle. Let $M = \{a_1 < a_2 < \dots < a_m\}$ be the set of all vertices participating in a cycle of \vec{G}_f . Notice that M is the maximal set such that $f|_M$ is a bijection. To get T_f , the tree corresponding to the function f , we first define a path by taking the vertices of M and adding the $m - 1$ edges of the form $\{f(a_i), f(a_{i+1})\}$. We then mark $f(a_1)$ as “left” and $f(a_m)$ as “right”. Finally we add the vertices in $[n] \setminus M$ with the edges $\{i, f(i)\}$ from \vec{G}_f (forgetting about directions).

Given a tree T with two such markings, we go back by defining M as the vertices in the path P connecting “left” and “right”, and directing all other vertices towards P . Sort the members of M according to their value and denote them by $a_1 < a_2 < \dots < a_m$. We define f as follows. If $i \in M$ is the j 'th vertex in the path then $f(i) = a_j$. If $i \notin M$ then there is one edge, (i, j) , emanating from i , and we set $f(i) = j$. It is easy to verify that this is indeed the inverse of the mapping described above.

Notice that vertices that are not in a cycle are left by the Joyal mapping as they were in \vec{G}_f , meaning that they will be incident with exactly the same edges as in the functional graph. In particular, edges with both endpoints being vertices that are not in a cycle of \vec{G}_f will touch the same edges in T_f as in \vec{G}_f . For our purpose, the fate of vertices lying in a cycle is irrelevant.

Direct the edges of R towards the root to get \vec{R} . Consider a random function f on $[n]$ and let X be the random variable counting the number of directed edges (u, v) in \vec{G}_f such that u, v and the ancestors of v in \vec{G}_f do not belong to any cycle in \vec{G}_f , and in addition, v and its ancestors form an isomorphic copy of \vec{R} .

Denote the vertices of \vec{R} by r_1, \dots, r_k , the root being r_k . Fix a $(k + 1)$ -tuple of vertices of \vec{G}_f , say $1, 2, \dots, k + 1$. The probability that the edge $(k, k + 1)$ meets the condition described above is at least the probability that $(k, k + 1) \in E(\vec{G}_f)$, the mapping $i \rightarrow r_i$ is an isomorphism between \vec{R} and $\vec{G}_f[\{1, \dots, k\}]$, and in addition, there are no other edges of \vec{G}_f incoming to $\{1, \dots, k + 1\}$. The latter is

$$\left(\frac{1}{n}\right)^k \left(\frac{n - k - 1}{n}\right)^{n-k}.$$

In order to see this simply notice that for $1 \leq i \leq k$ there is only one valid target for $f(i)$, while for $i \geq k + 1$ it is enough to require that f will map i outside of $\{1, 2, \dots, k + 1\}$. Therefore we get

$$EX \geq \binom{n}{k + 1} n^{-k} \left(\frac{n - k - 1}{n}\right)^{n-k},$$

which implies $EX = \Omega(n)$.

We want to show that X is concentrated around its mean. Consider the *value exposing* martingale, in which we expose the values of f one by one. Now, changing the value of f in

¹A *functional digraph* is a directed graph with all outdegrees equal one.

one coordinate, i , can ruin at most two copies of \vec{R} (one using the edge $(i, f(i))$ and another that now has an extra edge $(i, f'(i))$). Therefore the Lipschitz condition with constant two holds and we can apply the Azuma Inequality [2, 3] which yields $\Pr[X = 0] < e^{-cn}$ for some constant $c > 0$.

Observe that if $X(f) > 0$ then by the definition of X , the corresponding tree T_f contains the edge $\{u, v\}$ requested by the proposition.

As mentioned above, the Joyal correspondence is n^2 to one. If a labeled tree T does not contain an edge as required, then all its n^2 preimages f satisfy $X(f) = 0$. Therefore, the probability not to get a tree with a required edge is at most $\Pr[X = 0] < e^{-cn}$ as proven above. \square

Remark 4. Plugging the estimates in [2, Theorem 7.4.2] we can show that the constant c above is bigger than $(100k^{2k+1})^{-1}$.

The next argument of the proof states the existence of a *nullifying tree* Z (depending on F and m) such that if a tree T has a Z -leaf then $s(F, T) \equiv 0 \pmod{m}$.

Lemma 5. *Let F be a tree with at least one edge and let m be an integer. Then there exists a rooted tree Z such that, if $Z \cong T^{(u,v)}$ for some edge $\{u, v\} \in T$, then $s(F, T) \equiv 0 \pmod{m}$.*

Proof. The proof is constructive. By Proposition 6 to be proven below there exists a tree Y such that $s(F, Y) \equiv 0 \pmod{m}$.

Let $\Delta(F)$ be the maximal degree of F . To get Z take $\Delta(F) + 1$ copies of Y , add a new vertex r to be viewed as the root of Z , and connect r to a vertex of each Y (thus adding $\Delta(F) + 1$ edges).

Let T be a tree and assume that $Z \cong T^{(u,v)}$ for some edge $\{u, v\} \in T$. We wish to show that $s(F, T) \equiv 0 \pmod{m}$. There are finitely many ways in which one may cover v by a copy of F , and it may also be that v remains uncovered. We classify F -matchings in T into classes C_1, C_2, \dots, C_q according to the copy of F covering v , with the set of F -matchings not covering v being a separate class C_0 . We argue that the number of F -matchings in each such class is a zero residue. Indeed, the number of F -matchings in a given class C_i is precisely the number of F -matchings in the forest remaining from T after removing v and the copy covering it, if there is one. In fact, this number is the product of the number of F -matchings in every connected component of the forest. By our construction of Z , at least one of the trees in this forest is isomorphic to Y . Since $s(F, Y) \equiv 0 \pmod{m}$ we deduce that the number of F -matchings in the forest, and also in C_i , is zero modulo m . This is true for all C_i , and since $S(F, T) = \cup C_i$ one has $s(F, T) \equiv 0 \pmod{m}$. \square

Before stating and proving the next proposition we define some notation. Let F be a tree. Take a longest path in F and denote its vertices by u_1, u_2, \dots, u_{l+1} , where l is the diameter of F . If we disconnect all edges of the form $\{u_i, u_{i+1}\}$ we get $l + 1$ subtrees. Let b_i be the number of vertices in the subtree containing u_i . With this notation we have $|F| = \sum_{i=1}^{l+1} b_i$. Since $b_{l+1} = 1$ we may also write $|F| = 1 + \sum_{i=1}^l b_i$. We shall use this notation in the proof of the next proposition and in the proof of Proposition 9 as well.

Proposition 6. *Let F be a tree with at least one edge and let m be an integer. Then there exists a rooted tree Y such that $s(F, Y) \equiv 0 \pmod{m}$.*

Proof. Let W_t be a tree made of t copies of F in which we identify the vertex u_{l+1} of copy i with the vertex u_1 of copy $i + 1$ (for $1 \leq i \leq t - 1$). Let $P \subset W_t$ be the path in W_t connecting the first copy of u_1 to the last copy of u_{l+1} , and number its vertices by $1, \dots, lt + 1$ in the natural order, from the copy of u_1 in the first copy of F to the copy of u_{l+1} in the last copy of F . We want to have a direction of parenthood in W_t , so we set 1 to be the root. Notice that all connected components of $W_t \setminus V[P]$ are of size strictly less than $|F|$.

We are interested in embeddings of F in W_t , that is, in subgraphs of W_t that are isomorphic to F . Notice that every such embedding must have a vertex in P . Let C be an embedding of F in W_t . We call the vertex $\min\{C \cap P\}$ the *starting vertex* of C . Consider the set of all starting vertices in W_t . If $1 \leq i \leq (t - 2)l + 1$ is a starting vertex, then by symmetry so is $i + l$. Observe that trivially 1 is a starting vertex (and so are $l + 1, 2l + 1, \dots$). By the symmetry argument above, if there are d starting vertices between 1 and $l + 1$ (inclusive), then there are $1 + (t - 1)(d - 1)$ starting vertices in W_t . To see this recall that 1 is always a starting vertex, and each copy but the last adds $d - 1$ starting vertices; also, the last copy of F in W_t does not contain any starting vertices apart from $1 + l(t - 1)$ as deleting $1 + l(t - 1)$ leaves less than $|F|$ vertices to the right of it. Similarly, if i is a starting vertex then there are d starting vertices between i and $i + l$, inclusive.

Now we can define $\{Y_r\}$, a family of subtrees of W_t a member of which will eventually be the sought after tree. Set t to be large enough ($t = 1 + \lceil (r - 1)/(d - 1) \rceil$ will do). To get Y_r take the minimal subpath of $P \subset W_t$ containing the last r starting vertices and then append to each vertex in the subpath the subtree of its descendants through children outside P . For example, Y_1 is the single starting vertex $1 + l(t - 1)$ and Y_d is the next to the last copy of F in W_t .

Let $g(r)$ be the number of F -matchings in Y_r . We count such F -matchings by the membership of i , the first vertex in Y_r . If i is not covered by the matching, then the next embedding of F begins no earlier than the next starting vertex. This means that the number of F -matchings of Y_r in which i is not covered is $g(r - 1)$.

We argue now that if i is covered by the matching then the next $d - 1$ starting vertices are also covered. Let $\varphi: F \rightarrow Y_r$ be an embedding covering i . We claim that the next $d - 1$ starting vertices are also covered by φ . First, since the diameter of F is l , no vertex of P farther than $i + l$ (which is the starting vertex $d - 1$ away from i) is covered by φ . On the other hand, the path from i to $i + l - 1$ contains one copy of each u_i (not necessarily in the natural order). Thus, the number of vertices in the set containing $i, i + 1, \dots, i + l - 1$ and their descendants is exactly $\sum_{i=1}^l b_i$, hence φ extends also to $i + l$. Therefore, the other embeddings in the F -matching need to start after $i + l$. We get that the number of such matchings is exactly $g(r - d)$. This gives the recursion $g(r) = g(r - 1) + g(r - d)$.

Observe that the tree Y_r , $1 \leq r < d$, does not contain a copy of F , and thus the only F -matching in Y_r is the empty one, implying $g(r) = 1$ for every $1 \leq r < d$; also, $g(d) = 2$ as $Y_d = F$. We can extend the recursion backwards by defining $g(0) = 1$ and $g(-1) = 0$. By Claim 7 below there is an integer $r_0 > 0$ such that $g(r_0) \equiv 0 \pmod{m}$.

Define $Y = Y_{r_0}$. By the definition of $g(r)$ we have $s(F, Y) \equiv 0 \pmod{m}$. □

Claim 7. *Let $g(r): \mathbb{N} \rightarrow \mathbb{Z}$ be a sequence of integers obeying a recurrence relation with integer coefficients $g(r) = \sum_{i=1}^d c_i g(r-i)$. Assume that $g(0) = 0$ and $c_d = 1$. Then for every positive integer $m > 0$ there exists an index $r_0 = r_0(m) > 0$ such that $g(r_0) \equiv 0 \pmod{m}$.*

Proof. First we claim that $g(r) \pmod{m}$ is periodic. Indeed, since $g(r) \pmod{m}$ is determined by the d -tuple of the previous d values, and since modulo m there are at most m^d possible d -tuples, then after at most m^d steps the sequence $g(r) \pmod{m}$ must become periodic. Next we claim that $g(r) \pmod{m}$ is periodic from the beginning. To see this simply extend the sequence m^d steps backwards using the recurrence relation $g(r-d) = g(r) - \sum_{i=1}^{d-1} c_i g(r-i)$. The previous argument shows that the extended sequence is periodic starting at most at the m^d th element, which is the first element of the original sequence. Hence $g(r) \pmod{m}$ is periodic from its first element, $g(0) = 0$, and thus there is some $r_0 > 0$ such that $g(r_0) \equiv 0 \pmod{m}$. □

For more information on recurrence sequences modulo m , in particular for a better estimate of the index of the first zero residue element, see [4, Section 6.3].

3 The induced case

In this section we prove Theorem 2. The proof is similar to the proof of Theorem 1 and we shall focus on the differences between the proofs. As before, the proof is probabilistic. Lemma 3 is the probabilistic part here as well, but the deterministic part is replaced by Lemma 8 below.

We begin by constructing a nullifying rooted tree from copies of a tree Y' having $s'(F, Y') \equiv 0 \pmod{m}$.

Lemma 8. *Let F be a tree and let m be an integer. There exists a rooted tree Z' such that if $Z' \cong T^{(u,v)}$ for some edge $\{u, v\} \in T$, then $s'(F, T) \equiv 0 \pmod{m}$.*

Proof. By Proposition 9 below there exists a tree Y' such that $s'(F, Y') \equiv 0 \pmod{m}$. Construct Z' by taking $\Delta(F) + 2$ copies of Y' , adding a new vertex r to be viewed as the root of Z' , connecting one copy to r with a new edge and connecting the rest of the $\Delta(F) + 1$ copies to r via a path of length two.

Let T be a tree and assume that $Z' \cong T^{(u,v)}$ for some edge $\{u, v\} \in T$. We need to show that $s'(F, T) \equiv 0 \pmod{m}$.

There are finitely many ways in which v may be covered by a copy of F , if it is covered at all. We classify induced F -matchings according to the copy of F covering v . Denote these classes by C_1, \dots, C_k and let C_0 be the class of all induced F -matchings of T in which v is left uncovered. Clearly $S'(F, T) = \bigcup_{i=0}^k C_i$. We claim that $|C_i| \equiv 0 \pmod{m}$ for every $0 \leq i \leq k$.

Consider first the class C_0 of induced F -matchings in T that leave v uncovered. The number of such matchings is the number of matchings in the forest remaining after deleting

v . This forest has a component isomorphic to Y — the copy of Y that was connected to v by a single edge. The number of induced F -matchings in C_0 is then the product of the number of induced F -matchings in every connected component of the aforementioned forest which is zero modulo m .

Consider now the class C_i for $i > 0$. As before, there is a natural one to one correspondence between induced F -matchings in T that belong to C_i and induced F -matchings of the forest remaining after removing the copy of F covering v and all neighbors of vertices in that copy. Since v is covered by the matching, all of its neighbors that are not covered by the same copy of F must remain uncovered. Otherwise, an additional edge outside the copies of F would be spanned. This means that in the above forest at least one of the $\Delta(F) + 1$ copies that were connected to v by a path of length two will now remain as a connected component. Hence, the number of induced F -matchings in C_i is a zero residue.

Summing the sizes of the C_i 's we get that $m'(F, T) \equiv 0 \pmod{m}$. \square

Proposition 9. *Let F be a tree and let m be an integer. Then there exists a rooted tree Y' such that $s'(F, Y') \equiv 0 \pmod{m}$.*

Proof. The construction and the proof are similar to those in the proof of Proposition 6, and we shall use the notation defined just before it. We define W'_t as a collection of t disjoint copies of F , and we add an edge between the vertex u_{l+1} of the i 'th copy and the vertex u_1 of the $(i + 1)$ 'th copy. We think of the first copy of u_1 as the root of W'_t .

Let P be the path connecting the first copy of u_1 with the last copy of u_{l+1} and denote its vertices by $1, \dots, t(l + 1)$ in the natural order. We define starting vertices in the same manner as in the proof of Lemma 5. The symmetry argument still holds, only now the period is $l + 1$, that is, if $1 \leq i \leq (t - 2)(l + 1) + 1$ is a starting vertex then so is $i + l + 1$. Also, if there are d starting vertices between 1 and $l + 1$, then there are d starting vertices between every starting vertex i and $i + l$ and all in all there are $(t - 1)d + 1$ starting vertices in W'_t .

Let Y'_r be the subgraph of W'_t composed of the minimal path of P containing the last r starting vertices together with their descendants through vertices that are not in P . Hence, Y'_1 is a single vertex and Y'_{d+1} is a copy of F with an extra vertex connected to u_{l+1} . Finally we define $g'(r)$ as the number of induced F -matchings in Y'_r .

We wish to derive a recurrence formula for $g'(r)$. We count induced F -matchings of Y'_r by the membership of the first vertex. The number of induced F -matchings that do not cover the first vertex (who is also the first starting vertex) is exactly $g'(r - 1)$.

Consider matchings in which the first starting vertex i is covered. The embedding of F covering i can not cover vertices of P farther than $i + l$, since the diameter of F is l . On the other hand, the number of vertices in the subgraph made of the path connecting i to $i + l$ together with their descendants that are not in P is exactly $\sum b_i = |F|$. Hence $i + l$ is also covered by the same embedding that covers i . Now, if $i + l + 1$ is covered by another embedding of F , then $\{i + l, i + l + 1\}$ is spanned, which is forbidden, so $i + l + 1$ is not covered. Since there are d starting vertices between i and $i + l$, and since $i + l + 1$ is a starting vertex as well, we get that the number of such matchings is exactly $g'(r - d - 1)$. Therefore we have $g'(r) = g'(r - 1) + g'(r - d - 1)$.

Clearly $g'(r) = 1$ for every $1 \leq r \leq d-1$, as the number of vertices in Y_r' in these cases is smaller than $|F|$. The value of $g'(d)$ may be either 1 or 2, depending on whether F may be embedded into Y_d or not. The value of $g'(d+1)$ can also be one of a few options. Still, we extend g' backwards by defining $g'(0) = g'(d+1) - g'(d)$, $g'(-1) = g'(d) - g'(d-1)$, and $g'(-2) = g'(d-1) - g'(d-2) = 0$. We complete the proof by applying Claim 7. \square

4 Concluding discussion

Our initial objective was to provide a simple and intuitive explanation to the fact that almost all labeled trees have an even number of independent sets. Indeed, there are nullifying trees Z such that when a tree T has a Z -leaf, the number of independent sets in T is even. Also, every fixed tree Z appears as a Z -leaf in a random tree with n vertices with probability tending to one as n goes to infinity. Therefore almost all trees have an even number of independent sets.

The simplicity of the explanation allowed vast generalizations — Theorems 1 and 2 above. In fact, the proof also works in other scenarios. If a probability space of graphs has a property corresponding to the probabilistic part of the proof, then the number of (induced) F -matchings will be a zero residue in that probability space as well.

As a concrete example, let \mathcal{P}_n be the *random planar graph* of order n , that is, \mathcal{P}_n is the set of all simple labeled planar graphs with n vertices endowed with the uniform distribution. In [10] it is shown that with probability exponentially close to one, \mathcal{P}_n has an R -leaf for every fixed rooted tree R . Thus, by the above, the number of (induced) F -matchings is a zero residue in a random planar graph. Notice that \mathcal{P}_n is connected with probability at least $1/e$ as shown in [10], so a potential simpler strategy of proving the same result — showing the existence of a component having a zero residue number of (induced) F -matchings — will not suffice.

Similar results may be obtained for other random graphs models as well. On the other hand, if we consider dense random graphs then a different approach is required. For example, it is not clear how the number of independent sets typically behaves as a residue for the binomial random graph $G(n, 1/2)$. Moreover, it is not difficult to show that for $p = p(n)$ close to 1 in the range in which the maximum independent set of $G(n, p)$ is $\Theta(1) > 1$ asymptotically almost surely, the number of independent sets in $G(n, p)$ is nearly uniformly distributed modulo any constant m . See [13] for several related results.

Our proof implies that the number of F -matchings in a random tree of order n is typically zero modulo any constant m when the size of F grows slowly enough with n . It may be interesting to find the maximal rate of growth for which this property still holds.

In a recent paper [7] Law shows that for every two integers k, m , there exists a tree T such that the number of independent sets of T is congruent to k modulo m . One may ask if the same holds for general (induced) F -matchings as well. Is it true that for every tree F and integers k, m there exists a tree T such that the number of (induced) F -matchings of T is congruent to k modulo m ?

Using Remark 4 we can estimate the constants appearing in Theorems 1 and 2. Clearly the constant will depend on the size of the nullifying tree. No effort was made to find the

smallest nullifying tree.

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