Hitting time results for Maker-Breaker games

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Abstract

We study Maker-Breaker games played on the edge set of a random graph. Specifically, we analyze the moment a typical random graph process first becomes a Maker's win in a game in which Maker's goal is to build a graph which admits some monotone increasing property $\mathcal{P}$. We focus on three natural target properties for Maker's graph, namely being k-vertex-connected, admitting a perfect matching, and being Hamiltonian. We prove the following optimal hitting time results: with high probability Maker wins the k-vertex connectivity game exactly at the time the random graph process first reaches minimum degree $2k$; with high probability Maker wins the perfect matching game exactly at the time the random graph process first reaches minimum degree 2; with high probability Maker wins the Hamiltonicity game exactly at the time the random graph process first reaches minimum degree 4. The latter two statements settle conjectures of Stojaković and Szabó. We also prove generalizations of the latter two results; these generalizations partially strengthen some known results in the theory of random graphs.

An extended abstract of this paper was previously published in [4].

1 Introduction

Let X be a finite set and let $\mathcal{F} \subseteq 2^X$ be a family of subsets. In the positional game $(X, \mathcal{F})$, two players take turns in claiming one previously unclaimed element of X and the game ends when all of the elements of X have been claimed by either of the players. The set X is often referred to as the *board* of the game. Positional games have attracted a lot of attention in the past decade and a thorough introduction to this field with a plethora of results can be found in a recent monograph of Beck [3]. In a *Maker-Breaker*-type positional game, the two players are called *Maker* and *Breaker* and the members of $\mathcal{F}$ are referred to as the *winning sets*. Maker wins the game if he occupies all elements of some winning set; otherwise Breaker wins. We will always assume that Breaker starts the game. We say that a game $(X, \mathcal{F})$ is a *Maker's win* if Maker has a strategy (that can be adaptive to Breaker's moves) that ensures his win in this game against any strategy of Breaker, otherwise the game is a *Breaker's win*. Note that X and $\mathcal{F}$ alone determine whether the game

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is a Maker's win or a Breaker's win. A classical example of this Maker-Breaker setting is the popular board game HEX.

1.1 Maker-Breaker games on graphs

Let $G = (V, E)$ be a graph and let $\mathcal{P}$ be a monotone increasing graph property on V (a family of graphs on V, closed under isomorphism and addition of edges). Consider the Maker-Breaker game $(E, \mathcal{F}_{\mathcal{P}})$ whose board is E, the edge set of G. The game is won by Maker if and only if the graph spanned by the edges he claims throughout the game satisfies the property $\mathcal{P}$. We denote the family of graphs G for which the $(E(G), \mathcal{F}_{\mathcal{P}})$ game is a Maker's win by $\mathcal{M}_{\mathcal{P}}$. Although the property $\mathcal{M}_{\mathcal{P}}$ is described in game-theoretic terms, it should be noted that the games $(E(G), \mathcal{F}_{\mathcal{P}})$ are finite perfect information games with no chance moves, and thus $\mathcal{M}_{\mathcal{P}}$ is some graph property which clearly satisfies $\mathcal{M}_{\mathcal{P}} \subseteq \mathcal{P}$. Moreover, since $\mathcal{P}$ is monotone increasing, $\mathcal{M}_{\mathcal{P}}$ is clearly monotone increasing as well. By considering monotone increasing graph properties, the game can be terminated as soon as the graph spanned by Maker's edges satisfies the property, regardless of whether all edges have been claimed or not. This leads to several natural questions. First, how sparse can a graph $G \in \mathcal{M}_{\mathcal{P}}$ be? In this context, playing on random graphs (where the density of the graph is chosen according to the property at hand) becomes very natural. The systematic study of this setting was initiated in [26] by Stojaković and Szabó, and this current work is a further exploration of it. Second, one can also study the minimum number of moves needed for Maker in order to win the game (see e.g. [2, 24, 13, 15, 18]). Although "winning fast" is not in the focus of this current paper, we will see how an upper bound on the number of moves required by Maker to win the game helps, and is in fact crucial, in the analysis of Maker's winning strategy.

1.2 Random graphs

The most widely used random graph model is the Binomial random graph, $\mathcal{G}(n, p)$. In this model we start with n vertices, labeled, say, by $V = \{1, \dots, n\} = [n]$, and select a graph on these n vertices by going over all $\binom{n}{2}$ pairs of vertices, deciding independently with probability p for a pair to be an edge. The model $\mathcal{G}(n, p)$ is thus a probability space of all labeled graphs on the vertex set $[n]$ where the probability of such a graph, $G = ([n], E)$, to be selected is $p^{|E|}(1-p)^{\binom{n}{2}-|E|}$. This product probability space provides us with a wide variety of probabilistic tools for analyzing the behavior of various random graph properties. (See monographs [8] and [19] for a thorough introduction to the subject of random graphs). In the subsequent sections we will need at some point to employ a slightly generalized model. Let $F \subseteq \binom{V}{2}$ be an arbitrary subset and let $\mathcal{G}(n, p)_{-F} := \mathcal{G}(n, p) \setminus F$.

Although the Binomial random graph model is very natural and relatively easy to use, it was not the first model to be considered. In their seminal paper, Erdős and Rényi considered the uniform probability space over all graphs on a fixed set of vertices with exactly M edges, $\mathcal{G}(n, M)$. Note that for any value of p, if we condition the random graph $\mathcal{G}(n, p)$ to have exactly M edges, then we obtain exactly the Erdős-Rényi random graph model. The similarity of the two models enables us to prove the occurrence of events in the $\mathcal{G}(n, p)$ model and get the corresponding result in the $\mathcal{G}(n, M)$ model.

Proposition 1.1 ([19], Proposition 1.13)**.** *Let $\mathcal{P} = \mathcal{P}(n)$ be a sequence of monotone increasing graph properties, $0 \leq a \leq 1$ and $0 \leq M \leq \binom{n}{2}$ be an integer. If for every sequence $p = p(n) \in [0, 1]$ such that $p =$*

$M/\binom{n}{2} \pm O\left(M\left(\binom{n}{2}-M\right)/\binom{n}{2}^3\right)$ *it holds that* $\lim_{n\to\infty} \Pr[\mathcal{G}(n,p) \in \mathcal{P}] = a$*, then* $\lim_{n\to\infty} \Pr[\mathcal{G}(n,M) \in \mathcal{P}] = a$*.*

The converse result to Proposition 1.1 holds[1] as well (see e.g. Proposition 1.12 in [19]); this enables us to transfer results from one model to the other. Unfortunately, not all properties we will encounter and explore are monotone increasing, and hence Proposition 1.1 cannot be used in those cases. Nonetheless, we would like to take advantage of the "ease" of calculations in the $\mathcal{G}(n,p)$ model (due to the independence of appearance of its edges), and transfer the results to the $\mathcal{G}(n,M)$ model, for the appropriate values of M. To achieve this we will use this more crude estimate (see e.g. [19]), which will suffice for our purposes.

Claim 1.2 ([19], inequality (1.6))**.** *Let* $\mathcal{P}$ *be a property of graphs on* n *vertices and let* $1 \le M \le \binom{n}{2}$ *be an integer. Setting* $p = M/\binom{n}{2}$ *we have* $\Pr[\mathcal{G}(n,M) \in \mathcal{P}] \le 3\sqrt{M} \cdot \Pr[\mathcal{G}(n,p) \in \mathcal{P}]$.

Next, we consider the following generation process of graphs. Given a set V of n vertices and an ordering on the pairs of vertices $\pi : \binom{V}{2} \to \left[\binom{n}{2}\right]$, we define a *graph process* to be a sequence of graphs $\widetilde{G} = \widetilde{G}(\pi) = \{G_t\}_{t=0}^{\binom{n}{2}}$ on V. Starting with $G_0 = (V, \emptyset)$, for every integer $1 \le t \le \binom{n}{2}$, the graph G_t is defined by $G_t := G_{t-1} \cup \pi^{-1}(t)$. For a given graph process $\widetilde{G}$ on V, we define the *hitting time* of a monotone increasing graph property $\mathcal{P}$ on V as

$$\tau(\widetilde{G}; \mathcal{P}) = \min\{t \ : \ G_t \in \mathcal{P}\}. \tag{1}$$

When selecting π uniformly at random, the process $\widetilde{G}(\pi)$ is usually called the *random graph process*. If $\widetilde{G} = \{G_t\}_{t=0}^{\binom{n}{2}}$ is the random graph process, then, for every $0 \le M \le \binom{n}{2}$, the graph G_M is distributed according to $\mathcal{G}(n,M)$, that is, $G_M \sim \mathcal{G}(n,M)$. This shows that analyzing the hitting time of a monotone increasing property $\mathcal{P}$ is in fact a refinement of the study of values of M and p for which $\mathcal{G}(n,M) \in \mathcal{P}$ and $\mathcal{G}(n,p) \in \mathcal{P}$ respectively (where to get the values of p we employ the converse of Proposition 1.1 as stated above).

For every positive integer k let δ_k denote the graph property of having minimum degree at least k, let $\mathcal{EC}_k$ denote the graph property of being k-edge connected, let $\mathcal{VC}_k$ denote the graph property of being k-vertex connected, and let $\mathcal{HAM}$ denote the graph property of admitting a Hamilton cycle. Two cornerstone results in the theory of random graphs are that of Bollobás and Thomason [10] who proved that for every $1 \le k \le n-1$, with high probability (or w.h.p. for brevity)[2] $\tau(\widetilde{G}; \delta_k) = \tau(\widetilde{G}; \mathcal{EC}_k) = \tau(\widetilde{G}; \mathcal{VC}_k)$, and that of Komlós and Szemerédi [21] who proved that w.h.p. $\tau(\widetilde{G}; \delta_2) = \tau(\widetilde{G}; \mathcal{HAM})$ (see also [7]). Note that these two results (and many others which have succeeded) provide a very strong indication that the "bottleneck" for such properties in random graphs is in fact the vertices of minimum degree. The results of this paper are of the very same nature.

1.3 Motivation and previous results

Given a graph G with minimum degree at most $2k-1$, when playing on the board $E(G)$ Breaker can keep claiming edges incident to some vertex of minimum degree, and with the advantage of playing first will thus leave Maker with a graph containing a vertex of degree at most $k-1$. This implies that Breaker wins the

[1] In fact, when moving from $\mathcal{G}(n,M)$ to $\mathcal{G}(n,p)$ the monotonicity requirement is not necessary.

[2] In this paper, we say that a sequence of events $\mathcal{A}_n$ in a random graph model occurs w.h.p. if the probability of $\mathcal{A}_n$ tends to 1 as the number of vertices n tends to infinity.

k-edge-connectivity game $(E(G), \mathcal{F}_{\mathcal{EC}_k})$ for such graphs, and therefore $\tau(\widetilde{G}; \mathcal{M}_{\mathcal{EC}_k}) \geq \tau(\widetilde{G}; \delta_{2k})$ for every graph process $\widetilde{G}$. In [26] Stojaković and Szabó were the first to consider Maker-Breaker games played on random graphs. By combining theorems of Lehman [22] and of Palmer and Spencer [23], they observed that for every fixed positive integer k, if $\widetilde{G}$ is the random graph process, then w.h.p. $\tau(\widetilde{G}; \mathcal{M}_{\mathcal{EC}_k}) = \tau(\widetilde{G}; \delta_{2k})$, thus providing a very precise hitting time result for the edge-connectivity game[3]. Similarly to the edge-connectivity case we have that for every graph process $\widetilde{G}$

$$\tau(\widetilde{G}; \delta_{2k}) \leq \tau(\widetilde{G}; \mathcal{M}_{\mathcal{VC}_k}). \tag{2}$$

Let $\mathcal{PM}$ denote the graph property of admitting a matching of size $\lfloor n/2 \rfloor$ in a graph on n vertices. Every graph G on an *even* number of vertices with minimum degree at most 1 is a win for Breaker in the perfect matching game $(E(G), \mathcal{F}_{\mathcal{PM}})$. Hence, for every graph process $\widetilde{G}$ on an even number of vertices

$$\tau(\widetilde{G}; \delta_2) \leq \tau(\widetilde{G}; \mathcal{M}_{\mathcal{PM}}). \tag{3}$$

In [26] Stojaković and Szabó conjectured that if $\widetilde{G}$ is the random graph process, then w.h.p. equality holds in (3). Although they did not prove this conjecture, in [26] they proved that if $p > \frac{64 \ln n}{n}$, then w.h.p. $\mathcal{G}(n,p) \in \mathcal{M}_{\mathcal{PM}}$. Note that this result is optimal in p up to multiplicative constant factor, for if $p \leq \frac{\ln n + \ln\ln n - \omega(1)}{n}$, where $\omega(1)$ is some function which tends to infinity with n arbitrarily slowly, then w.h.p. $\delta(\mathcal{G}(n,p)) \leq 1$, and hence by (3), w.h.p. $\mathcal{G}(n,p) \notin \mathcal{M}_{\mathcal{PM}}$.

Clearly, every graph G with minimum degree at most 3 is a win for Breaker in the Hamiltonicity game $(E(G), \mathcal{F}_{\mathcal{HAM}})$. Hence, we have that for every graph process $\widetilde{G}$

$$\tau(\widetilde{G}; \delta_4) \leq \tau(\widetilde{G}; \mathcal{M}_{\mathcal{HAM}}). \tag{4}$$

In [26] Stojaković and Szabó conjectured that if $\widetilde{G}$ is the random graph process, then w.h.p. equality holds in (4).

One of the first results in the field of Maker-Breaker games on graphs is due to Chvátal and Erdős in their seminal paper [11], which states that $K_n \in \mathcal{M}_{\mathcal{HAM}}$ for sufficiently large values of n (in [18] the third author and Stich proved that $n \geq 38$ suffices). The problem of finding sparse graphs which are a win for Maker was addressed by Hefetz et. al. [17] where they showed that, for sufficiently large values of n, there exists a graph $G \in \mathcal{M}_{\mathcal{HAM}}$ on n vertices with $e(G) \leq 21n$. Playing the Hamiltonicity game $(E(G), \mathcal{F}_{\mathcal{HAM}})$ on the random graph $\mathcal{G}(n,p)$ was first considered in the original paper of Stojaković and Szabó [26] where they proved that if $p > \frac{32 \ln n}{\sqrt{n}}$, then w.h.p. $\mathcal{G}(n,p) \in \mathcal{M}_{\mathcal{HAM}}$. Later, Stojaković [25] found the correct order of magnitude proving that $p > 5.4 \ln n/n$ suffices for $\mathcal{G}(n,p)$ to be w.h.p. Maker's win in the Hamiltonicity game. This requirement on p was subsequently improved to $p \geq \frac{\ln n + (\ln\ln n)^s}{n}$, where s is some large but fixed constant, by Hefetz et. al. [16]. Note that this result is very close to being optimal, for if $p = \frac{\ln n + 3\ln\ln n - \omega(1)}{n}$, where $\omega(1)$ is some function which tends to infinity with n arbitrarily slowly, then w.h.p. $\delta(\mathcal{G}(n,p)) < 4$ and hence by (4) w.h.p. $\mathcal{G}(n,p) \notin \mathcal{M}_{\mathcal{HAM}}$. Lastly, in [5] the first and fourth authors with Sudakov studied the Hamiltonicity game played on the edges of random regular graphs (the uniform probability measure over all d-regular graphs on a fixed vertex set) and proved that for large enough constant values of d this game is Maker's win.

[3] In [26] only the case of $k = 1$ is explicitly mentioned, but it can be generalized for any positive integer k in a straightforward manner.

1.4 Our results

In this paper we address the aforementioned Maker-Breaker games on random graphs, namely when Maker's goal is to build graphs which satisfy the properties of being k-vertex connected, admitting a perfect matching, and being Hamiltonian. Specifically, the main objective of this paper is to prove that the trivial minimum degree requirement as stated in (2), (3), and (4) is actually the bottleneck for a typical random graph to be a win for Maker in all of the above mentioned games. The following results will thus be proved.

Theorem 1. *For every fixed integer $k \geq 1$, if $\widetilde{G}$ is the random graph process, then w.h.p.*

$$\tau(\widetilde{G}; \mathcal{M}_{\mathcal{VC}_k}) = \tau(\widetilde{G}; \delta_{2k}).$$

For every positive integer k it holds that $\mathcal{VC}_k \subseteq \mathcal{EC}_k$, hence Theorem 1 is in fact an improvement of the aforementioned result of Stojaković and Szabó in [26].

The following result for the prefect matching game is also proved.

Theorem 2. *If $\widetilde{G}$ is the random graph process on an even number of vertices, then w.h.p.*

$$\tau(\widetilde{G}; \mathcal{M}_{\mathcal{PM}}) = \tau(\widetilde{G}; \delta_2).$$

Theorem 2 settles a conjecture raised in [26]. By the connection between the random graph models as described in Section 1.2 and by known results on the distribution of the minimum degree of $\mathcal{G}(n,p)$, Theorem 2 implies that w.h.p. $\mathcal{G}(n,p) \in \mathcal{M}_{\mathcal{PM}}$ for every $p \geq \frac{\ln n + \ln \ln n + \omega(1)}{n}$, improving on the result of Stojaković and Szabó in [26].

Theorem 3. *If $\widetilde{G}$ is the random graph process, then w.h.p.*

$$\tau(\widetilde{G}; \mathcal{M}_{\mathcal{HAM}}) = \tau(\widetilde{G}; \delta_4).$$

Theorem 3 settles a conjecture raised in [26]. Moreover, similarly to the above, Theorem 3 improves on the result of Hefetz et. al. in [16] by implying that w.h.p. $\mathcal{G}(n,p) \in \mathcal{M}_{\mathcal{HAM}}$ for every $p \geq \frac{\ln n + 3 \ln \ln n + \omega(1)}{n}$.

1.4.1 From Maker-Breaker games to general random graphs

We stress that using some simple observations, all of the above results have implications for the general framework of random graphs, hence implying that, in a sense, Theorems 1, 2, and 3 in fact partially strengthen some of the classical results of random graph theory.

Lehman's Theorem [22] states that $G \in \mathcal{M}_{\mathcal{EC}_k}$ if and only if G admits $2k$ pairwise edge-disjoint spanning trees [4]. We note that the assertion of Theorem 1 combined with Lehman's Theorem implies that for fixed k w.h.p. the hitting time of $\widetilde{G}$ for admitting $2k$ pairwise edge disjoint spanning trees is precisely the time the process first hits minimum degree $2k$. Hence, we recover a result of Palmer and Spencer [23] for even values of k.

Next, we stress that by using a *strategy stealing* argument, all of the above results can be transferred from the Maker-Breaker setting to statements about *graph packing*. Indeed, let G be a graph and let $\mathcal{P}$ be some

[4] in fact, Lehman stated his theorem only for the case $k = 1$, but it is straightforward to generalize it to handle every positive integer k.

monotone increasing graph property for which we know that $G \in \mathcal{M}_{\mathcal{P}}$. Let $\mathcal{S}_M$ be some winning strategy of Maker for the game $(E(G), \mathcal{F}_{\mathcal{P}})$. Breaker, who is the first player under our assumptions, can choose to *steal* Maker's winning strategy as follows. He starts by claiming an arbitrary edge of G. He then pretends that the game starts with Maker's first move; he thus assumes the role of the second player. He responds to each of Maker's moves according to $\mathcal{S}_M$. If at any point during the game, the strategy $\mathcal{S}_M$ requires Breaker to claim an edge that he has already claimed, Breaker simply claims some other arbitrary edge which was not previously claimed by either of the players. Since Breaker is following a winning strategy for $(E(G), \mathcal{F}_{\mathcal{P}})$ and since $\mathcal{P}$ is a monotone increasing graph property (that is, extra edges cannot "hurt" Breaker), he will build a subgraph which satisfies $\mathcal{P}$ by the end of the game. On the other hand, we can assume that Maker follows $\mathcal{S}_M$ as well. It follows that Maker will also build a subgraph which satisfies $\mathcal{P}$ (as his winning strategy can beat any strategy chosen by Breaker). Clearly, at the end of the game we have found two edge-disjoint subgraphs, each satisfying $\mathcal{P}$.

For every positive integer $k \geq 1$, let $\mathcal{PM}^k$ and $\mathcal{HAM}^k$ denote the graph properties of admitting k pairwise edge-disjoint perfect matchings, and k pairwise edge-disjoint Hamilton cycles respectively. Applying the aforementioned strategy stealing argument to the assertions of Theorems 2 and 3 implies the following: if $\widetilde{G}$ is the random graph process, then w.h.p. $\tau(\widetilde{G}; \mathcal{PM}^2) = \tau(\widetilde{G}; \delta_2)$ and $\tau(\widetilde{G}; \mathcal{HAM}^2) = \tau(\widetilde{G}; \delta_4)$. In Section 7 we discuss some generalizations of Theorems 2 and 3 which in turn imply that $\tau(\widetilde{G}; \mathcal{PM}^{2k}) = \tau(\widetilde{G}; \delta_{2k})$ and $\tau(\widetilde{G}; \mathcal{HAM}^{2k}) = \tau(\widetilde{G}; \delta_{4k})$ for a fixed integer $k \geq 1$. This is in fact a classical theorem (for fixed even minimum degree) of Bollobás and Frieze [9] for optimal packing of perfect matchings and Hamilton cycles in sparse random graphs (see also further extensions to random graphs of non-constant minimum degree [14, 6, 20]).

1.5 Organization

The rest of the paper is organized as follows. In Section 2 we provide some preliminary technical results about positional games, expanders, and random graphs, which will be needed in the course of our proofs. Section 3 is devoted to the analysis of a general game in which Maker's goal is to build an expander graph. This will give us a framework from which we can build on to prove the concrete results on the more natural games mentioned above. In Section 4 we prove some properties of random graphs and random graph processes that will be useful in the proofs of our main results. We then move on to provide the full proofs of Theorems 1 and 2 in Section 5. These proofs will rely heavily on the general expander game and the properties of random graphs and random graph processes which we discussed in the preceding two sections. In Section 6 we move on to the proof of Theorem 3, which is more delicate than the previous two and requires some more ideas to get the result in full. Lastly, we discuss some further generalizations and sketch their proofs in Section 7.

2 Preliminaries

In this section we cite some tools which we will make use of in the succeeding sections. First, we will need to employ bounds on large deviations of random variables. We will mostly use the following well-known bound on the lower and the upper tails of the Binomial distribution due to Chernoff (see e.g. [1, Appendix A]).

Theorem 2.1 (Chernoff bounds)**.** *If $X \sim B(n,p)$ then*

1. $\Pr[X < (1-\varepsilon)np] < \exp(-\frac{\varepsilon^2 np}{2})$ *for every* $\varepsilon > 0$*;*

2. $\Pr[X > (1+\varepsilon)np] < \exp(-\frac{np}{3})$ *for every* $\varepsilon \geq 1$*.*

It will sometimes be more convenient to use the following bound on the upper tail of the Binomial distribution.

Lemma 2.2. *If* $X \sim Bin(n,p)$ *and* $k \geq np$*, then* $\Pr[X \geq k] \leq (enp/k)^k$*.*

Note that the bound given in Lemma 2.2 is especially useful when k is "much larger" than np.

For the sake of simplicity and clarity of presentation, we do not make a particular effort to optimize the constants obtained in our proofs. We also omit floor and ceiling signs whenever these are not crucial. Most of our results are asymptotic in nature and whenever necessary we assume that n is sufficiently large.

2.1 Notation

Our graph-theoretic notation is standard and follows that of [27]. In particular, we use the following. For a graph G, let $V(G)$ and $E(G)$ denote its sets of vertices and edges respectively, and let $e(G) = |E(G)|$. For a set $A \subseteq V(G)$, let $E_G(A)$ denote the set of edges of G with both endpoints in A, and let $e_G(A) = |E_G(A)|$. For disjoint sets $A, B \subseteq V(G)$, let $E_G(A,B)$ denote the set of edges of G with one endpoint in A and the other in B, and let $e_G(A,B) = |E_G(A,B)|$. For a set $S \subseteq V(G)$, let $N_G(S) = \{u \in V(G) \setminus S : \exists v \in S, \{u,v\} \in E(G)\}$ denote the set of neighbors of S in $V(G) \setminus S$. For a vertex $w \in V(G)$, we abbreviate $N_G(\{w\})$ to $N_G(w)$. For a vertex $w \in V(G) \setminus S$ let $d_G(w,S) = |\{u \in S : \{u,w\} \in E(G)\}|$ denote the number of vertices of S that are adjacent to w in G. We abbreviate $d_G(w, V \setminus \{w\})$ to $d_G(w)$ which denotes the degree of w in G. The minimum vertex degree in G is denoted by $\delta(G)$. For a set $S \subseteq V(G)$ let $G[S]$ denote the subgraph of G with vertex set S and edge set $E_G(S)$. Let $conn(G)$ and $odd(G)$ respectively denote the number of connected components and the number of connected components of odd cardinality in G. Lastly, we will denote by $\ell(G)$ the length of a longest path in G, where the length of a path is the number of its edges.

2.2 Basic positional games results

The following theorem is a classical result of Erdős and Selfridge [12] which provides a useful sufficient condition for Breaker's win in the $(X, \mathcal{F})$ game.

Theorem 2.3 (Erdős and Selfridge [12])**.** *For any hypergraph* $(X, \mathcal{F})$*, if*

$$\sum_{A \in \mathcal{F}} 2^{-|A|} < \frac{1}{2},$$

then Breaker, playing as the first or second player, has a winning strategy for the $(X, \mathcal{F})$ *game.*

The following simple lemma is useful when a player is trying to ensure expansion of small sets. A similar lemma appeared in [16].

Lemma 2.4. *For every integer* $k > 0$*, if* H *is a graph on* n *vertices with minimum degree* $\delta(H) \geq 4k$*, then* $H \in \mathcal{M}_{\delta_k}$*. Moreover, Maker can win the minimum degree* k *game on the edge set of* H *in at most* kn *moves.*

Proof. We define a new graph H^*, where $H^* = H$ if all the degrees in H are even, and otherwise H^* is the graph obtained from H by adding a new vertex v^* and connecting it to every vertex of odd degree in H. Since all degrees of H^* are even, it admits an Eulerian orientation $\overrightarrow{H}^*$. For every $v \in V(H)$, let $E(v) = \{\{v, u\} \in E(H) \ : \ \overrightarrow{(v, u)} \in E(\overrightarrow{H}^*)\}$. Clearly, $|E(v)| \geq \left\lceil \frac{d_H(v)-1}{2} \right\rceil$ and the sets $\{E(v)\}_{v \in V(H)}$ are pairwise disjoint. In every round, if Breaker claims an edge of $E(v)$, then Maker responds by claiming an edge from $E(v)$, unless he already has k edges incident with v in which case Maker proceeds by claiming an edge from $E(u)$, where u is some vertex such that Maker did not yet claim k of its incident edges (if no such vertex exists, then the game was already won by Maker). Note that as $\left\lceil \frac{|E(v)|-1}{2} \right\rceil \geq \left\lceil \frac{d_H(v)-2}{4} \right\rceil \geq \left\lceil \frac{4k-2}{4} \right\rceil \geq k$, Maker can always play according to this strategy until he claims k edges incident with v. Disregarding the orientation, after at most kn moves, the graph spanned by Maker's edges has minimum degree at least k as claimed. □

2.3 (R, c)-expanders

Let us first define the type of expanders we wish to study.

Definition 2.5. For every $c > 0$ and every positive integer R we say that a graph $G = (V, E)$ is an (R, c)-*expander* if every subset of vertices $U \subseteq V$ of cardinality $|U| \leq R$ satisfies $|N_G(U)| \geq c \cdot |U|$. We denote the graph property of being an (R, c)-expander by $\mathcal{X}_{R,c}$.

Remark 2.6. From the above definition it clearly follows that for every $c > 0$ and every positive integer R (both c and R can be functions of the number of vertices of the graph in question), the graph property $\mathcal{X}_{R,c}$ is monotone increasing.

Next, we consider some structural properties of (R, c)-expanders. The following two claims show that the removal or addition of subsets that satisfy certain properties result in graphs that are still expanders. These properties will allow us to slightly modify certain expanders without losing their expansion properties.

Claim 2.7. *If $G = (V, E)$ is an (R, c)-expander and $U \subseteq V$ is a subset of vertices such that no two vertices of U have a common neighbor in G, then $G[V \setminus U]$ is an $(R, c-1)$-expander.*

Proof. Let $S \subseteq V \setminus U$ be a set of cardinality $|S| \leq R$. It follows by our assumption on U that $|N_G(v) \cap U| \leq 1$ holds for every vertex $v \in S$. Hence $|N_{G[V \setminus U]}(S)| \geq |N_G(S)| - |S| \geq (c-1)|S|$. □

Claim 2.8. *Let $G = (V, E)$ be a graph, let $c > 0$, and let R be a positive integer. Let $U \subseteq V$ be a subset of vertices such that $d_G(u) \geq (c-1)$ for every $u \in U$, and, moreover, there is no path of length at most 4 in G whose (possibly identical) endpoints lie in U. If $G[V \setminus U]$ is an (R, c)-expander, then G is an $(R, c-1)$-expander.*

Proof. Let $V' = V \setminus U$ and let $H = G[V']$. Let $S \subseteq V$ be of cardinality $s \leq R$, and let $S_1 = S \cap U$ and let $S_2 = S \setminus S_1$ with respective cardinalities s_1 and $s_2 = s - s_1$. Our assumptions on U imply that it is independent and that for every $U' \subseteq U$ we have that $|N_G(U')| \geq (c-1)|U'|$. It follows that $N_G(S_1) \subseteq V \setminus U$. Furthermore, $N_G(S_1)$ can contain at most one vertex from each set $\{\{t\} \cup N_H(t)\}_{t \in V'}$, and hence $|N_G(S_1) \cap (S_2 \cup N_H(S_2))| \leq |S_2|$. It follows that $N_G(S) \supseteq N_H(S_2) \cup (N_G(S_1) \setminus (N_H(S_2) \cup S_2))$, which implies $|N_G(S)| \geq c \cdot s_2 + (c-1)s_1 - s_2 = (c-1)s$ as claimed. □

Next, we describe some sufficient conditions for a graph $G = (V, E)$ to be an expander (with appropriate parameters). Define:

(M1) $e_G(U) \leq \frac{\delta(G)|U|}{2(c+1)}$ for every subset of vertices $U \subseteq V$ of cardinality $1 \leq |U| < (c+1)r$;

(M2) $e_G(U, W) > 0$ for every pair of disjoint subsets of vertices $U, W \subseteq V$ of cardinality $|U| = |W| = r$.

Lemma 2.9. *For every $c > 0$, if $G = (V, E)$ is a graph which satisfies properties **M1** and **M2** for some positive integer $r \leq \frac{|V|}{c+2}$, then G is a $(\frac{|V|-r}{c+1}, c)$-expander.*

Proof. Set $R = \frac{|V|-r}{c+1}$; note that $R \geq r$ holds by the assumption of the lemma. Assume for the sake of contradiction that there exists a set $S \subseteq V$ of cardinality $|S| \leq R$ for which $|N_G(S)| < c|S|$. Let $T = S \cup N_G(S)$, then $|T| < (c+1)|S|$. If $1 \leq |S| \leq r$, then $|T| < (c+1)r$. Moreover, since all edges that have at least one endpoint in S are spanned by the vertices of T, it follows that $e_G(T) \geq \frac{\delta(G)|S|}{2} > \frac{\delta(G)|T|}{2(c+1)}$, which contradicts property **M1**. If $r < |S| \leq R$, then, since $e_G(S, V \setminus T) = 0$ and $|V \setminus T| > |V| - (c+1)|S| \geq |V| - (c+1)R = r$, we obtain a contradiction to property **M2**. This concludes the proof of the lemma. □

The reason we study (R, c)-expanders is the fact that they entail some pseudo-random properties from which (under some conditions on R and c) some of the natural properties that are considered in this paper follow. We will provide a sufficient conditions for an (R, c)-expander to be k-vertex connected and to admit a perfect matching. Hence by playing for an (R, c)-expander, Maker will be able to win the two games whose goals are the aforementioned two properties (each posing different conditions on R and c). The sufficient condition for a graph to be Hamiltonian, that we will use in the course of the proof, is more delicate than the conditions for k-vertex connectivity and for admitting a perfect matching, and requires some additional ideas, but the heart of the proof will still rely on expanders, and the same expander game.

3 An expander game on pseudo-random graphs

The main object of this section is to describe a general Maker-Breaker game which will reside in the core of all of our proofs. Specifically, the goal of this section is to provide sufficient conditions for $G \in \mathcal{M}_{\mathcal{X}_{R,c}}$, or namely, for a graph G to be Maker's win when Maker's goal is to build an (R, c)-expander. Although this game may seem at first to be an unnatural and artificial game to study, it turns out that this game will lie in the heart of our proofs of all of the results presented in this paper. Given parameters $c > 0$, $0 < \varepsilon < 1$, $K > 0$ and a positive integer $r \leq \frac{|V|}{c+1}$, we define the following two properties of a graph $H = (V, E)$ on n' vertices. These properties, which are closely related to properties **M1** and **M2**, will be needed in the proof of the main result of this section. Define:

(Q1) $e_H(U) \leq \frac{\varepsilon\delta(H)|U|}{10(c+1)}$ for every subset of vertices $U \subseteq V$ of cardinality $1 \leq |U| < (c+1)r$;

(Q2) $e_H(U, W) \geq Kr \ln\left(\frac{n'}{r}\right)$ for every pair of disjoint subsets of vertices $U, W \subseteq V$ of cardinality $|U| = |W| = r$.

Remark 3.1. Whenever we will cite property **Q2** we will give an explicit expression for K which will not necessarily be a constant.

Theorem 3.2. *There exists an integer $n_0 > 0$ such that for every graph $G' = (V, E)$ on $n' \geq n_0$ vertices with minimum degree $\delta(G') > 0$ and for every choice of parameters $\frac{1}{2\delta(G')} < \varepsilon < \frac{1}{2}$, $c > 0$, and integer $0 < r \leq \min\{\frac{n'}{c+2}, \frac{n'}{e^{30}}\}$ for which G' satisfies properties **Q1** and **Q2** with $K = \frac{n'}{r(1-2\varepsilon)}$, Maker can win the $(\frac{n'-r}{c+1}, c)$-expander game on G', that is, $G' \in \mathcal{M}_{\mathcal{X}_{R,c}}$ with $R = \frac{n'-r}{c+1}$.*

Our proof of this theorem will be presented as a series of three lemmata whose composition implies Theorem 3.2 directly.

Lemma 3.3. *There exists an integer $n_0 > 0$ such that for every graph $G' = (V, E)$ on $n' \geq n_0$ vertices with minimum degree $\delta(G') > 0$ and for every choice of parameters $\frac{1}{2\delta(G')} < \varepsilon < \frac{1}{2}$ and integer $0 < r \leq n'/e^4$ for which G' satisfies property **Q2** with $K = \frac{n'}{r(1-2\varepsilon)}$, the edge set E can be split into two disjoint subsets $E = E_1 \cup E_2$ such that the graph $G_1 = (V, E_1)$ has minimum degree $\delta(G_1) \geq \varepsilon\delta(G')$ and the graph $G_2 = (V, E_2)$ satisfies property **Q2** with $K = 3$.*

Proof. Pick every edge of G' to be an edge in G_1 with probability 2ε independently of all other choices. The degree in G_1 of every vertex $v \in V$ is binomially distributed, that is, $d_{G_1}(v) \sim \text{Bin}(d_{G'}(v), 2\varepsilon)$ and thus its median is at least $\lfloor 2\varepsilon\delta(G') \rfloor$. By our choice of ε we have that $\lfloor 2\varepsilon\delta(G') \rfloor > \varepsilon\delta(G')$ and therefore $\Pr[d_{G_1}(v) \geq \varepsilon\delta(G')] > 1/2$. Since the degrees of every two vertices are positively correlated, by the FKG inequality (see e.g. [1, Chapter 6]) we have that

$$\Pr[\delta(G_1) \geq \varepsilon\delta(G')] > 2^{-n'}.$$

Let U, W be a pair of disjoint subsets of vertices of cardinality $|U| = |W| = r$. By our assumption on G' we have that $e_{G'}(U, W) \geq \frac{n' \ln\left(\frac{n'}{r}\right)}{1-2\varepsilon}$. As $e_{G_2}(U, W) \sim \text{Bin}(e_{G'}(U, W), 1-2\varepsilon)$ we have $\mathbf{E}[e_{G_2}(U, W)] \geq n' \ln\left(\frac{n'}{r}\right)$. Applying Theorem 2.1 we have

$$\Pr\left[e_{G_2}(U, W) < 3r \ln\left(\frac{n'}{r}\right)\right] \leq \exp\left(-\frac{\left(1 - \frac{3r}{n'}\right)^2 n' \ln\left(\frac{n'}{r}\right)}{2}\right) \leq \exp\left(-\frac{n' \ln\left(\frac{n'}{r}\right)}{3}\right).$$

By applying the union bound over all pairs of disjoint subsets of vertices of cardinality r each, we conclude that the probability that G_2 violates property **Q2** with $K = 3$ is at most

$$\begin{aligned}
\binom{n'}{r}\binom{n'-r}{r} \exp\left(-\frac{n' \ln\left(\frac{n'}{r}\right)}{3}\right) &\leq \left(\frac{en'}{r}\right)^{2r} \cdot \exp\left(-\frac{n' \ln\left(\frac{n'}{r}\right)}{3}\right) \\
&= \exp\left(2r\left(1 + \ln\left(\frac{n'}{r}\right)\right) - \frac{n' \ln\left(\frac{n'}{r}\right)}{3}\right) \\
&\leq \exp\left(-\frac{n' \ln\left(\frac{n'}{r}\right)}{4}\right) \\
&< 2^{-n'},
\end{aligned}$$

and therefore there exists a partition of G' as claimed. □

The following lemma provides a sufficient condition for a graph $G = (V, E)$ to be a Maker's win in the game $(E, \mathcal{F}_{M2})$, that is, the game on the edge set of G in which Maker's goal is to build a subgraph which satisfies the (monotone increasing) property **M2**. In order to prove this result, we invoke a rather standard technique of studying a dual game in which the roles of Maker and Breaker are exchanged. Note that in the dual game, Breaker (which was the original Maker) is the second player.

Lemma 3.4. *There exists an integer $n_0 > 0$ such that for every graph $G_2 = (V, E_2)$ on $n' \geq n_0$ vertices and for every integer $0 < r \leq n'/e^{30}$ for which G_2 satisfies property **Q2** with $K = 3$, playing on E_2 Maker can build a subgraph of G_2 which satisfies property **M2**.*

Proof. Let G_2 be any graph with vertex set V. In order for Maker to build a graph which satisfies property **M2**, he can adopt the role of Breaker in the game $(E_2, \mathcal{L})$, where $\mathcal{L}$ is the family of edge-sets of all induced bipartite subgraphs of G_2 with both parts of size r. Recall that, by property **Q2** with $K = 3$, the size of every such winning set $L \in \mathcal{L}$ is at least $3r \ln\left(\frac{n'}{r}\right)$. It follows that

$$\begin{aligned}
\sum_{L \in \mathcal{L}} 2^{-|L|} &\leq \sum_{\substack{U \subseteq V \\ |U|=r}} \sum_{\substack{W \subseteq V \setminus U \\ |W|=r}} 2^{-e_{G_2}(U,W)} \\
&\leq \binom{n'}{r}\binom{n'-r}{r} \cdot \exp\left(-3r \ln\left(\frac{n'}{r}\right) \ln 2\right) \\
&\leq \left(\frac{en'}{r}\right)^{2r} \cdot \exp\left(-3r \ln\left(\frac{n'}{r}\right) \ln 2\right) \\
&\leq \exp\left(r \cdot \left(2 \ln\left(\frac{n'}{r}\right) + 2 - \ln 2 \cdot 3 \ln\left(\frac{n'}{r}\right)\right)\right) \\
&< \frac{1}{2}.
\end{aligned}$$

The assertion of the lemma readily follows from Theorem 2.3. □

Lemma 3.5. *There exists an integer $n_0 > 0$ such that for every graph $G' = (V, E)$ on $n' \geq n_0$ vertices and for every choice of parameters $0 < \varepsilon < 1$, $c > 0$ and integer $0 < r \leq \frac{n'}{c+2}$ for which G' satisfies property **Q1** and whose edge set can be partitioned into two disjoint sets $E = E_1 \cup E_2$ where $G_1 = (V, E_1)$ is of minimum degree $\delta(G_1) \geq \varepsilon \cdot \delta(G')$, and $G_2 = (V, E_2)$ satisfies **Q2** with $K = 3$, Maker can win the $(\frac{n'-r}{c+1}, c)$-expander game, that is, $G' \in \mathcal{M}_{\mathcal{X}_{R,c}}$ with $R = \frac{n'-r}{c+1}$.*

Proof. Before the game starts, Maker splits the board into two parts, $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ as indicated in the lemma. Maker then plays two separate games in parallel, one on E_1 and the other on E_2. In every turn in which Breaker claims some edge of E_i, for $i = 1, 2$, Maker responds by claiming an edge of E_i as well (except for maybe once if Breaker has claimed the last edge of E_i). Let H denote the graph built by Maker by the end of the game and set $H_1 = (V, E(H) \cap E_1)$ and $H_2 = (V, E(H) \cap E_2)$.

The game on E_1 is played according to Lemma 2.4. Hence, by the end of the game, Maker's graph H_1 will have minimum degree at least $\delta(H_1) \geq \frac{\delta(G_1)}{5}$. Since G' satisfies property **Q1** and $\delta(G_1) \geq \varepsilon\delta(G')$ it follows that, for every $U \subseteq V$ of cardinality $1 \leq |U| < (c+1)r$, the number of Maker's edges with both endpoints in U is $e_H(U) \leq e_{G'}(U) \leq \frac{\varepsilon\delta(G')|U|}{10(c+1)} \leq \frac{\delta(G_1)|U|}{10(c+1)} \leq \frac{\delta(H_1)|U|}{2(c+1)} \leq \frac{\delta(H)|U|}{2(c+1)}$. Hence, H satisfies property **M1**.

The game on E_2 is played according to Lemma 3.4. Hence, by the end of the game, Maker will build a graph H_2 which satisfies property **M2**. By the monotonicity of **M2**, this property also holds for H. Noting that H, n', r and c satisfy the conditions of Lemma 2.9, we deduce that $H \in \mathcal{M}_{\mathcal{X}_{R,c}}$, that is, Maker's graph is an (R, c)-expander as claimed. □

4 Properties of random graphs and random graph processes

We consider the random graph model we are interested in, the random graph process. For every fixed integer $k \geq 1$ we define two functions as follows:

$$m_k = \binom{n}{2} \frac{\ln n + (k-1) \ln \ln n - \ln \ln \ln n}{n}; \quad (5)$$

$$M_k = \binom{n}{2} \frac{\ln n + (k-1) \ln \ln n + \ln \ln \ln n}{n}. \quad (6)$$

The following lemma (see e.g. [8]) describes a fairly precise behavior of the minimum degree of the random graph process.

Lemma 4.1. *For every fixed integer $k \geq 1$, if $\widetilde{G}$ is the random graph process, then w.h.p.*

$$m_k < \tau(\widetilde{G}; \delta_k) < M_k.$$

Let $G = (V, E)$ be a graph on n vertices and, for a positive integer t, let

$$\mathcal{D}_t = \mathcal{D}_t(G) = \{v \in V \;:\; d_G(v) < t\}. \quad (7)$$

Remark 4.2. Let $\widetilde{G} = \{G_i\}_{i=0}^{\binom{n}{2}}$ be the random graph process, then $\mathcal{D}_t(G_{i-1}) \supseteq \mathcal{D}_t(G_i)$ holds for every $1 \leq i \leq \binom{n}{2}$.

Next, we prove and cite some structural properties of the set $\mathcal{D}_t(\mathcal{G}(n, M)) = \mathcal{D}_t(G_M)$. In order to prove these results, we resort to the use of $\mathcal{G}(n, p)$, where the analysis is much simpler, and then use Claim 1.2 to transfer the results to the random graph model $\mathcal{G}(n, M)$.

Claim 4.3. *For every integer $t \leq \ln^{0.9} n$, if $\widetilde{G} = \{G_i\}_{i=0}^{\binom{n}{2}}$ is the random graph process and $M \geq m_1$, then w.h.p. $|\mathcal{D}_t(G_M)| \leq n^{0.3}$.*

Proof. Set $p = M/\binom{n}{2}$ and let $G \sim \mathcal{G}(n, p)$. By (5) we have that $p > 0.9 \ln n/n$. Fix a subset $U \subseteq V(G)$ of cardinality $|U| = \lfloor n^{0.3} \rfloor$. We upper bound the probability that every vertex of U has strictly less than t neighbors outside of U. Let $N = |V(G) \setminus U| = (1 - o(1))n$ and let $u \in U$ be an arbitrary vertex, then $e_G(u, V \setminus U) \sim \text{Bin}(N, p)$, and therefore

$$\begin{aligned}
\Pr\left[e_G(u, V(G) \setminus U) < t\right] &\leq \sum_{i=0}^{t} \binom{N}{i} p^i (1-p)^{N-i} \\
&\leq \sum_{i=0}^{t} \binom{n}{i} p^i (1-p)^{n-i} \\
&\leq \sum_{i=0}^{t} \exp\left\{i \cdot \ln(np) - p(n-i)\right\} \\
&\leq n^{-0.89}
\end{aligned}$$

Since the numbers of edges emitting out of U from each vertex of U are independent random variables (each counting the appearance of edges in a set disjoint of all others), it follows that the probability that every vertex of U has strictly less than t neighbors outside of U, is at most $n^{-0.89|U|}$. There are $\binom{n}{|U|}$ subsets of this cardinality. Hence, applying the union bound over all of these sets entails that the probability there exists such a set U is at most

$$\binom{n}{|U|} \cdot n^{-0.89|U|} \leq \exp\left(|U| \cdot (1 + \ln \frac{n}{|U|} - 0.89 \ln n)\right) \leq \exp\left(-0.18|U| \ln n)\right) \leq e^{-n^{0.3}}.$$

By the definition of $\mathcal{D}_t(G)$ all its vertices have less than t edges emitting out of it, hence the probability that $|\mathcal{D}_t(G)| > n^{0.3}$ is at most $e^{-n^{0.3}}$. Applying Claim 1.2 we have that

$$\Pr\left[|\mathcal{D}_t(G_M)| > n^{0.3}\right] \leq 3\sqrt{M} \cdot \exp\left(-n^{0.3}\right) < 2\sqrt{n \ln n} \cdot \exp\left(-n^{0.3}\right) = o(1).$$

This concludes the proof of the claim. □

Claim 4.4. *For every fixed integer $k \geq 1$ and for every integer $t \leq \ln^{0.9} n$, if $\widetilde{G} = \{G_i\}_{i=0}^{\binom{n}{2}}$ is the random graph process and $M = \tau(\widetilde{G}; \delta_k)$, then w.h.p $G = G_M$ does not contain a non-empty path of length at most 4 such that both of its (possibly identical) endpoints lie in $\mathcal{D}_t(G_M)$.*

Proof. Clearly, it suffices to consider the case $t = \ln^{0.9} n$. We will prove the claim for two distinct endpoints in $\mathcal{D}_t(G_M)$, and for paths of length $2 \leq r \leq 4$ between them, where the other cases are similar (and a little simpler). By Lemma 4.1 we can assume that $m_k < \tau(\widetilde{G}; \delta_k) < M_k$, and hence it follows by Remark 4.2 that $\mathcal{D}_t(G_M) \subseteq \mathcal{D}_t(G_{m_k}) := D$. Our analysis will consist of two stages. First we show that w.h.p. there is no path of length at most 4 that connects two vertices of D. Second we show that w.h.p. none of the edges that were added during the random graph process between time m_k and time M_k, that is, the edges in $E(G_{M_k}) \setminus E(G_{m_k})$, create such a path.

Let $G = G_{m_k} \sim \mathcal{G}(n, m_k)$ and let $P = (v_0, \ldots, v_r)$ be a sequence of $r + 1$ vertices of $V(G)$, where $2 \leq r \leq 4$. Our first goal is to bound the probability that P forms a path of length r in G such that v_0 and v_r are both in D. Denote by $\mathcal{A}_P$ the event $\{v_i, v_{i+1}\} \in E(G)$ for every $0 \leq i \leq r - 1$, we have

$$\Pr\left[\mathcal{A}_P \wedge \{v_0, v_r\} \subseteq D\right] = \Pr\left[\mathcal{A}_P\right] \cdot \Pr\left[\{v_0, v_r\} \subseteq D \mid \mathcal{A}_P\right].$$

Denoting $N = \binom{n}{2}$ and recalling (5) we have

$$\Pr\left[\mathcal{A}_P\right] = \frac{\binom{N-r}{m_k - r}}{\binom{N}{m_k}} < 1.01 \left(\frac{\ln n}{n}\right)^r. \tag{8}$$

Next, we note that $\Pr\left[\{v_0, v_r\} \subseteq D \mid \mathcal{A}_P\right] \leq \Pr\left[e_G(\{v_0, v_r\}, V \setminus \{v_0, v_r\}) \leq 2t \mid \mathcal{A}_P\right]$. Conditioning on $\mathcal{A}_P$ implies that the two edges $\{v_0, v_1\}$ and $\{v_{r-1}, v_r\}$ are present in G. It follows that $[e_G(\{v_0, v_r\}, V \setminus \{v_0, v_r\})| \mathcal{A}_P] - 2$ is distributed according to the hypergeometric distribution with parameters $N - r$, $m_k - r$, and $2n - 6$.

Putting everything together we conclude that

$$\begin{aligned}
\Pr\left[\{v_0, v_r\} \subseteq D \mid \mathcal{A}_P\right] &\leq \Pr\left[e_G(\{v_0, v_r\}, V \setminus \{v_0, v_r\}) - 2 \leq 2t - 2 \mid \mathcal{A}_P\right] \\
&\leq \sum_{j=0}^{2t-2} \binom{2n-6}{j} \cdot \frac{\binom{N-r-2n+6}{m_k-r-j}}{\binom{N-r}{m_k-r}} \\
&\leq 2t \cdot \binom{2n}{2t} \cdot \left(\frac{m_k - r}{N - m_k + 2t}\right)^{2t} \cdot \left(\frac{N-r-2n+6}{N-r}\right)^{m_k-r-2t} \\
&\leq 2t \cdot \left(\frac{en(m_k - r)}{t(N - m_k + 2t)}\right)^{2t} \cdot \exp\left(-(m_k - r - 2t) \cdot \frac{2n-6}{N-r}\right) \\
&\leq 2t \cdot \exp\left(-1.9 \ln n\right) \\
&\leq n^{-1.8}.
\end{aligned}$$

Hence, applying a union bound argument over all such sequences of $r+1$ vertices, we conclude that the probability there exists a path in G of length $2 \leq r \leq 4$ connecting two distinct vertices of D is at most $\sum_{r=2}^{4} n^{r+1} \cdot 1.01 \left(\frac{\ln n}{n}\right)^r \cdot n^{-1.8} = o(1)$.

In light of the above, we can assume that after m_k steps the random graph process does not admit a short path connecting two vertices of D. Moreover, by Claim 4.3 we can assume that $|D| \leq n^{0.3}$. Now, let $m_k < M' \leq M_k$, set $H = G_{M'-1}$, and let e be the edge added at step M' (that is, $e = E(G_{M'}) \setminus E(H)$). We upper bound the probability that e creates a short path which connects two vertices of $D \supseteq \mathcal{D}_t(H)$. We note that a standard application of the Chernoff bound (Theorem 2.1) in conjunction with Claim 1.2 implies that w.h.p. the maximum degree of H satisfies $\Delta(H) \leq 2 \ln n$. Let U be the set of vertices at distance at most 3 from $\mathcal{D}_t(H)$. If e closes a short path that connects two vertices of $\mathcal{D}_t(H)$, then both endpoints of e must lie in U. Clearly $|U| \leq |D| \cdot \Delta(H)^3 \leq 8n^{0.3} \ln^3 n$. It follows that the probability that e is chosen within this set is at most $\frac{\binom{|U|}{2}}{N-M'} = o(n^{-1.3})$. Since $M_k - m_k = O(n \ln \ln \ln n)$, applying a union bound argument over all integral values of $m_k \leq M' \leq M_k$, implies that the probability that such an edge is selected is $o(1)$. This concludes the proof of the claim. $\square$

Claim 4.5. *For every fixed integer $k \geq 2$, if $\widetilde{G} = \{G_i\}_{i=0}^{\binom{n}{2}}$ is the random graph process and $M = \tau(\widetilde{G}; \delta_k)$, then w.h.p. $G_M = (V, E)$ is such that $e_{G_M}(U) < |U| \ln^{0.8} n$ for every subset of vertices $U \subseteq V$ of cardinality $1 \leq |U| \leq \frac{n}{\ln^{0.3} n}$.*

Proof. By Lemma 4.1 we can assume that $M < M_k$. As the complement of the property at hand is monotone increasing, it follows by Proposition 1.1 that it suffices to prove that, if $p = p(n) \leq 2 \ln n/n$ and $G \sim \mathcal{G}(n, p)$, then the probability that there exists a subset $U \subseteq V$ of cardinality $1 \leq |U| \leq \frac{n}{\ln^{0.3} n}$ such that $e_G(U) \geq |U| \ln^{0.8} n$, tends to 0 as n tends to infinity. Fix a subset U of cardinality $1 \leq u \leq n \cdot \ln^{-0.3} n$, then $e_G(U) \sim Bin(\binom{u}{2}, p)$. Since $u \cdot \ln^{0.8} n \geq \binom{u}{2} \cdot p$, we can apply Lemma 2.2 to upper bound the probability that $e_G(U)$ is too large. We can then upper bound the probability of the claim being violated, by applying a union bound argument as follows

$$\begin{aligned}
\sum_{u=1}^{n\cdot\ln^{-0.3} n} \binom{n}{u} \Pr\left[e_G(U) \geq u \cdot \ln^{0.8} n\right] &\leq \sum_{u=1}^{n\cdot\ln^{-0.3} n} \left(\frac{en}{u}\right)^u \cdot \left(\frac{e\binom{u}{2}p}{u \cdot \ln^{0.8} n}\right)^{u\cdot\ln^{0.8} n} \\
&\leq \sum_{u=1}^{n\cdot\ln^{-0.3} n} \left(e^{\ln^{0.8} n+1} \cdot \left(\frac{u}{n}\right)^{\ln^{0.8} n-1} \cdot (\ln^{0.2} n)^{\ln^{0.8} n}\right)^u \\
&\leq \sum_{u=1}^{n\cdot\ln^{-0.3} n} \left(4 \cdot \left(\frac{u}{n}\right)^{0.99} \cdot (\ln^{0.2} n)\right)^{u\ln^{0.8} n} \\
&\leq \sum_{u=1}^{n\cdot\ln^{-0.3} n} \left(\ln^{-0.09} n\right)^{u\ln^{0.8} n} \\
&= o(1),
\end{aligned}$$

where the last equality follows from the fact that we are summing a geometric series with a first element and quotient both being $o(1)$. This concludes the proof of the claim. $\square$

Claim 4.6. *For every fixed integer $k \geq 1$ and for an integer $r = \frac{n}{2\ln^{0.4} n}$, if $\widetilde{G} = \{G_i\}_{i=0}^{\binom{n}{2}}$ is the random graph process and $M = \tau(\widetilde{G}; \delta_k)$, then w.h.p. $e_{G_M}(U, W) \geq n \ln^{0.1} n$ for every pair of disjoint subsets $U, W \subseteq V(G_M)$ of cardinality $|U| = |W| = r$.*

Proof. By Lemma 4.1 we can assume that $M > m_k$. As the property at hand is monotone increasing, it follows by Proposition 1.1 that it suffices to prove the claim for $G \sim \mathcal{G}(n, p)$ with $p \geq \frac{\ln n}{n}$. Fix a pair of disjoint subsets $U, W \subseteq V(G)$ of cardinality r each. Then $e_G(U, W) \sim \text{Bin}(r^2, p)$, and thus $\mathbf{E}\left[e_G(U, W)\right] \geq \frac{n\ln^{0.2} n}{4}$. We upper bound the probability that $e_G(U, W)$ is too small using Theorem 2.1. We can then upper bound the probability of the claim being violated, by applying a union bound argument as follows

$$\begin{aligned}
\binom{n}{r}\binom{n-r}{r} \Pr\left[e_G(U, W) < n\ln^{0.1} n\right] &\leq \left(\frac{en}{r}\right)^{2r} \exp\left(-\frac{\left(1 - \frac{4}{\ln^{0.1} n}\right)^2 r^2 p}{2}\right) \\
&\leq \exp\left(r\left(2 + \ln\ln n - \frac{\ln^{0.2} n}{10}\right)\right) \\
&= o(1).
\end{aligned}$$

This concludes the proof of the claim. $\square$

Finally, we prove that removing vertices of small degree from a random graph with an appropriate number of edges typically results in a graph on which Maker can win the expander game. In fact, we even show that Maker can win the game when this graph is thinned substantially (that is, the vast majority of edges are removed). This stronger property will play a crucial role in the proof of Theorem 3. Our proof will make use, in particular, of results we have obtained in Claims 4.3, 4.5, 4.6 and in Theorem 3.2.

Lemma 4.7. *For every $\alpha > 0$ and for every fixed integer $k \geq 2$, if $\widetilde{G} = \{G_i\}_{i=0}^{\binom{n}{2}}$ is the random graph process and $M = \tau(\widetilde{G}; \delta_k)$, then w.h.p. $G' = (V', E') := G_M \setminus \mathcal{D}_{\ln^{0.9} n}(G_M)$ on n' vertices contains a spanning subgraph $\widehat{G} \subseteq G'$ with at most $2n' \ln^{0.97} n'$ edges, such that $\widehat{G} \in \mathcal{M}_{\mathcal{X}_{R,c}}$ for every $0 < c \leq \ln^{0.02} n'$ and $R \leq (1 - \alpha)\frac{n'}{c+1}$.*

Remark 4.8. As was noted in Remark 2.6, by the monotonicity of $\mathcal{X}_{R,c}$, the above lemma can be used to deduce that $G' \in \mathcal{M}_{\mathcal{X}_{R,c}}$.

Proof. Pick every edge of G' to be an edge of $\widehat{G}$ with probability $\gamma = \ln^{-0.03} n$, independently of all other choices. Our goal is to prove that, with positive probability, $\widehat{G}$ satisfies the conditions of Theorem 3.2, with parameters

$$\varepsilon = \gamma \qquad \text{and} \qquad r = \frac{n'}{\ln^{0.4} n'}.$$

Based on typical properties of the random graph process, we can assume that G' satisfies the following properties:

1) $\delta(G') \geq \ln^{0.9} n$;
2) $e(G') \leq e(G_M) \leq e(G_{M_k}) \leq (1+o(1))\frac{n \ln n}{2}$ (Lemma 4.1);
3) $|\mathcal{D}_{\ln^{0.9} n}(G_M)| \leq n^{0.3}$, and therefore $n' \geq n(1-n^{-0.7})$ (Claim 4.3);
4) Every set $U \subseteq V'$ of cardinality $|U| \leq (c+1)r \leq \frac{n}{\ln^{0.3} n}$ satisfies $e_{G'}(U) = e_{G_M}(U) \leq |U| \ln^{0.8} n \leq |U| \ln^{0.81} n'$ (Claim 4.5);
5) Every pair of disjoint subsets $U, W \subseteq V'$ of cardinality $|U| = |W| = r \geq \frac{n}{2\ln^{0.4} n}$ satisfies $e_{G'}(U,W) \geq n \ln^{0.1} n$ (Claim 4.6).

It follows that our choice of parameters meets the requirements on ε, c and r, made in Theorem 3.2.

We proceed to prove that, with a "not too small" probability, $\widehat{G}$ satisfies property **Q1**. First note that every set $U \subseteq V'$ of cardinality $|U| \leq (c+1)r$ satisfies $e_{\widehat{G}}(U) \leq e_{G'}(U) \leq |U| \ln^{0.81} n'$. The degree in $\widehat{G}$ of every vertex $v \in V'$ is binomially distributed, $d_{\widehat{G}}(v) \sim \mathrm{Bin}(d_{G'}(v), \gamma)$, with median at least $\lfloor \gamma\delta(G') \rfloor$. Therefore $\Pr\left[d_{\widehat{G}}(v) \geq \lfloor \gamma\delta(G') \rfloor\right] \geq 1/2$. Since $\delta(G') \geq \ln^{0.9} n$ and since the degrees of every two vertices are positively correlated, using the FKG inequality (see e.g. [1, Chapter 6]) we have that

$$\Pr\left[\delta(\widehat{G}) \geq \lfloor \ln^{0.87} n \rfloor\right] \geq \Pr\left[\delta(\widehat{G}) \geq \lfloor \gamma\delta(G') \rfloor\right] \geq 2^{-n'}.$$

It follows that with probability at least $2^{-n'}$ we have $\frac{\varepsilon\delta(\widehat{G})}{10(c+1)} > \ln^{0.81} n'$, and thus $\widehat{G}$ satisfies property **Q1** with probability at least $2^{-n'}$.

Next, we prove that, with "very large" probability, $\widehat{G}$ satisfies property **Q2**. Fixing a pair of disjoint sets of vertices $U, W \subseteq V'$ of cardinality r each, it clearly follows that $e_{\widehat{G}}(U,W) \sim \mathrm{Bin}(e_{G'}(U,W), \gamma)$, and thus $\mathbf{E}\left[e_{\widehat{G}}(U,W)\right] \geq n \ln^{0.07} n > n' \ln^{0.07} n'$. Since $\frac{n'}{(1-2\varepsilon)} \ln\left(\frac{n'}{r}\right) \leq n' \ln^{0.05} n'$, we can upper bound the probability that the pair U, W does not satisfy property **Q2** with $K = \frac{n'}{r(1-2\varepsilon)}$, using Theorem 2.1 as follows.

$$\Pr\left[e_{\widehat{G}}(U,W) < n' \ln^{0.05} n'\right] \leq \exp\left(-\frac{(1-\ln^{-0.02} n')^2 n' \ln^{0.07} n'}{2}\right) \leq \exp\left(-\frac{n' \ln^{0.07} n'}{3}\right).$$

Applying a simple union bound argument we deduce that the probability that there exists a pair of disjoint subsets of vertices, of cardinality r each, which does not satisfy property **Q2** with $K = \frac{n'}{r(1-2\varepsilon)}$ is at most

$$\binom{n'}{r} \cdot \binom{n'-r}{r} \cdot \exp\left(-\frac{n' \ln^{0.07} n'}{3}\right) \leq \exp\left(2r \ln\left(\frac{en'}{r}\right) - \frac{n' \ln^{0.07} n'}{3}\right) \leq \exp\left(-\frac{n' \ln^{0.07} n'}{4}\right).$$

Finally, note that $e(\widehat{G}) \sim \text{Bin}(e(G'), \gamma)$ and thus $\mathbf{E}\left[e(\widehat{G})\right] = (1-o(1))\frac{n' \ln^{0.97} n'}{2}$. Hence, using Theorem 2.1 we deduce that

$$\Pr\left[e(\widehat{G}) > 2n' \ln^{0.97} n'\right] < \exp\left(-\frac{(1-o(1))n' \ln^{0.97} n'}{6}\right).$$

Putting everything together we conclude that $\exp\left(-\frac{(1-o(1))n' \ln^{0.97} n'}{6}\right) + \exp\left(-\frac{n' \ln^{0.07} n'}{4}\right) < 2^{-n'}$. Hence, there exists a subgraph $\widehat{G} \subseteq G'$ with at most $2n' \ln^{0.97} n'$ edges which satisfies the conditions of Theorem 3.2. It follows that $\widehat{G} \in \mathcal{M}_{\mathcal{X}_{R,c}}$ as claimed. □

5 Hitting time of the k-vertex connectivity and perfect matching games

This short section is devoted to the proofs of Theorems 1 and 2. These two theorems are simple corollaries of the results presented in the previous sections.

5.1 k-vertex connectivity

As already mentioned in Section 2 we will provide a sufficient condition on R and c such that an (R,c)-expander will surely be k-vertex connected.

Lemma 5.1. *For every positive integer k, if $G=(V,E)$ is an (R,c)-expander such that $c \geq k$, and $Rc \geq \frac{1}{2}(|V|+k)$, then $G \in \mathcal{VC}_k$.*

Proof. Assume for the sake of contradiction that there exists some set $S \subseteq V$ of size $|S| \leq k-1$ whose removal disconnects G. Denote the connected components of $G \setminus S$ by $S_1, \ldots, S_t$, where $t \geq 2$ and $1 \leq |S_1| \leq \ldots \leq |S_t|$. If $|S_1| \leq R$, then $k-1 \geq |S| \geq |N_G(S_1)| \geq c|S_1| \geq c \geq k$, which is clearly a contradiction. Assume then that $|S_1| > R$. For $i \in \{1,2\}$, let $A_i \subseteq S_i$ be an arbitrary subset of size R. It follows that $|V| \geq |S_1 \cup S_2 \cup N_G(S_1) \cup N_G(S_2)| \geq |N_G(A_1) \cup N_G(A_2)| = |N_G(A_1)| + |N_G(A_2)| - |N_G(A_1) \cap N_G(A_2)| \geq 2Rc - |S| \geq |V|+1$, which is clearly a contradiction. It follows that G is k-vertex-connected as claimed. □

In order to prove Theorem 1 it thus suffices to show that w.h.p. at the moment the random graph process first reaches minimum degree $2k$, Maker has a winning strategy for the (R,c)-expander game for suitably chosen values of R and c. In doing so we will heavily rely on Theorem 3.2.

Proof of Theorem 1. Fix some positive integer $k \geq 1$ and let $\widetilde{G} = \{G_i\}_{i=0}^{\binom{n}{2}}$ denote the random graph process. Set $M = \tau(\widetilde{G}; \delta_{2k})$, let $G = G_M$, $\textsc{Small} = \mathcal{D}_{\ln^{0.9} n}(G)$, $G' = G[V \setminus \textsc{Small}]$ and denote by n' the number of vertices in G'. Setting $c = k+2$, and $R = \frac{n'}{k+4}$, the conditions of Lemma 4.7 are met, and thus $G' \in \mathcal{M}_{\mathcal{X}_{\frac{n'}{k+4}, k+2}}$.

Maker's strategy will consist of splitting the board into $F_1 = E(G')$ and $F_2 = E_G(\textsc{Small}, V \setminus \textsc{Small})$, and playing the corresponding two games in parallel, that is, in each move Maker will claim an edge of the board Breaker chose his last edge from (except for possibly his last move in one of the two games). Playing on the edges of F_1, Maker aims to build an $(\frac{n'}{k+4}, k+2)$-expander. As noted above, Maker has a winning strategy for this game. Playing on the edges of F_2, Maker follows a simple pairing strategy which guarantees that, by the end of the game, the graph H which Maker constructs will satisfy $d_H(v) \geq \lfloor d_G(v)/2 \rfloor$ for every $v \in$ SMALL. To achieve this goal, whenever Breaker claims an edge which is incident with some vertex $v \in$ SMALL, Maker responds by claiming a different edge incident with v if such an edge exists, and otherwise he claims an arbitrary free edge of $F_1 \cup F_2$. Since the minimum degree in G is $2k$, it follows by Maker's strategy for the game on F_2 and by Claim 4.4, that in Maker's graph H, the vertices of SMALL form an independent set with k edges emitting out of each vertex. Since the graph $H' = H[V \setminus \text{SMALL}]$ is an $(\frac{n'}{k+4}, k+2)$-expander, and since $(k+2) \cdot \frac{n'}{k+4} \geq \frac{1}{2}(n+k)$ holds for every $k \geq 1$ by Claim 4.3, Lemma 5.1 implies that $H' \in \mathcal{VC}_k$. Adding to H' the vertices of SMALL with their incident edges clearly keeps the k-vertex connectivity property, as connecting a new vertex to at least k vertices of a k-vertex connected graph produces a k-vertex connected graph. This concludes the proof of the theorem. □

5.2 Perfect matching

Next, in order to show that expansion entails admitting a perfect matching, we make use of the well-known Berge-Tutte formula for the size of a maximum matching in a graph (see e.g. [27, Corollary 3.3.7]).

Theorem 5.2 (Berge-Tutte)**.** *The maximum number of vertices which are saturated by a matching in a graph $G = (V, E)$ is $\min_{S \subseteq V} \{|V| + |S| - odd(G - S)\}$.*

The following lemma is applicable regardless of the parity of the number of vertices in the graph.

Lemma 5.3. *If $G = (V, E)$ is an (R, c)-expander such that $c \geq 2$ and $(c+1)R \leq |V| < 2Rc - 8c$, then $G \in \mathcal{PM}$.*

Proof. From the conditions on R and c it follows that $Rc > |V|/2$ and, combined with G being an (R, c)-expander, this trivially implies that the graph G must be connected. Setting $S = \emptyset$, we have that $odd(G-S) = 1$ for odd $|V|$, and that $odd(G - S) = 0$ for even $|V|$. By Theorem 5.2 we can thus assume that $S \neq \emptyset$. We will in fact prove that $|S| \geq conn(G - S)$ holds for every non-empty $S \subseteq V$. It clearly suffices to prove this for every $\emptyset \neq S \subseteq V$ of cardinality $|S| \leq |V|/2$. Let S be such a set, let $t = conn(G - S)$, and let $S_1, \ldots, S_t$ denote the connected components of $G - S$, where $1 \leq |S_1| \leq \ldots \leq |S_t|$. Assume first that there exists a set $A \subseteq \{1, \ldots, t\}$ such that $|S|/c < \left|\bigcup_{i \in A} S_i\right| \leq R$. By definition we have $N_G(\bigcup_{i \in A} S_i) \subseteq S$. It follows that $|S| \geq |N_G(\bigcup_{i \in A} S_i)| \geq c \left|\bigcup_{i \in A} S_i\right| > |S|$, which is clearly a contradiction. Hence, no such $A \subseteq \{1, \ldots, t\}$ exists. It follows that there must exist some $0 \leq j^* \leq t$ such that $\sum_{i=1}^{j^*} |S_i| \leq \lfloor |S|/c \rfloor$ and $|S_i| > R - |S|/c$ for every $j^* < i \leq t$. If $j^* \geq t-1$, then, since $|S_i| \geq 1$ for every $1 \leq i \leq t$, it follows that $t \leq \sum_{i=1}^{t-1} |S_i| + 1 \leq \lfloor |S|/c \rfloor + 1 \leq |S|$. Hence, we can assume that $j^* \leq t-2$. We split our analysis of this case into two subcases. First, assume that $1 \leq |S| < \frac{Rc}{2}$ or equivalently, that $c(R - |S|/c) > |S|$. If $R - |S|/c \leq |S_{j^*+1}| \leq R$, then, as $S \supseteq N_G(S_{j^*+1})$ we have that $|S| \geq |N_G(S_{j^*+1})| \geq c(R - |S|/c) > |S|$, a contradiction. Therefore, in this subcase $|S_i| > R$ holds for every $j^* < i \leq t$. Since $j^* \leq t-2$, for $i \in \{t-1, t\}$, we can choose $A_i \subseteq S_i$ to be an arbitrary subset of size R. It follows that $|V| = \sum_{i=1}^{t} |S_i| + |S| \geq t - 2 + |S_{t-1} \cup S_t \cup N_G(S_{t-1}) \cup N_G(S_t)| \geq t - 2 + |N_G(A_{t-1}) \cup N_G(A_t)| = t-2+|N_G(A_{t-1})|+|N_G(A_t)|-|N_G(A_{t-1}) \cap N_G(A_t)| \geq t-2+2Rc-|S| > |V|+8c+t-2-|S|$, which implies $|S| > t+8c-2 > t$. This completes the proof of the first subcase. Second, we assume that $Rc/2 \leq |S| \leq |V|/2$; it follows that $|S| > |V|/4$. Note that under our assumption on R and c we have that $R - |S|/c > 4$, and therefore $|S_i| \geq 5$ for every $j^* < i \leq t$. Moreover, since $|S_i| \geq 1$ holds for every $1 \leq i \leq j^*$, it follows that

$j^* \leq |S|/c$. Putting everything together we have that $|S| > \frac{1}{3}\sum_{i=1}^{t}|S_i| \geq \frac{j^*+(t-j^*)(R-|S|/c)}{3} \geq \frac{5t-4j^*}{3}$, and therefore $t < \frac{|S|}{5}(3+\frac{4}{c}) \leq |S|$. This concludes the proof of the lemma. $\square$

In order to prove Theorem 2 we proceed very similarly to the proof of Theorem 1.

Proof of Theorem 2. Let $\widetilde{G} = \{G_i\}_{i=0}^{\binom{n}{2}}$ denote the random graph process. Set $M = \tau(\widetilde{G}; \delta_2)$, let $G = G_M$, $\text{SMALL} = \mathcal{D}_{\ln^{0.9} n}(G)$, $G' = G[V \setminus \text{SMALL}]$ and denote by n' the number of vertices in G'. Setting $c = 8$, and $R = \frac{n'}{10}$, the conditions of Lemma 4.7 are met, and thus $G' \in \mathcal{M}_{\mathcal{X}_{\frac{n'}{10},8}}$.

Maker's strategy is quite similar to the one presented in the proof of Theorem 1. He splits the board into $F_1 = E(G')$ and $F_2 = E_G(\text{SMALL}, V \setminus \text{SMALL})$, and plays the corresponding two games in parallel, that is, in each move Maker will claim an edge of the board Breaker chose his last edge from (except for possibly his last move in one of the two games). Playing on the edges of F_1, Maker aims to build an $(n'/10, 8)$-expander. As noted above, Maker has a winning strategy for this game. We denote the restriction of the graph built by Maker by the end of the game to the edges of F_1 by H_1. Playing on the edges of F_2, Maker follows a simple pairing strategy which guarantees that, by the end of the game, the graph H_2 which Maker constructs will satisfy $d_{H_2}(v) \geq \lfloor d_G(v)/2 \rfloor$ for every $v \in \text{SMALL}$. To achieve this goal, whenever Breaker claims an edge which is incident with some vertex $v \in \text{SMALL}$, Maker responds by claiming a different edge incident with v if such an edge exists, and otherwise he claims an arbitrary free edge of $F_1 \cup F_2$. Recalling Claim 4.4 we can assume that SMALL is an independent set in G and that no two vertices in SMALL share a common neighbor. As the minimum degree in G is 2, Maker's graph, $H = H_1 \cup H_2$, will contain at least one edge emitting out of every vertex in SMALL, each incident with a different vertex of $V \setminus \text{SMALL}$. Therefore, there exists a matching $\mathcal{M}$ which covers all vertices of SMALL. Let T denote the set of vertices of $V \setminus \text{SMALL}$ which are covered by $\mathcal{M}$. Again, by Claim 4.4 we can assume that no two vertices in T share a common neighbor (as this would create a path of length 4 between two vertices in SMALL). Since, the graph H_1 is an $(n'/10, 8)$-expander, it follows by Claim 2.7 that the graph $H' = H_1 \setminus T$ is an $(n'/10, 7)$-expander. The values $R = n'/10$ and $c = 7$ satisfy the condition of Lemma 5.3, implying that $H' \in \mathcal{PM}$. Let $\mathcal{M}'$ be some perfect matching of H', then $\mathcal{M} \cup \mathcal{M}'$ is a perfect matching of H. This concludes the proof of the theorem. $\square$

6 Hitting time of the Hamiltonicity game

Our proof of Theorem 3 is fairly similar to the two proofs presented in the previous section. However, having built an appropriate expander, Maker will need to claim additional edges in order to transform his expander into a Hamiltonian graph. In order to describe the relevant connection between Hamiltonicity and (R, c)-expanders, we require the notion of *boosters*.

Definition 6.1. For every graph G, we say that a non-edge $\{u, v\} \notin E(G)$ is a *booster* with respect to G, if either $G \cup \{u, v\}$ is Hamiltonian or $\ell(G \cup \{u, v\}) > \ell(G)$. We denote by $\mathcal{B}_G$ the set of boosters with respect to G.

The following is a well-known property of $(R, 2)$-expanders (see e.g. [14]).

Lemma 6.2. *If G is a connected non-Hamiltonian $(R, 2)$-expander, then $|\mathcal{B}_G| \geq R^2/2$.*

Our goal is to show that during a game on an appropriate graph G, assuming Maker can build a subgraph of G which is an (R,c)-expander, he can also claim sufficiently many such boosters, so that his (R,c)-expander becomes Hamiltonian. In order to do so, we further analyze the structure of the random graph process.

Lemma 6.3. *If $\widetilde{G} = \{G_i\}_{i=0}^{\binom{n}{2}}$ is the random graph process and $M = \tau(\widetilde{G}; \delta_4)$, then w.h.p. G_M does not contain a connected non-Hamiltonian $(n/5,2)$-expander Γ with at most $n\ln^{0.98} n$ edges such that $|E(G_M) \cap \mathcal{B}_\Gamma| \le \frac{n\ln n}{100}$.*

Proof. First we note that any $(n/5,2)$-expander must be connected, as each connected component must be of size at least $n/5 + 2n/5 > n/2$. Let $m_4 \le M' \le M_4$ be an integer, let $p = M'/\binom{n}{2} > \frac{\ln n}{n}$, and let $G = (V,E) \sim \mathcal{G}(n,p)$. Our goal is to prove that the probability that G contains a connected non-Hamiltonian $(n/5,2)$-expander subgraph Γ with at most $n\ln^{0.98} n$ edges such that $|E \cap \mathcal{B}_\Gamma| \le \frac{n\ln n}{100}$ is "much smaller" than the probability that $e(G) = M'$. Applying Claim 1.2 for every integer $m_4 \le M' \le M_4$ and then summing over all such integers, will enable us to complete the proof.

Let $\mathcal{S}$ denote the set of all labeled non-Hamiltonian $(n/5,2)$-expanders on the vertex set V which have at most $n\ln^{0.98} n$ edges. Fix a graph $\Gamma = (V,F) \in \mathcal{S}$, then clearly $\Pr[\Gamma \subseteq G] = p^{|F|}$. Now, let $G' = (V, E\setminus F) \sim \mathcal{G}(n,p)_{-F}$. By definition, every booster with respect to Γ is a non-edge in Γ, hence $\mathcal{B}_\Gamma$ is a subset of the potential pairs of the graph G'. Lemma 6.2 implies that $|\mathcal{B}_\Gamma| \ge n^2/50$, and since $|E(G') \cap \mathcal{B}_\Gamma| \sim \text{Bin}(|\mathcal{B}_\Gamma|, p)$, it follows that $\mathbf{E}[|E(G') \cap \mathcal{B}_\Gamma|] \ge \frac{n^2 p}{50} > \frac{n\ln n}{50}$. Applying Theorem 2.1 we have

$$\Pr\left[|E(G') \cap \mathcal{B}_\Gamma| \le \frac{n\ln n}{100}\right] \le \exp\left(-\frac{\left(1-\frac{50}{100}\right)^2 n^2 p}{100}\right) = \exp\left(-\frac{n^2 p}{400}\right).$$

Next, we note that by the independence of appearance of edges in $\mathcal{G}(n,p)$, the event $\Gamma \subseteq G$ and the event that some booster e with respect to Γ was chosen among the edges of G', are independent events. It follows that the probability that G contains a connected non-Hamiltonian $(n/5,2)$-expander Γ with $m \le n\ln^{0.98} n$ edges, such that $|E \cap \mathcal{B}_\Gamma| \le \frac{n\ln n}{100}$ is at most $\binom{\binom{n}{2}}{m} p^m \cdot \exp\left(-\frac{n^2p}{400}\right)$. Applying a union bound argument over all integers $1 \le m \le n\ln^{0.98} n$ we obtain

$$\begin{aligned} & \sum_{m=1}^{n\ln^{0.98} n} \binom{\binom{n}{2}}{m} p^m \cdot \exp\left(-\frac{n^2p}{400}\right) \\ \le \; & \sum_{m=1}^{n\ln^{0.98} n} \left(\frac{en^2p}{2m}\right)^m \cdot \exp\left(-\frac{n^2p}{400}\right) \\ \le \; & \sum_{m=1}^{n\ln^{0.98} n} \exp\left(m \cdot \left(1 + \ln\left(\frac{n^2p}{2m}\right)\right) - \frac{n^2p}{400}\right) \\ \le \; & \exp\left(-\frac{n^2p}{401}\right). \end{aligned}$$

Using Claim 1.2, the above calculation implies that the same event, with $G \sim \mathcal{G}(n,M')$, is upper bounded by $3\sqrt{M'} \cdot \exp\left(-\frac{n^2p}{401}\right) \le \exp\left(\frac{-n\ln n}{402}\right)$. Taking the union bound over all integral values of $m_4 \le M' \le M_4$, we conclude that the probability there exists such an integer M' for which $G_{M'}$ violates the claim is at most $(M_4 - m_4 + 1) \cdot \exp\left(\frac{-n\ln n}{402}\right) \le n\ln\ln\ln n \cdot \exp\left(\frac{-n\ln n}{402}\right) = o(1)$. □

We are now ready to present the full proof of Theorem 3.

Proof of Theorem 3. Let $\widetilde{G} = \{G_i\}_{i=0}^{\binom{n}{2}}$ denote the random graph process. Set $M = \tau(\widetilde{G}; \delta_4)$, let $G = G_M$, $\textsc{Small} = \mathcal{D}_{\ln^{0.9} n}(G)$, $G' = G[V \setminus \textsc{Small}]$ and denote by n' the number of vertices in G'. By Claim 4.3 we can assume that $|\textsc{Small}| \leq n^{0.3}$. Setting $c = 3$, and $R = \frac{9n'}{40}$, the conditions of Lemma 4.7 are met, and thus there exists a subgraph $\widehat{G} \subseteq G'$ such that $\widehat{G} \in \mathcal{M}_{\mathcal{X}_{\frac{9n'}{40},3}}$ and $e(\widehat{G}) \leq 2n' \ln^{0.97} n'$.

Maker's strategy consists of two phases. Let e_i denote the edge selected by Maker in his ith move and let $H_i = (V, \{e_1, \ldots, e_i\})$ denote Maker's graph immediately after his ith move. Let H' denote Maker's graph at the end of the first phase and let H denote Maker's graph at the end of the second phase, that is, Maker's final graph. Before the game starts, Maker splits the board $E(G)$ into three parts $F_1 = E(\widehat{G})$, $F_2 = E_G(\textsc{Small}, V \setminus \textsc{Small})$ and $F_3 = E(G' \setminus \widehat{G})$. During the first phase, Maker plays two games in parallel, one on F_1 and the other on F_2. For every $j \geq 1$, on his jth move of the first phase, Maker claims an edge of $F_1 \cup F_2$, according to his strategy for each of the two games. If on his jth move Breaker claims an edge of F_i, for some $i \in \{1, 2\}$, then Maker claims an edge of F_i as well (unless he has already achieved his goal in the game on F_i). If Breaker claims an edge of F_3, then Maker claims an edge of $F_1 \cup F_2$ which brings him closer to his goal in the corresponding game. Playing on the edge set of F_1, Maker aims to build a $(9n'/40, 3)$-expander H_1'. As noted above, Maker has a winning strategy for this game. Moreover, since $|F_1| \leq 2n' \ln^{0.97} n'$, Maker can build such an expander within at most $t_{1,1} := n' \ln^{0.97} n'$ moves. Playing on the edges of F_2, Maker follows a simple pairing strategy which guarantees that, by the end of the game, the graph H_2' which Maker constructs, will satisfy $d_{H_2'}(v) \geq 2$ for every $v \in \textsc{Small}$. For every edge which is incident with some vertex $v \in \textsc{Small}$ that Breaker claims, Maker responds by claiming a different edge incident with v. Note that if Maker's current graph already contains two edges incident with v he can simply claim another free edge of $F_1 \cup F_2$ which brings him closer to his goal in the corresponding game. Hence, the number of moves required for Maker to reach his goal in the game on F_2 is at most $t_{1,2} := 2|\textsc{Small}| \leq 2n^{0.3}$. It follows by Claim 4.4 that $\textsc{Small}$ is an independent set and that no two edges emitting from $\textsc{Small}$ are incident with the same vertex of $V \setminus \textsc{Small}$. Hence, Maker's graph H_2', satisfies $N_{H_2'}(U') \geq 2|U'|$ for every $U' \subseteq \textsc{Small}$. Applying Claim 2.8 and noting that $9n'/40 \geq n/5$, it follows that $H' = H_1' \cup H_2'$ is an $(n/5, 2)$-expander. Clearly, Maker's final graph H is an $(n/5, 2)$-expander as well. A crucial point to keep in mind is that the number of moves required for Maker to construct his $(n/5, 2)$-expander H', is $t_1 = t_{1,1} + t_{1,2} = o(n \ln^{0.98} n)$.

After having completed the construction of H', Maker proceeds to the second phase of his strategy. Let $t_2 \leq n$ denote the number of moves Maker plays during the second phase. For every $t_1 < j \leq t_1 + t_2$, on his jth move, Maker claims an edge of G which is a booster with respect to H_{j-1}. This is possible since, throughout the game Breaker claims at most $t_1 + t_2 \leq t_1 + n$ edges of G, but by Lemma 6.3, w.h.p. either H_{j-1} is Hamiltonian or it has at least $n \ln n/100 > t_1 + n$ boosters among the edges of G. It follows by the definition of a booster that either H_j is Hamiltonian or $\ell(H_j) > \ell(H_{j-1})$. Repeating the same argument $t_2 \leq n$ times, we conclude that H is Hamiltonian as claimed. $\square$

7 Remarks on possible generalizations

We note that, by using a slight modification of our proofs, Theorems 2 and 3 can in fact be extended. Recall that for every positive integer $k \geq 1$, $\mathcal{PM}^k$ and $\mathcal{HAM}^k$ denote the graph properties of admitting k pairwise edge-disjoint perfect matchings, and k pairwise edge-disjoint Hamilton cycles respectively.

Theorem 4. *For every fixed integer $k \geq 1$, if $\widetilde{G}$ is the random graph process, then w.h.p.*

$$\tau(\widetilde{G}; \mathcal{M}_{\mathcal{PM}^k}) = \tau(\widetilde{G}; \delta_{2k}).$$

Theorem 5. *For every fixed integer $k \geq 1$, if $\widetilde{G}$ is the random graph process, then w.h.p.*

$$\tau(\widetilde{G}; \mathcal{M}_{\mathcal{HAM}^k}) = \tau(\widetilde{G}; \delta_{4k}).$$

Theorem 5 can be viewed as a Combinatorial game analog of the classical result of Bollobás and Frieze [9] who proved that w.h.p. $\tau(\widetilde{G}; \mathcal{HAM}^k) = \tau(\widetilde{G}; \delta_{2k})$ (see also [14] for an extension to non-constant minimum degree in the $\mathcal{G}(n,p)$ model). Moreover, as noted in Subsection 1.4.1, Theorem 5 entails this result of Bollobás and Frieze when k is even.

We now sketch how the proof of Theorem 3 can be adapted so as to entail Theorem 5. Similarly, the proof of Theorem 4 can be obtained using appropriate modifications to the proof of Theorem 2, but as this case is simpler, we omit the details.

It suffices to prove that when removing all vertices of degree at most $\ln^{0.9} n$ from the random graph $\mathcal{G}(n, M)$, where $M = \tau(\widetilde{G}; \delta_{4k})$, playing on this subgraph G' on n' vertices, w.h.p. Maker can quickly (that is, within $o(n' \ln n')$ moves) build a $(9n'/40k, 3k)$-expander H' for which the property **M2** with $r = n'/\ln^{0.4} n'$ holds. Moreover, at the same time, Maker can ensure that the minimum degree of his graph will be at least $2k$. After the removal of $0 \leq i \leq k-1$ edge-disjoint Hamilton cycles from the original graph we have removed a $2i$-regular graph from H' and are left with a graph $\widehat{H}_i$ (which is spanned by the vertices which are not in Small) for which $|N_{\widehat{H}_i}(U)| \geq 3k|U| - 2i|U| \geq (k+2)|U|$ for every $U \subseteq V(H')$ of cardinality $|U| \leq 9n'/40k$. To complete the proof it is left to note that the choice of the parameter r guarantees that between sets of linear size there is a super-linear number of edges. It is not hard to see that adding back the vertices of Small, each of which is incident with at least $2k - 2i \geq 2$ edges, results in a connected $(n/5, 2)$-expander. This graph has more boosters than the number of moves played so far. It follows that Breaker could not have claimed all of them. Maker can thus continue playing for another Hamilton cycle using the boosters left in the graph. As there is a super-linear number of boosters and Breaker can claim at most n of them per Hamilton cycle, Maker can keep playing this way until he completely saturates his vertices of minimum degree.

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