

# Fast construction on a restricted budget

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## Abstract

We introduce a model of a controlled random graph process. In this model, the edges of the complete graph  $K_n$  are ordered randomly and then revealed, one by one, to a player called Builder. He must decide, immediately and irrevocably, whether to purchase each observed edge. The observation time is bounded by parameter  $t$ , and the total budget of purchased edges is bounded by parameter  $b$ . Builder's goal is to devise a strategy that, with high probability, allows him to construct a graph of purchased edges possessing a target graph property  $\mathcal{P}$ , all within the limitations of observation time and total budget. We show the following:

- Builder has a strategy to achieve minimum degree  $k$  at the hitting time for this property by purchasing at most  $c_k n$  edges for an explicit  $c_k < k$ ; and a strategy to achieve it (slightly) after the threshold for minimum degree  $k$  by purchasing at most  $(1 + \varepsilon)kn/2$  edges (which is optimal);
- Builder has a strategy to create a Hamilton cycle if either  $t \geq (1 + \varepsilon)n \log n/2$  and  $b \geq Cn$ , or  $t \geq Cn \log n$  and  $b \geq (1 + \varepsilon)n$ , for some  $C = C(\varepsilon)$ ; similar results hold for perfect matchings;
- Builder has a strategy to create a copy of a given  $k$ -vertex tree if  $t \geq b \gg \{(n/t)^{k-2}, 1\}$ , and this is optimal;
- For  $\ell = 2k + 1$  or  $\ell = 2k + 2$ , Builder has a strategy to create a copy of a cycle of length  $\ell$  if  $b \gg \max\{n^{k+2}/t^{k+1}, n/\sqrt{t}\}$ , and this is optimal.

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# 1 Introduction

## 1.1 The model

The random graph process, introduced by Erdős and Rényi [20, 21], is a stochastic process that starts with an empty  $n$ -vertex graph and, at each step, gains a new uniformly selected random edge. At any fixed time  $t$ , the process is distributed as the uniform random graph  $G(n, t)$ . A **graph property** is a family of graphs that is closed under isomorphisms. It is **monotone** if it is closed under addition of edges. A vast body of literature is concerned with finding *thresholds* for various monotone graph properties in the random graph process, namely, with finding time  $t_c$  such that the random graph process belongs to  $\mathcal{P}$  with high probability (**whp**; that is, with probability tending to 1 as  $n \rightarrow \infty$ ) whenever  $t$  is much (or somewhat) larger than  $t_c$  and does not belong to  $\mathcal{P}$  **whp** if  $t$  is much (or somewhat) smaller than  $t_c$ .

In many cases, if one observes the random graph process at time  $t$  above the threshold, the graph has the desired property but, in fact, contains a much sparser subgraph that has the property. For example, one of the outstanding results in this model regards a threshold for Hamiltonicity: Komlós and Szemerédi [34] and, independently, Bollobás [15] showed that if  $2t/n - \log n - \log \log n$  tends to infinity, then the random graph process, at time  $t$ , contains a Hamilton **whp**. Evidently, not all  $\sim n \log n/2$  observed edges must be included in the resulting graph for it to be Hamiltonian (as any Hamilton cycle only uses  $n$  edges). Nevertheless, when an edge arrives, it is generally hard to determine whether it will be crucial for Hamiltonicity.

The above motivates the following “online” version of building a subgraph of the random graph process. We think of it as a one-player game, where the player (“**Builder**”) has a limited “budget”. Edges “arrive” one at a time in random order and are presented to Builder. Whenever he *observes* an edge, he must immediately and irrevocably decide whether to *purchase* it. A non-purchased edge is thrown away and never reappears. The **time** (total number of presented edges) and the **budget** (maximum number of edges Builder can purchase) are both capped. The question is whether, under the given time and budget constraints, Builder has a strategy that allows him to obtain, **whp**, a particular monotone graph property in the graph of purchased edges.

To demonstrate the model, consider the property of connectedness. Let  $\tau_C$  be the (random) time at which the random graph process becomes connected. Evidently, if Builder wishes to construct a connected subgraph of the random graph process, he must purchase at least  $n - 1$  edges. However, in this case, purchasing  $n - 1$  edges suffices: Builder’s strategy would be to purchase an edge if and only if it decreases the number of connected components in his graph. That way, Builder maintains a forest, which becomes connected exactly at time  $\tau_C$ . Therefore, in this example, Builder does not have to “pay” for having to make decisions online. However, this is not always the case. For example, if Builder wants to purchase a triangle and wishes to do so while observing  $o(n^2)$  edges, he must purchase (much) more than three edges (see Theorem 6).

We denote the underlying random graph process at time  $t$  (namely, after  $t$  edges have been presented to Builder) by  $G_t$  (where the number of vertices,  $n$ , is implicit). The **hitting time** for a monotone graph property  $\mathcal{P}$  is the (random) minimum time  $t$  for which  $G_t$  has  $\mathcal{P}$ . Builder’s graph at time  $t$ , denoted  $B_t$ , is a subgraph of  $G_t$  on the same vertex set that consists of the edges purchased by Builder by time  $t$ . A  $(t, b)$ -**strategy** of Builder is a (potentially random) function that, for any  $s \leq t$ , decides whether to purchase the edge presented at time  $s$ , given  $B_{s-1}$ , under the limitation that  $B_s$  has at most  $b$  edges.

## 1.2 Our results

Our first results discuss strategies for constructing a subgraph with a given minimum degree. For every positive integer  $k$ , denote by  $\tau_k$  the hitting time for minimum degree  $k$  in the random graph process. It was proved by Erdős and Rényi [22] that  $\tau_k \sim \frac{n}{2}(\log n + (k-1) \log \log n + \Theta(1))$  **whp**.

**Theorem 1 (Minimum degree at the hitting time).** *For every positive integer  $k$  there exist constants  $o_k \in [k/2, k)$  and  $c_k$  such that the following holds. If  $b \geq o_k n + c_k \sqrt{n} \log n$  then there exists a  $(\tau_k, b)$ -strategy  $B$  of Builder such that*

$$\lim_{n \rightarrow \infty} \mathbb{P}(\delta(B_{\tau_k}) \geq k) = 1.$$

The constant  $o_k$  in Theorem 1 is explicit and computable (see Section 2.1). We believe that at the hitting time  $\tau_k$ , that constant is optimal (see Conjecture 7). However, if we allow the time to be only asymptotically optimal, then an asymptotically optimal budget suffices.

**Theorem 2 (Minimum degree).** *Let  $k$  be a positive integer and let  $\varepsilon > 0$ . If  $t \geq (1 + \varepsilon)n \log n/2$  and  $b \geq (1 + \varepsilon)kn/2$  then there exists a  $(t, b)$ -strategy  $B$  of Builder such that*

$$\lim_{n \rightarrow \infty} \mathbb{P}(\delta(B_t) \geq k) = 1.$$

The above theorem is tight in the following sense: if  $t \leq (1 - \varepsilon)n \log n/2$  then the underlying random graph process has, **whp**, isolated vertices; and if  $b < kn/2$  then every strategy will fail since Builder will not be able to purchase enough edges to obtain the required minimum degree. We believe a similar statement should hold for  $k$ -connectivity instead of minimum degree (see Conjecture 8).

We continue to Hamiltonicity. The next theorem shows that in asymptotically optimal time, a constant-inflated budget suffices, and in constant-inflated time, asymptotically optimal budget suffices.

**Theorem 3 (Hamiltonicity).** *For every  $\varepsilon > 0$  there exists  $C > 0$  such that the following hold.*

1. *If  $t \geq (1 + \varepsilon)n \log n/2$  and  $b \geq 9n$  then there exists a  $(t, b)$ -strategy  $B$  of Builder such that*

$$\lim_{n \rightarrow \infty} \mathbb{P}(B_t \text{ is Hamiltonian}) = 1.$$

2. *If  $t \geq Cn \log n$  and  $b \geq (1 + \varepsilon)n$  then there exists a  $(t, b)$ -strategy  $B$  of Builder such that*

$$\lim_{n \rightarrow \infty} \mathbb{P}(B_t \text{ is Hamiltonian}) = 1.$$

It is not clear to us whether the constant  $C$  that appears in Part (2) of the theorem must depend on  $\varepsilon$ . We believe, however, that it cannot be close to 1 (see Conjecture 9).

We have an analogous result for perfect matchings.

**Theorem 4 (Perfect matchings).** *Suppose  $n$  is even. For every  $\varepsilon > 0$  there exists  $C > 0$  such that the following hold.*

1. *If  $t \geq (1 + \varepsilon)n \log n/2$  and  $b \geq 9n$  then there exists a  $(t, b)$ -strategy  $B$  of Builder such that*

$$\lim_{n \rightarrow \infty} \mathbb{P}(B_t \text{ has a perfect matching}) = 1.$$

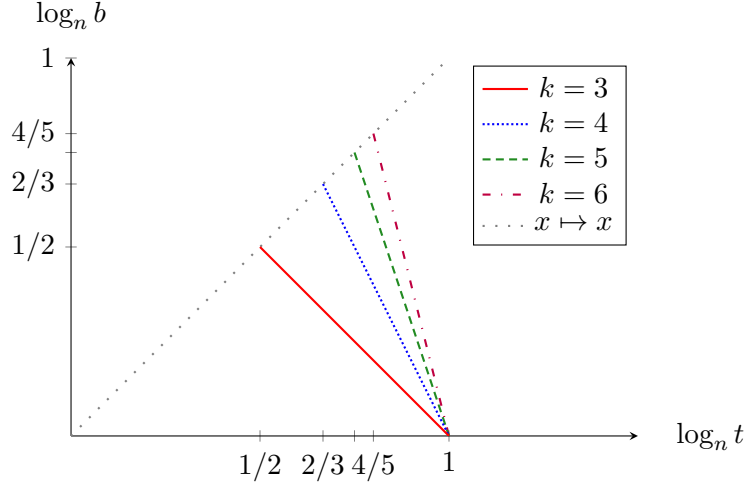


Figure 1: Budget thresholds for  $k$ -vertex trees.

2. If  $t \geq Cn \log n$  and  $b \geq (1 + \varepsilon)n/2$  then there exists a  $(t, b)$ -strategy  $B$  of Builder such that

$$\lim_{n \rightarrow \infty} \mathbb{P}(B_t \text{ has a perfect matching}) = 1.$$

Note that Part (1) of Theorem 4 is a direct corollary of Part (1) of Theorem 3. Part (2) requires additional work. Again, it is not clear to us whether the constant  $C$  that appears in Part (2) of the theorem must depend on  $\varepsilon$ .

The next two theorems discuss optimal strategies for purchasing small subgraphs. We resolve the problem whenever the target subgraph is a fixed tree or a cycle.

**Theorem 5 (Small trees).** Let  $k \geq 3$  be an integer and let  $T$  be a  $k$ -vertex tree. If  $t \geq b \gg \max\{(n/t)^{k-2}, 1\}$  then there exists a  $(t, b)$ -strategy  $B$  of Builder such that

$$\lim_{n \rightarrow \infty} \mathbb{P}(T \subseteq B_t) = 1$$

and if  $b \ll (n/t)^{k-2}$  then for any  $(t, b)$ -strategy  $B$  of Builder,

$$\lim_{n \rightarrow \infty} \mathbb{P}(T \subseteq B_t) = 0.$$

For a visualization of Theorem 5 see Fig. 1.

**Remark.** We show in the proof of Theorem 5 that if  $t \gg n$  then, in fact, there exists a  $(t, b)$ -strategy for  $b = k - 1$  that succeeds **whp** in building a copy of a  $k$ -vertex tree.

**Theorem 6 (Short cycles).** Let  $k \geq 1$  be an integer and let  $H = C_{2k+1}$  or  $H = C_{2k+2}$ . Write  $b^* = b^*(n, t, k) = \max\{n^{k+2}/t^{k+1}, n/\sqrt{t}\}$ . If  $t \gg n \log n$  and  $b \gg b^*$  then there exists a  $(t, b)$ -strategy  $B$  of Builder such that

$$\lim_{n \rightarrow \infty} \mathbb{P}(H \subseteq B_t) = 1$$

and if  $t \ll n$  or  $b \ll b^*$  then for any  $(t, b)$ -strategy  $B$  of Builder,

$$\lim_{n \rightarrow \infty} \mathbb{P}(H \subseteq B_t) = 0.$$

For a visualization of Theorem 6 see Fig. 2. For discussion on the difficulty arising in handling graphs of larger excess see Section 5.

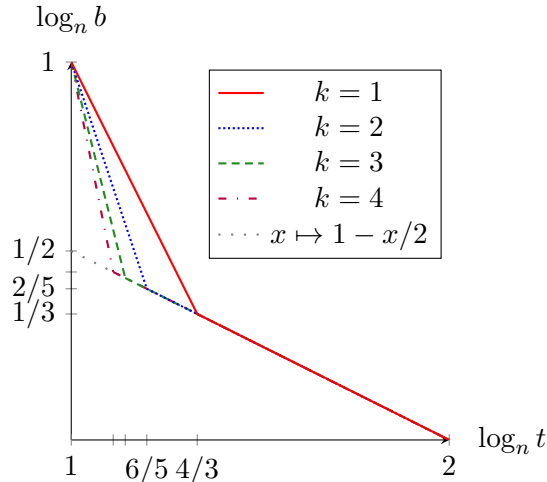


Figure 2: Budget thresholds for  $C_{2k+1}, C_{2k+2}$ .

### 1.3 Tools and techniques

**Expanders** *Expanders* are graphs in which (sufficiently) small sets expand. Namely, these are graphs in which the neighbourhood of each small set is larger than that set by a constant factor<sup>1</sup>. It is well known that connected expanders are helpful in finding Hamilton cycles (see Lemma 3.1); more concretely, connected non-hamiltonian expanders have many “boosters”, namely, non-edges whose addition to the graph creates a graph that is either Hamiltonian or whose longest path is longer (for a comprehensive account, we refer the reader to [35]). Thus, expanders will play a crucial role in the proof of Theorem 3: Builder will attempt to construct (sparse, and therefore cheap) connected expanders within the random graph process. A standard method for obtaining sparse expanders in random graphs is choosing an appropriate random (sub)graph model. Natural candidates are discussed in the following paragraphs.

**Random  $k$ -out graphs** In the goals described in Theorems 1 to 4, Builder must achieve a certain minimum degree in his (spanning) graph. The (standard) random graph process is quite wasteful in this regard, as to avoid isolated vertices, a superlinear number of edges must arrive. Thus, a wise Builder would instead construct a much sparser subgraph of the random graph process with the desired minimum degree. A classical sparse graph with a given minimum degree is the *random regular graph* (see, e.g., in [27]). However, such a graph is generally very hard to construct online. A much simpler alternative is the so-called *random  $k$ -out graph* (see, e.g., in [27]). In the  $k$ -out graph, each vertex chooses, random, independently, and without repetitions,  $k$  neighbours to connect to. Thus, the total number of edges in a  $k$ -out graph is at most  $kn$ , and the minimum degree is at least  $k$ . Unfortunately,  $k$ -out graphs are not generally subgraphs of the random graph process just after the threshold for minimum degree  $k$ . We will therefore analyse a different model, which can be considered the undirected counterpart of the  $k$ -out graph (see the next paragraph). Nevertheless, we will exploit the simplicity of the  $k$ -out graph to analyse its (slightly more complicated) undirected counterpart.

**Random  $k$ -nearest neighbour graphs** Suppose the edges of the complete graph are endowed with (potentially random) “lengths”. The graph made up only of the  $k$  shortest edges incident

<sup>1</sup>We need, and hence use, a rather weak notion of expansion; see in Section 2.1.

with each vertex is called the *random  $k$ -nearest neighbour graph*, and has been studied in various, primarily geometric, contexts (see, e.g., [19]). When the weights are independent uniform random variables supported on  $[0, 1]$ , the model becomes a symmetric random graph<sup>2</sup>, which we denote by  $\mathcal{O}_k$ . To prove Theorem 1, we devise a simple strategy that emulates  $\mathcal{O}_k$ . Since the minimum degree of  $\mathcal{O}_k$  is obviously  $k$ , the statement would follow from a theorem of Cooper and Frieze [18] according to which the number of edges of  $\mathcal{O}_k$  is at most  $cn$  for some constant  $c < k$  (see Theorem 2.3).

By observing that  $\mathcal{O}_k$  is stochastically dominated by the random  $k$ -out graph (Observation 2.5), and using the result of Cooper and Frieze regarding the (typical) connectivity of  $\mathcal{O}_3$  (see Theorem 2.4), we show that  $\mathcal{O}_8$  is (typically) an expander (Proposition 2.6). We use that helpful fact in several places: when creating a Hamilton cycle in optimal time, we spend most of the time constructing  $\mathcal{O}_8$  and the remaining time hitting boosters; when creating a Hamilton cycle under an optimal budget, we emulate small copies of  $\mathcal{O}_8$  in small parts of the graph, to allow us to *absorb* a long path into a Hamilton cycle; finally, we use a similar approach for the construction of perfect matchings.

**High-level arguments for the containment of fixed graphs** We give brief proof outlines for Theorems 5 and 6. The proof for the upper bound on the budget threshold for trees is essentially inductive: given a fixed  $k$ -vertex tree  $T$ , we let  $T'$  be obtained from  $T$  by removing a leaf. The inductive argument is that Builder can construct sufficiently many copies of  $T'$  while leaving enough time to extend some of them into copies of  $T$ . The proof for the lower bound is based on a similar idea: we show that Builder cannot construct a connected component of size at least  $k$ . In order to construct such a component, he needs to construct enough smaller components so that in the remaining time, he will, **whp**, connect two of them so that the resulting component will be large enough.

The strategy for obtaining a cycle of length  $\ell = 2k + 1$  goes through the construction of many “traps”, namely, non-edges whose addition to Builder’s graph would create the desired cycle. An optimised way to construct many such traps quickly is by constructing  $r$   $d$ -ary trees of depth  $k$  (for a correct choice of  $r, d$ ). If  $d$  is large, most pairs of leaves in such a tree are connected by a path of length  $2k$ , and thus form a trap. The argument for a cycle of length  $\ell = 2k + 2$  is similar. We complement the upper bound with two lower bounds; each matches the upper bound in a different regime. We first show a “universal” lower bound, based on the straightforward observation that the number of traps Builder has in his graph is bounded by  $O(b^2)$ , as he has at most  $2b$  vertices of positive degree in his graph. A (slightly) more involved lower bound fits the earlier regimes (lower values of  $t$ ). To this aim, we observe that Builder’s graph is typically  $\Theta(1)$ -degenerate. Then, we use this observation, together with an estimate on the maximum degree of the underlying graph process, to bound from above the number of paths of length  $\ell$  (and hence the number of traps).

## 1.4 Related work

The study of random graph processes is at the heart of the theory of random graphs, providing a more dynamic point of view on their evolution. One of the main questions is to determine the thresholds of monotone increasing graph properties or their (more refined) hitting times. Classical results of this sort include the thresholds of minimum degree and vertex connectivity  $k$  [22], the appearance of a “giant” component [21], Hamiltonicity [34, 15] and the containment of fixed subgraphs [21, 14]. For a comprehensive coverage of the topic, we refer the reader to the books of Bollobás [16], Janson, Łuczak and Ruciński [33], and Frieze and Karoński [27]. In our context, an

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<sup>2</sup>The choice of uniform distribution supported on  $[0, 1]$  is arbitrary; any distribution without atoms would yield the same random graph.

obvious necessary condition for the existence of a winning strategy for Builder is that the “time” is above the threshold (or at least at the hitting time), as otherwise, the underlying graph process is not guaranteed to have the desired property **whp**. Evidently, if  $b = t$  (namely, if the budget equals the time), and  $t$  is (sufficiently) above the threshold of the target property, then Builder has a winning strategy: the naive one that purchases each observed edge. Since in many cases Builder can do better, the model can be seen as an extension of the “standard” random graph process.

In the last couple of decades, partly inspired by the remarkable work of Azar, Broder, Karlin and Upfal [2] on *balanced allocations*, there has been a growing interest in *controlled* random processes. In the context of graph processes, an algorithm is provided with a random flow of edges (usually, but not always, the random graph process) and with (the *offline* version) or without (the *online* version) “peeking into the future”, makes a decision that alters that flow by accepting/rejecting edges, by colouring them, or by other means. We mention several related models that fall into this category, mainly in the online version.

The *Achlioptas process*, proposed by Dimitris Achlioptas in 2000, is perhaps the most studied controlled random graph process. In this process, the algorithm is fed by a stream of random  $k$ -sets of edges (with or without repetitions) and should pick, immediately and irrevocably (“online”), a single edge to accept, rejecting the others. The algorithm’s goal is to make the graph of accepted edges satisfy some monotone increasing (decreasing) graph property while minimising (maximising) the total number of rounds. Early works on this model [8, 9, 11] treated the original question posed by Achlioptas of *avoiding* a giant component for as long as possible. Other works [25, 13] considered the opposite of the original question, namely, the question of accelerating the appearance of the giant (see also [43] for a general framework). The model has also been studied for other objectives, such as avoiding [36, 41] or creating [38] a fixed subgraph or obtaining a Hamilton cycle [37]. Several variants have also been studied, such as an *offline* version, in which the algorithm sees all  $k$ -sets of edges at the beginning of the process (see, e.g., [12]); and a *memoryless* version, in which the algorithm’s decision may not depend on its previous decisions (see [7]).

The *semi-random process*, proposed by the third author in 2016, is a variation of the Achlioptas process in which the algorithm is fed by a stream of random spanning stars (with repetitions) instead of  $k$ -sets of edges. As with the Achlioptas process, the algorithm must immediately and irrevocably pick a single edge to accept. Works on this model treat monotone properties such as containment of fixed subgraphs [6, 3], minimum degree and vertex connectivity [6], perfect matchings [31], Hamilton cycles [30, 32] and bounded degree spanning graphs [5].

Another model that fits in this setting was studied by Frieze and Pegden [28, 29] and by Anastos [1] under the name “purchasing under uncertainty”. In their model, whenever an edge arrives, it is given an independent random “cost”, and the algorithm has to decide whether to purchase that edge, aiming to pay the minimum total cost required to obtain a desired graph property.

In a Ramsey-type version of controlled random processes, incoming edges are coloured (irrevocably) by an online algorithm. The algorithm aims to avoid, or to create, a monochromatic property. In [26], the triangle-avoidance game for up to 3 colours is discussed. This was extended to any fixed cyclic graph and any number of colours in [39, 40]. The model was further studied in the contexts of the giant component [10, 44] and Hamilton cycles [17].

Finally, we would like to mention a related adaptation of the *two-stage optimisation with recourse* framework for the *minimum spanning tree* model [24]. Here, every edge of the complete graph is given an independent random “Monday” cost and another independent random “Tuesday” cost. The algorithm sees all Monday costs and decides (immediately and irrevocably) which edges to incorporate into his (future) spanning tree. Then, Tuesday costs are revealed, and the algorithm uses them to complete his (current) forest into a spanning tree. The algorithm’s goal is to minimise the total cost of edges in his constructed tree.

## 1.5 Paper organisation and notation

The rest of the paper is organised as follows. In Section 2 we introduce some preliminaries. Section 3 contains the proofs of Theorems 1 to 4, and Section 4 contains the proofs of Theorems 5 and 6. In the final section, Section 5, we mention a few relevant open problems.

Throughout the paper, all logarithms are in the natural basis. If  $f, g$  are functions of  $n$  we write  $f \preceq g$  if  $f = O(g)$ ,  $f \ll g$  if  $f = o(g)$ ,  $f \asymp g$  if  $f = \Theta(g)$ , and  $f \sim g$  if  $f = (1 + o(1))g$ . For simplicity and clarity of presentation, we often make no particular effort to optimise the constants obtained in our proofs and omit floor and ceiling signs whenever they are not crucial.

## 2 Preliminaries

We will make use of the following version of Chernoff bounds (see, e.g., in [33, Chapter 2]).

**Theorem 2.1** (Chernoff bounds). *Let  $n \geq 1$  be an integer and let  $p \in [0, 1]$ , let  $x \sim \text{Bin}(n, p)$ , and let  $\mu = \mathbb{E}x = np$ . Then, for every  $a > 0$ ,*

$$\mathbb{P}(x \leq \mu - a) \leq \exp\left(-\frac{a^2}{2\mu}\right), \quad \mathbb{P}(x \geq \mu + a) \leq \exp\left(-\frac{a^2}{2(\mu + a/3)}\right).$$

We will also make use of the following standard estimate for the binomial coefficient:

**Lemma 2.2.** *For every  $1 \leq k \leq n$ ,*

$$\binom{n}{k} \leq \left(\frac{en}{k}\right)^k.$$

### 2.1 The random $k$ -nearest neighbour graph and the random $k$ -out graph

Suppose the edges of the complete graph are endowed with independent uniform random “lengths” in  $[0, 1]$ . The (random) graph obtained by retaining (only) the  $k$  shortest edges incident with each vertex is called the **random  $k$ -nearest neighbour graph** (see [18]). We denote it by  $\mathcal{O}_k$ . The following theorem of Cooper and Frieze [18] (stated here in a weaker form) will be useful for us in several places in this paper.

**Theorem 2.3** ([18, Theorem 1.1]). *For every constant  $k$  there exists  $k/2 \leq o_k < k$  such that the number of edges of  $\mathcal{O}_k$  is **whp**  $o_k n + O(\sqrt{n} \log n)$ .*

We wish to show that  $\mathcal{O}_k$ , for sufficiently large  $k$ , is a (connected) sparse expander. To this end, we need the following result from [18] regarding the connectivity of  $\mathcal{O}_3$  (and hence of  $\mathcal{O}_k$  for  $k \geq 3$ ).

**Theorem 2.4** ([18, Theorem 1.4]).  *$\mathcal{O}_3$  is **whp** connected.*

The next helpful observation that we will need for this goal is that  $\mathcal{O}_k$  is stochastically dominated by the **random  $k$ -out graph** (see, e.g., in [27]): the random graph whose edges are generated independently for each vertex by a random choice of  $k$  distinct edges incident to that vertex. For brevity, denote it by  $\mathcal{G}_k$ .

**Observation 2.5.**  *$\mathcal{O}_k$  is stochastically dominated by  $\mathcal{G}_k$ .*



*Proof.* For every ordered pair of distinct vertices  $u, v$  let  $\mathbf{w}(u, v)$  be a uniform  $[0, 1]$  random weight assigned to the ordered pair  $(u, v)$ . Then,  $k$ -out is obtained by adding the edge  $\{u, v\}$  whenever  $\mathbf{w}(u, v)$  is one of the  $k$  smallest weights in the family  $\{\mathbf{w}(u, w)\}_w$ . For every (unordered) pair of distinct vertices  $u, v$  let  $\mathbf{x}\{u, v\} = \min\{\mathbf{w}(u, v), \mathbf{w}(v, u)\}$ . Observe crucially that while  $\mathbf{x}$  is not uniform, it is continuous on  $[0, 1]$  and, more importantly, independent for distinct edges. Thus,  $\mathcal{O}_k$  is obtained by including the edge  $\{u, v\}$  whenever  $\mathbf{x}\{u, v\}$  is one of the  $k$  smallest weights in the family  $\{\mathbf{x}\{u, w\}\}_w$ . Let us now show that under this coupling,  $\mathcal{O}_k \subseteq \mathcal{G}_k$ . Let  $\{u, v\} \in \mathcal{O}_k$  and assume without loss of generality that  $\mathbf{x}\{u, v\} = \mathbf{w}(u, v)$ . Pick a vertex  $w$  for which  $\mathbf{x}\{u, v\} < \mathbf{x}\{u, w\}$  (there are at least  $n - 1 - k$  such vertices). Then,  $\mathbf{w}(u, v) = \mathbf{x}\{u, v\} < \mathbf{x}\{u, w\} \leq \mathbf{w}(u, w)$ . It follows that there are at least  $n - 1 - k$  vertices  $w$  for which  $\mathbf{w}(u, v) < \mathbf{w}(u, w)$ , hence  $\{u, v\} \in \mathcal{G}_k$ .  $\square$

We are now ready to show that  $\mathcal{O}_8$  is, **whp**, an expander. To be precise, say that a graph  $G = (V, E)$  is an  $R$ -expander if every set  $U \subseteq V$  with  $|U| \leq R$  has  $|N(U)| \geq 2|U|$ .

**Proposition 2.6.**  $\mathcal{O}_8$  is **whp** a  $\beta n$ -expander for  $0 < \beta < 1/10^9$ .

*Proof.* We first prove that, **whp**, no vertex set  $S$  in  $\mathcal{O}_4$  of cardinality  $s \leq \gamma n$ , for  $\gamma < (1/400)^3$ , spans more than  $\frac{4}{3} \cdot s$  edges. Since (by Observation 2.5)  $\mathcal{O}_8$  is stochastically dominated by  $\mathcal{G}_8$ , we may prove that statement for  $\mathcal{G}_8$ . By the union bound over the choice of  $S$  and the choices of the spanned edges, and using Lemma 2.2, that probability is at most

$$\sum_{s=2}^{\gamma n} \binom{n}{s} \binom{\binom{s}{2}}{\frac{4}{3} \cdot s} \left(\frac{16}{n}\right)^{\frac{4}{3} \cdot s} \leq \sum_{s=2}^{\gamma n} \left[ \frac{en}{s} \cdot \left(\frac{e \binom{s}{2}}{\frac{4}{3} \cdot s} \cdot \frac{16}{n}\right)^{4/3} \right]^s \leq \sum_{s=2}^{\gamma n} \left[ 400 \cdot \left(\frac{s}{n}\right)^{1/3} \right]^s.$$

If  $n/\log n \leq s \leq \gamma n$  then the summand is at most  $c^{n/\log n}$  for some  $c < 1$ . Otherwise, it is  $(o(1))^s$ . Thus, the sum is  $o(1)$ . Back to  $\mathcal{O}_8$ , if there exists a set  $A$  of size at most  $\beta n$  for  $\beta < 1/10^9$  such that  $|N(A)| \leq 2|A|$ , then taking  $S = A \cup N(A)$  we obtain a set of size  $s \leq 3\beta n \leq \gamma n$  that spans at least  $4\beta n$  edges, and this does not happen **whp**.  $\square$

## 2.2 Greedy $k$ -matchings

We show that the greedy purchasing strategy works “well” for purchasing a large  $k$ -matching; namely, a subgraph of maximum degree  $k$  in which all but a few vertices are of degree  $k$ . This will be useful in the proofs of Theorem 2 and Theorem 4.

**Lemma 2.7.** Let  $k$  be a positive integer and let  $\varepsilon > 0$ . Then, there exists  $C = C(k, \varepsilon)$  such that if  $t \geq Cn$  and  $b \geq (k - \varepsilon)n/2$  then there exists a  $(t, b)$ -strategy of Builder that purchases a graph with maximum degree  $k$  in which all but at most  $\varepsilon n$  vertices are of degree  $k$ .

*Proof.* Builder follows the greedy strategy; that is, he purchases every edge both of whose endpoints are of degree below  $k$ . Let  $U$  denote the (dynamic) set of vertices of degree below  $k$ . Let  $C = k\varepsilon^{-2}$  and let  $t \geq Cn$ . For  $i = 1, \dots, t$  let  $x_i$  be the indicator of the event that the  $i$ 'th edge arriving is contained in  $U$  (and thus purchased by Builder) or  $|U| \leq \varepsilon n$ . The probability for this event is at least  $\sim \varepsilon^2$ . Thus,  $x := \sum x_i$  stochastically dominates a binomial random variable with mean  $\sim \varepsilon^2 Cn = kn$ . Therefore, the probability that  $x < (k - \varepsilon)n/2$  is, by Chernoff bounds (Theorem 2.1),  $o(1)$ . On the event that  $x \geq (k - \varepsilon)n/2$ , either  $|U| \leq \varepsilon n$ , or Builder has purchased at least  $(k - \varepsilon)n/2$  edges, in which case it follows that  $|U| \leq \varepsilon n$ . Obviously, by that time Builder has purchased at most  $(k - \varepsilon)n/2$  edges.  $\square$

## 3 Spanning subgraphs

### 3.1 Minimum degree

It follows from Theorem 2.3 that Builder can construct a graph with minimum degree  $k$  by purchasing (sufficiently) more than  $o_k n$  edges, while observing only enough edges to guarantee a sufficient minimum degree in the underlying random graph process.

*Proof of Theorem 1.* We describe Builder's strategy. Builder purchases any edge touching at least one vertex of degree less than  $k$ . Since by time  $t$  the random graph process has, by definition, minimum degree of at least  $k$ , the obtained graph is exactly  $\mathcal{O}_k$ . We conclude by observing that by Theorem 2.3 Builder purchased, **whp**, at most  $o_k n + O(\sqrt{n} \log n)$  edges.  $\square$

*Proof of Theorem 2.* Let  $\varepsilon' = \varepsilon/2$ . We describe Builder's strategy in stages.

#### Stage I (Constructing a $k$ -matching)

**Time:**  $Cn$  — **Budget:**  $(k - \varepsilon')n/2$

During the first  $Cn$  steps, for  $C = C(\varepsilon', k)$ , Builder constructs a  $k$ -matching, in which all but at most  $\varepsilon'n$  vertices are of degree  $k$ . This is possible, **whp**, by Lemma 2.7.

#### Stage II (Handling low-degree vertices)

**Time:**  $(1 + \varepsilon')n \log n$  — **Budget:**  $\varepsilon'kn$

Denote by  $V_0$  the set of vertices of degree smaller than  $k$  in Builder's graph at the end of Stage I. By the above, **whp**,  $|V_0| \leq \varepsilon'n$ . Builder now emulates the  $\mathcal{O}_k$  model with respect to these vertices; namely, Builder purchases any edge contained in  $V_0$  that is incident to at least one vertex of degree less than  $k$  in  $G[V_0]$ . Since in the random graph process, by time  $t$ , the minimum degree is logarithmic in  $n$ , Builder will observe at least  $k$  additional edges incident to each vertex of  $V_0$ . He will purchase a subset of these (thus, at most  $\varepsilon'kn$  edges).  $\square$

## 3.2 Hamilton cycles

### 3.2.1 Optimal time, inflated budget

We describe Builder's strategy in (two) stages. At the first stage Builder observes  $(1 + \varepsilon/2)n \log n$  edges and purchases at most  $8n$  edges and at the second stage Builder observes  $\frac{\varepsilon}{2}n \log n$  edges and purchases at most  $n$  edges.

#### Stage I (Constructing an expander)

**Time:**  $(1 + \frac{\varepsilon}{2})n \log n$  — **Budget:**  $8n$

At the first at most  $t_1 = (1 + \varepsilon/2)n \log n$  observed edges, Builder constructs  $\mathcal{O}_8$ . By Theorem 2.3, this is achieved, **whp**, in time, and with purchasing less than  $8n$  edges.

#### Stage II (Hitting boosters)

**Time:**  $\frac{\varepsilon}{2}n \log n$  — **Budget:**  $n$

A non-edge  $\{x, y\}$  of  $G$  is called a **booster** if adding  $\{x, y\}$  to  $G$  creates a graph which is either Hamiltonian or whose longest path is longer than that of  $G$ . We wish to argue that if  $\mathcal{O}_8$  is not already Hamiltonian, then it must contain many boosters. Builder will try to collect these boosters at the second stage of his strategy. To this end, we will use the following known lemma, whose proof can be found, e.g., in [35].

**Lemma 3.1.** *Let  $G$  be a connected  $R$ -expander which contains no Hamilton cycle. Then  $G$  has at least  $(R + 1)^2/2$  boosters.*

At the next  $t_2 = \lceil \varepsilon n \log n / 2 \rceil$  observed edges, Builder purchases any observed booster. According to Proposition 2.6 and Lemma 3.1, Builder’s graph at the beginning of this stage has **whp**, at least  $\beta' n^2$  boosters for some  $\beta' > 0$ . In fact, since expansion is monotone w.r.t. addition of edges, then Proposition 2.6 implies that after purchasing a booster, Builder’s graph is either Hamiltonian or it still contains at least  $\beta' n$  boosters. For every  $i = 1, \dots, t_2$ , let  $x_i$  be the indicator of the event “the  $i$ ’th observed edge is a booster” (w.r.t. Builder’s current graph). For convenience, if the graph is already Hamiltonian, and hence no boosters exist, we set  $x_i$  to 1. It follows from the discussion above that  $x := \sum_{i=1}^{t_2} x_i$  stochastically dominates a binomial random variable with  $t_2$  attempts and success probability  $p_2 > \beta'$ . It follows from Chernoff bounds (Theorem 2.1) that  $x \geq n$  **whp**, implying that Builder obtains, **whp**, a Hamilton cycle during the second stage.  $\square$

### 3.2.2 Inflated time, optimal budget

We describe Builder’s strategy in (five) stages. In the first stage, Builder grows many sublinear paths, covering together  $(1 - \varepsilon)n$  vertices.

#### Stage I (Growing disjoint paths)

**Time:**  $\frac{3}{\varepsilon} n \log n$  — **Budget:**  $(1 - \varepsilon)n$

Let  $C = 3/\varepsilon$ . Let  $\sigma \in (0, 1/2)$  and let  $s \sim n/\log^\sigma n$  be an integer. Builder grows simultaneously paths  $P_1, \dots, P_s$  as follows. He begins by letting each  $P_i$  be a (distinct) vertex  $v_0^i$ . He then claims every edge that extends one of the paths without intersecting with other paths. Formally, if at a given stage Builder has the paths  $(v_0^1, \dots, v_{\ell_1}^1), \dots, (v_0^s, \dots, v_{\ell_s}^s)$  on the vertex set  $V_P$  then he claims an observed edge if and only if it is of the form  $\{v_{\ell_j}^j, w\}$  for  $j = 1, \dots, s$  and  $w \notin V_P$ . Builder stops if  $|V_P| \geq (1 - \varepsilon)n$  (in which case the stage is “successful”) or when he has observed  $Cn \log n$  edges (in which case the stage “fails”), whichever comes first. Let  $t_1 = \lceil Cn \log n \rceil$ . For  $1 \leq j \leq s$  and  $1 \leq i \leq t_1$ , let  $y_i^j$  be the indicator that Builder has purchased the  $i$ ’th observed edge, and that this edge extends the path  $P_j$ . For convenience, if  $|V_P| \geq (1 - \varepsilon)n$  we set  $y_i^j$  to 1. Observe that for every  $j, i$ ,  $\mathbb{P}(y_i^j = 1) \geq (n - |V_P|) / \binom{n}{2} > \varepsilon/n$ . Write  $y^j = \sum_{i=1}^{t_1} y_i^j$ . Thus,  $y^j$  stochastically dominates a binomial random variable with  $t_1$  attempts and success probability  $p_1 = \varepsilon/n$ . Set  $\mu = t_1 p_1 \sim 3 \log n$  and  $\nu = (1 - \varepsilon)n/s \asymp \log^\sigma n \ll \log n$ . By Chernoff bounds (Theorem 2.1), noting that  $\log s < \log n$  and  $\nu^2/\mu \asymp \log^{2\sigma} n / \log n \ll \log n$ ,

$$s \cdot \mathbb{P}(y^j \leq (1 - \varepsilon)n/s) \leq \exp\left(\log s - \frac{\mu}{2} + \nu - \frac{\nu^2}{2\mu}\right) = o(1).$$

Thus, by the union bound, **whp** all paths are of length at least  $(1 - \varepsilon)n/s$ .

#### Stage II (Connecting the paths)

**Time:**  $n \log^\eta n$  — **Budget:**  $o(n)$

At this stage Builder tries to connect most of the paths to each other, eventually obtaining an almost-spanning path in his graph. Let  $t_2 \sim n \log^\eta n$  for some  $\sigma < \eta < 2\sigma < 1$ . During the next  $t_2$  observed edges, Builder purchases an edge connecting the suffix of one of his paths to the prefix of another. Formally, he purchases every observed edge of the form  $\{v_{\ell_i}^i, v_0^j\}$  for  $i \neq j$ . We will need the following lemma:

**Lemma 3.2** ([4, Lemma 4.4]). *Let  $s, k \geq 1$  and let  $D$  be an  $s$ -vertex digraph in which for every two disjoint  $A, B \subseteq V(D)$  with  $|A|, |B| \geq k$ ,  $D$  contains an edge between  $A$  and  $B$ . Then,  $D$  contains a directed path of length  $s - 2k + 1$ .*

We use the lemma on an auxiliary digraph  $D$  on the vertex set  $[s]$  that is defined as follows. We observe  $t_2$  random edges. Whenever Builder observes (and purchases) an edge of the form  $\{v_{\ell_i}^i, v_0^j\}$ ,

we add the edge  $(i, j)$  to  $D$ . We now have to claim that  $D$  satisfies the requirements of the lemma for  $k = \varepsilon s$ .

**Claim 3.3.** *With high probability, every two disjoint  $A, B \subseteq [s]$  with  $|A|, |B| \geq \varepsilon s$  satisfy  $E_D(A, B) \neq \emptyset$ .*

*Proof.* Fix disjoint  $A, B$  with  $|A|, |B| \geq \varepsilon s$ . The event that  $E_D(A, B) = \emptyset$  implies that none of the  $t_2$  observed edges hits a pair  $\{v_{\ell_i}^i, v_0^j\}$  for  $i \in A$  and  $j \in B$ . There are at least  $\varepsilon^2 s^2$  such edges, hence that probability is at most

$$\binom{\binom{n}{2} - \varepsilon^2 s^2}{t_2} / \binom{\binom{n}{2}}{t_2} \leq \exp\left(-\varepsilon^2 s^2 t_2 / \binom{n}{2}\right) = \exp(-\Theta(n \log^{\eta-2\sigma} n)).$$

By the union bound over the choices of  $A, B$ , the probability that the event in the statement does not hold is at most

$$\binom{n}{\varepsilon s}^2 \exp(-\Theta(n \log^{\eta-2\sigma} n)) \leq \exp(3\varepsilon \sigma s \log \log n - \Theta(n \log^{\eta-2\sigma} n)) = o(1),$$

using Lemma 2.2, and since  $\eta > \sigma$ . □

Claim 3.3 and Lemma 3.2 imply that  $D$  has, **whp**, a directed path of length  $(1 - 2\varepsilon)s$ . By the way we have defined  $D$ , this implies the existence of a path  $Q$  of length  $(1 - 3\varepsilon)n$  in Builder's graph. To analyse the number of purchased edges, we note that any presented edge is purchased with probability  $\asymp s^2/n^2$ , hence the expected number of purchased edges is  $\asymp t_2 s^2/n^2 \asymp n/\log^{2\sigma-\eta}$ . By Chernoff bounds (Theorem 2.1), the number of purchased edges is sublinear **whp**.

### Stage III (Preparing the ground)

Time:  $\frac{3}{\varepsilon}n \log n$  — Budget:  $24\varepsilon n$

Let  $q_1, q_2$  be the endpoints of  $Q$ . Let  $V_1 \cup V_2 = V \setminus V(Q)$  be a partition of the vertices outside  $Q$  (so  $|V_i| = 3\varepsilon n/2$  for  $i = 1, 2$ ). Builder performs the next two tasks simultaneously.

**(Connecting the endpoints of  $Q$  to  $V_1, V_2$ )** Builder claims the first observed edge from  $\{q_i\}$  to  $V_i$  for each  $i = 1, 2$  (unless such an edge already exists). After  $cn \log n$  steps, for any  $c > 0$ , there will be a neighbour of each of  $q_i$  in  $V_i$ . Call this neighbour  $w_i$ .

**(Constructing expanders on  $V_1, V_2$ )** Builder purchases a copy of  $\mathcal{O}_8$  in  $V_i$ . Observing that every new edge falls inside  $V_i$  with probability at least  $\varepsilon^2$ , we conclude, using Theorems 2.1 and 2.3, that this could be done **whp** by observing at most  $3\varepsilon^{-1}n \log n$  edges and purchasing at most  $\frac{3 \cdot o_8}{2}\varepsilon n$  of them (where  $o_8 < 8$ ). In addition, by Theorem 2.4 and Proposition 2.6 we know that the obtained graphs are connected  $\beta n$ -expanders for sufficiently small  $\beta = \beta(\varepsilon) > 0$ .

### Stage IV (Hitting boosters)

Time:  $n \log n$  — Budget:  $2\varepsilon n$

For  $i = 1, 2$  let  $B_i = B[V_i]$  denote Builder's graph on the vertex set  $V_i$ . From the previous stage we know that both  $B_i$  are connected  $\beta n$ -expanders. At this stage Builder purchases any booster with respect to any of these expanders. Repeating the arguments in Section 3.2.1 (Stage II), we see that, **whp**, by observing at most  $n \log n$  edges and purchasing at most  $3\varepsilon n$  of them, both  $B_i$  become Hamiltonian. In fact, at that time, **whp**, more is true.

**Claim 3.4.** *For every  $i = 1, 2$ , the number of endpoints of Hamilton paths of  $B_i$  whose other endpoint is  $w_i$  is at least  $\beta n$ .*

The proof of Claim 3.4 relies on the so-called *rotation–extension* technique of Pósa [42]. Given a longest path  $P = (v_0, \dots, v_t)$  we say that  $P'$  obtained from  $P$  by an **elementary rotation** of  $P$  (with  $v_1$  fixed) if  $P' = (v_0, \dots, v_{i-1}, v_i, v_j, v_{j-1}, \dots, v_{i+1})$  ( $j > i$ ). We denote by  $\mathcal{R}(P)$  the set of endpoints of paths obtained from  $P$  by a (finite) sequence of elementary rotations. We will use the following classical lemma of Pósa:

**Lemma 3.5** (Pósa’s lemma [42]). *Let  $G$  be a graph and let  $P$  be a longest path in  $G$ . Then  $|N(\mathcal{R}(P))| \leq 2|\mathcal{R}(P)| - 1$ .*

*Proof of Claim 3.4.* Fix  $i \in \{1, 2\}$ . Since  $B_i$  is Hamiltonian, there exists a Hamilton path  $P = (w_i = v_0, v_1, \dots, v_t)$ . Recall that  $B_i$  is a  $\beta n$ -expander for some  $\beta > 0$ . By Lemma 3.5,  $|N_{B_i}(\mathcal{R}(P))| \leq 2|\mathcal{R}(P)| - 1$ ; thus  $|\mathcal{R}(P)| > \beta n$ .  $\square$

Denote the sets of endpoints of Hamilton cycles in  $B_i$  whose other endpoint is  $w_i$  by  $Y_i$  for  $i = 1, 2$ .

### Stage V (Closing a Hamilton cycle)

Time:  $\log n$  — Budget: 1

To close a Hamilton cycle, Builder purchases an edge between  $Y_1$  and  $Y_2$ . The probability of failing to do so after observing  $\log n$  steps is  $o(1)$ .

## 3.3 Perfect matchings

*Proof of Theorem 4.* As we mentioned earlier, Part (1) of Theorem 4 follows immediately from Part (1) of Theorem 3, as a Hamilton cycle contains a perfect matching. For Part (2), we describe Builder’s strategy (in stages). Let  $\varepsilon' = \varepsilon/20$ .

### Stage I (Constructing a partial matching)

Time:  $Cn$  — Budget:  $(1 - \varepsilon')n/2$

During the first  $Cn$  steps, for  $C = C(\varepsilon')$ , Builder constructs a matching of size  $(1 - \varepsilon')n/2$ , as described in Lemma 2.7 (for  $k = 1$ ). This succeeds **whp**.

### Stage II (Expanding on isolated vertices)

Time:  $(1 + \varepsilon')n \log n$  — Budget:  $8\varepsilon'n$

Denote by  $V_0$  the set of vertices not covered by Builder’s current matching. By the above, **whp**,  $|V_0| \leq \varepsilon'n$ . Builder now emulates the  $\mathcal{O}_8$  model with respect to these vertices; namely, Builder purchases any edge contained in  $V_0$  that is incident to at least one vertex of degree less than 8 in  $G[V_0]$ . Since in the random graph process, by time  $t$ , the minimum degree is logarithmic in  $n$ , Builder will observe at least 8 additional edges incident to each vertex of  $V_0$ , and he will purchase a subset of these (thus, at most  $8\varepsilon'n$  edges).

### Stage III (Hitting boosters)

Time:  $\varepsilon'n \log n$  — Budget:  $\varepsilon'n$

By Theorem 2.4 and Proposition 2.6 we know that Builder’s graph on  $V_0$  is, **whp**, a connected  $\beta n$ -expander for sufficiently small  $\beta = \beta(\alpha)$ . At this stage Builder purchases any booster with respect to his expander in  $V_0$ . As described in Stage II of Section 3.2.1, observing at most  $\varepsilon'n \log n$  edges and purchasing at most  $\varepsilon'n$  edges suffices to make  $G[V_0]$  Hamiltonian. Since  $V_0$  is of even size, it contains a perfect matching, which, together with the matching covering  $V \setminus V_0$ , is a perfect matching in Builder’s graph.  $\square$

## 4 Small subgraphs

We introduce the following terminology. For a fixed graph  $H$  we call a non-edge  $e$  in Builder's graph an  $H$ -**trap** (or, simply, a trap) if  $B + e$  contains a copy of  $H$ . Following the terminology of "hitting" boosters, we say that Builder **hits** a trap if he encounters (and therefore purchases) such a non-edge.

### 4.1 Trees

#### 1-statement

*Proof of the 1-statement of Theorem 5.* Let  $T_1 \subseteq \dots \subseteq T_k$  be a sequence of subtrees of  $T$ ,  $T_i$  having  $i$  vertices (so  $T_k = T$ ). We first note that if  $t \gg n$  then a budget  $b = k - 1$  suffices with high probability. Indeed, suppose Builder has constructed  $T_i$  (note that he builds  $T_1$  with a zero budget). To build  $T_{i+1}$ , Builder waits for one of at least  $\sim n$  edges whose addition to his current copy of  $T_i$  would create a copy of  $T_{i+1}$ . By time  $t/(k-1)$  Builder observes (and purchases) such an edge **whp**. Thus, by time  $t$  Builder constructs, **whp**, a copy of  $T$ .

We may thus assume from now on that  $t \leq cn$  for  $0 < c < 1/5$  and  $t \geq b \gg (n/t)^{k-2}$ . In particular,  $t \gg n^{1-1/(k-1)}$ . For  $i = 1, \dots, k$  define

$$s_i = \frac{b}{k-1} \cdot \left( \frac{t}{(k-1)n} \right)^{i-2},$$

and note that  $s_i$  is a strictly monotone decreasing sequence with  $n \geq s_1 > s_k \gg 1$ . We describe Builder's strategy in stages (for convenience, the first stage will be indexed by 2). We assume that at the beginning of stage  $i$ ,  $i = 2, \dots, k$ , Builder has built  $s_{i-1}$  vertex-disjoint copies of  $T_{i-1}$ . The assumption trivially holds for  $i = 2$ , since  $s_1 \leq n$ . At stage  $i \geq 2$ , Builder purchases any edge extending one of the  $s_{i-1}$  copies of  $T_{i-1}$  he currently has to a copy of  $T_i$ , so that the resulting trees are still vertex-disjoint. He continues doing so only as long as he does not have  $s_i$  copies of  $T_i$  (in which case this stage is successful) or if he had observed more than  $t/(k-1)$  edges during this stage (in which case this stage fails, and thus the entire strategy fails). For stage  $i$  and the  $1 \leq j \leq t/(k-1)$  let  $y_j^i$  denote be the event that the  $j$ 'th observed edge of stage  $i$  is being purchased, or that  $s_i$  copies of  $T_i$  have already been built. Being at stage  $i$ , there are at least  $s_{i-1} - s_i$  potential trees to extend, each can be extended by observing one of at least  $n - 2b$  edges. Thus, noting that  $s_{i-1} - s_i \geq s_{i-1}(1 - c/(k-1))$  and  $n - 2b \geq n - 2t \geq n(1 - 2c)$ , we have

$$\mathbb{P}(y_j^i) \geq \frac{(s_{i-1} - s_i)(n - 2b)}{\binom{n}{2}} \geq \frac{2(1 - \frac{c}{k-1})s_{i-1} \cdot (1 - 2c)}{n}.$$

Write  $c' = 2(1 - c/(k-1))(1 - 2c)$  and note that for small enough  $c > 0$  ( $c < 1/5$  suffices) we have  $c' > 1$ . Therefore,  $y^i = \sum_j y_j^i$  stochastically dominates a binomial random variable with mean  $\frac{t}{k-1} \cdot \frac{c's_{i-1}}{n} = c's_i$ . By Chernoff bounds (Theorem 2.1), the stage is successful **whp**. Thus, Builder builds, **whp**,  $s_k \geq 1$  copies of  $T_k = T$  by the end of  $k - 1$  stages, that is, by time  $t$ .  $\square$

#### 0-statement

*Proof of the 0-statement of Theorem 5.* We assume  $b \ll (n/t)^{k-2}$ . It follows that  $t \ll n$ . We may also assume that  $b \ll t$ ; indeed, if  $b \asymp t$  then  $t \ll (n/t)^{k-2}$ , hence  $t \ll n^{1-1/(k-1)}$ . This implies that **whp** no tree on  $k$  vertices exists in the graph of the observed edges. We prove that any  $(t, b)$ -strategy of Builder fails, **whp**, to build a connected component of size  $k$ . Define a sequence

$(r_i)_{i=1}^{k-1}$  by  $r_1 = n$  and  $r_i = b(3kt/n)^{i-2}$  for  $i = 2, \dots, k-1$ . We argue that **whp** Builder cannot build more than  $r_i$  connected components of size  $i$ . The proof is by induction on  $i$ . The case  $i = 1$  is trivial. The case  $i = 2$  follows deterministically by the budget restriction, as  $r_2 = b$ . To build a connected component of size  $2 < i < k$ , Builder must observe an edge connecting a connected component of size  $1 \leq j < i$  with a connected component of size  $i - j \geq j$ . By induction, the probability that the next observed edge is such is at most

$$\frac{\sum_{j=1}^{\lfloor i/2 \rfloor} (jr_j \cdot (i-j)r_{i-j})}{\binom{n}{2} - t} \leq \frac{inr_{i-1} + \Theta(b^2(t/n)^{i-4})}{\binom{n}{2} - t} \leq \frac{2.5kr_{i-1}}{n}.$$

Here we used the fact that  $b \ll t$ , hence  $b^2(t/n)^{i-4} \ll nr_{i-1}$ . Thus, by Chernoff bounds (Theorem 2.1), during  $t$  rounds the number of such edges is, **whp**, smaller than  $r_{i-1} \cdot 3kt/n = r_i$ . To build a tree of order  $k$ , Builder must construct a connected component of size at least  $k$ . For that, Builder must observe an edge connecting a connected component of size  $1 \leq i \leq k-1$  with a connected component of size  $\max\{i, k-i\} \leq j \leq k-1$  (in his graph). The probability that the next observed edge is such is at most

$$\frac{\sum_{i=1}^{k-1} \sum_{j=\max\{i, k-i\}}^{k-1} (ir_i \cdot jr_j)}{\binom{n}{2} - t} \leq \frac{r_{k-1}}{n}.$$

Hence, during  $t$  rounds the expected number of edges Builder observes that create such a connected component is  $\leq r_{k-1}t/n \ll 1$ ; thus, Builder fails to construct such a component **whp**.  $\square$

## 4.2 Cycles

### 1-statement

*Proof of the 1-statement of Theorem 6.* Let  $b \gg b^* = b^*(n, t, k)$ . Choose  $n \log n \ll t' \ll t$  such that it still holds that  $b \gg b' := b^*(n, t', k)$ . We may then take  $b$  to be small enough to satisfy  $\log b \ll (b')^{1/k}$  while still  $b \gg b'$ . Note that  $b \ll n$ . Set  $s = n^2/(b't') = \min\{(t'/n)^k, n/\sqrt{t'}\}$ , and note that  $s \ll b$ . Indeed,  $b/s \gg b'/s = n^2/(s^2t') \geq 1$ . Denote  $d = cs^{1/k}$  for  $c < 1/(1+k)$ , and note that  $d \gg \log b$ . Indeed, if  $s = (t'/n)^k$  then  $d \asymp t'/n \gg \log n \geq \log b$ , and if  $s = n/\sqrt{t'}$  then  $d \asymp (b')^{1/k} \gg \log b$ . We describe a (2-stage) strategy for odd cycles, and then explain briefly how to modify it for even cycles.

#### Stage I (Growing $d$ -ary trees)

**Time:**  $t'$  — **Budget:**  $b/2$

Set  $r = b/s \gg 1$ . Builder chooses  $r$  vertices arbitrarily and call them *roots*. Denote by  $L_0$  the set of these roots. His goal is to grow a vertex-disjoint  $d$ -ary tree of depth  $k$  from each of the roots. He is doing it in  $k$  rounds. At round  $i$ ,  $0 \leq i < k$ , each tree is of size  $\sim d^i < s/(2d)$ , hence the total number of vertices in all trees is at most  $b/(2d)$ . Let  $L_i$  be the set of vertices of the trees at distance  $i$  from their roots,  $0 \leq i < k$ , and write  $\ell_i = |L_i|$ . Write  $M_i = V \setminus (L_0 \cup \dots \cup L_i)$ . Setting  $\ell = rs/(2d) = b/(2d)$ , it follows that  $\ell_i \leq \ell$ . We get  $d|L_i| \leq d\ell = b/2 \ll n$ . We also observe that  $\log \ell \leq \log b \ll d$ . During the next  $dn$  steps we consider, for each  $v \in L_i$ , the random variable  $x_v$  that counts the number of edges from  $v$  to  $M_i$ , that do not intersect with previously presented edges between  $L_i$  and  $M_i$ . We note that the probability for an edge to satisfy this is at least  $\sim 2/n$ . Thus,  $x_v$  stochastically dominates a binomial random variable with  $dn$  attempts and success probability  $1.5/n$ . By Chernoff bounds (Theorem 2.1), the probability that  $x_v < d$  is at most  $\exp(-d/12) \ll \ell^{-1}$ . Thus, by the union bound,  $x_v \geq d$  for all  $v \in L_i$  **whp**. We can, therefore, and **whp**, grow one level in each of the trees, keeping the trees disjoint. The total time

that passes until the end of this stage is at most  $kdn \leq ckt' < t' \ll t$ , and the total budget is at most  $rd^k = c^krs < b/2$ .

## Stage II (Connecting leaves)

Time:  $t - t'$  — Budget: 1

For a leaf  $v$  in one of the constructed trees, denote by  $P_v$  the unique path from that leaf to its root. Note that if  $u, v$  are two leaves in the same tree for which the paths  $P_u, P_v$  are edge disjoint, then by adding the edge  $\{u, v\}$  a cycle of length  $\ell$  is formed. Observe that for each leaf  $u$  there exist exactly  $(d-1)d^{k-1}$  other leaves  $v$  in the same tree for which  $P_u, P_v$  are edge disjoint. Thus, the number of traps is  $rd^k(d-1)d^{k-1}/2 \asymp rs^2 = bs \gg n^2/t'$ . Thus, after time  $t - t' \gg t'$  we hit a trap **whp**.

**Even cycles** We briefly discuss the modifications needed to obtain the result for cycles of length  $2k+2$ . After choosing  $r$  roots  $\rho_1, \dots, \rho_r$ , Builder waits for the first edge touching each of these roots. Denote the other end of the edge incident to  $\rho_j$  by  $\sigma_j$ . Builder's goal is to grow vertex-disjoint  $d$ -ary trees of length  $k$  from each of  $\rho_j, \sigma_j$ , and this is done using the same strategy described above. This step requires observing at most  $t'$  edges and purchasing less than  $b$ . At this point, every pair of leaves, one rooted at  $\rho_j$  and one at  $\sigma_j$ , forms a trap. Thus, the number of traps is  $rd^{2k} \gg n^2/t'$ , hence by time  $t$  we hit a trap **whp**.  $\square$

## 0-statement

Let  $H = C_\ell$  for  $\ell = 2k+1$  or  $\ell = 2k+2$ . The 0-statement in Theorem 6 follows from combining the next two claims together with the known fact that if  $t \ll n$  then  $G_{n,t}$ , **whp**, contains no copy of  $H$ .

**Claim 4.1** (Cycles: universal lower bound). *If  $b \ll n/\sqrt{t}$  then for any  $(t, b)$ -strategy  $B$  of Builder,*

$$\lim_{n \rightarrow \infty} \mathbb{P}(H \subseteq B_t) = 0.$$

**Claim 4.2** (Cycles: specific lower bound). *If  $b \ll n^{k+2}/t^{k+1}$  then for any  $(t, b)$ -strategy  $B$  of Builder,*

$$\lim_{n \rightarrow \infty} \mathbb{P}(H \subseteq B_t) = 0.$$

*Proof of Claim 4.1.* We may assume  $t \ll n^2$ . Consider a strategy for building  $H$ . Let  $B'$  be Builder's graph before obtaining the first copy of  $H$ . We show that the last step in this strategy, namely, the number of edges Builder needs to observe between his last two purchases, is, typically, much larger than  $t$ . Let  $X$  denote the number of  $H$ -traps at this point. Observe that every  $H$ -trap is a non-edge of  $B'$ , both whose endpoints are vertices of positive degree. Evidently, the number of vertices of positive degree is at most  $2b$ , thus  $X \leq 4b^2$ . Thus, the probability of hitting a trap in  $t$  steps is at most  $tX/(\binom{n}{2} - t) \leq tb^2/n^2 \ll 1$ .  $\square$

For the proof of Claim 4.2 we will need a couple of lemmas. In what follows, we may assume that  $t \leq n^{(k+1)/(k+1/2)}$ , as otherwise  $n^{k+2}/t^{k+1} \leq n/\sqrt{t}$  and the claim follows from Claim 4.1.

**Lemma 4.3.** *Suppose  $a = a(n) \gg 1$  and  $m = m(n) \gg n$  satisfy  $a \leq n^3/m^2$ . Then, in  $G_m$ , **whp**, every vertex set of size  $q \leq a$  spans at most  $3q$  edges.*



*Proof.* Let  $N = \binom{n}{2}$  and  $p = m/N$ . For a set  $S$  of  $s$  edges we have

$$\mathbb{P}(S \subseteq G_m) = \frac{\binom{N-s}{m-s}}{\binom{N}{m}} \leq (1 + o(1))p^s.$$

Let  $U$  be a vertex set of size  $q$ . The probability that  $U$  spans at least  $3q$  edges is at most

$$(1 + o(1)) \binom{\binom{q}{2}}{3q} p^{3q}.$$

Thus, by the union bound, the probability that there exists a set  $U$  of size  $q$  that spans at least  $3q$  edges is at most

$$(1 + o(1)) \binom{n}{q} \binom{\binom{q}{2}}{3q} p^{3q} \leq \left( \frac{3n}{q} \cdot (qp)^3 \right)^q \leq (3na^2p^3)^q.$$

Here we have used Lemma 2.2. Noting that  $na^2p^3 \leq n/m \ll 1$ , the claim follows by the union bound over all sizes  $2 \leq q \leq a$ .  $\square$

**Claim 4.4.** *With high probability, every subgraph of  $G_t$  with at most  $b$  edges is 6-degenerate.*

*Proof.* Let  $B$  be a subgraph of  $G_t$  with at most  $b$  edges. Thus,  $B$  has at most  $2b$  non-isolated vertices. Apply Lemma 4.3 with  $a = 2b$  and  $m = t$  (we can do so as  $b \ll n^{k+2}/t^{k+1} \leq n^3/t^2$ ). We therefore obtain that, with high probability, every subgraph of  $B$  on  $q$  vertices spans at most  $3q$  edges, and hence has minimum degree of at most 6, and the result follows.  $\square$

We turn to count the number of copies of a path of length  $\ell$  in a  $z$ -degenerate graph  $G_0$  with maximum degree  $\Delta$ .

**Lemma 4.5.** *Let  $\ell, z \geq 1$  be integers. Let  $G_0$  be a  $z$ -degenerate  $q$ -vertex graph with maximum degree  $\Delta$ . Then, the number of paths of length  $\ell$  in  $G_0$  is at most  $q \cdot 2^\ell z^{\lceil \ell/2 \rceil} \Delta^{\lfloor \ell/2 \rfloor}$ .*

*Proof.* Let  $v_1, \dots, v_q$  be an ordering of the vertices of  $G_0$  such that for  $1 \leq i \leq q$ ,  $v_i$  has at most  $z$  neighbours among  $v_1, \dots, v_{i-1}$ . Given a path  $P$  of length  $\ell$ , one of its consistent orientations has at least  $\ell/2$  edges going backwards (w.r.t. the orientation). Thus, to count the number of copies of  $P$  in  $G_0$ , it suffices to count the number of *directed* paths of length  $\ell$  with at least  $\lceil \ell/2 \rceil$  edges going backwards. To count those, first choose a starting vertex ( $n$  options), then choose the locations of the edges going back (at most  $2^\ell$  options), and finally choose from at most  $z$  options for an edge pointing back, or at most  $\Delta$  options otherwise. Altogether, there are at most  $q \cdot 2^\ell z^{\lceil \ell/2 \rceil} \Delta^{\lfloor \ell/2 \rfloor}$  such paths.  $\square$

*Proof of Claim 4.2.* Consider a strategy for building  $H$ . Let  $B'$  be Builder's graph before obtaining the first copy of  $H$ . We show that the last step in this strategy, namely, the number of edges Builder needs to observe between his last two purchases, is, typically, much larger than  $t$ . Let  $X$  denote the number of  $H$ -traps at this point. Observe that  $X$  is bounded from above by the number of paths of length  $\ell - 1$ . By Claim 4.4,  $B'$  is, **whp**, 6-degenerate. By standard estimates on the maximum degree of a random graph (e.g., by Chernoff bounds),  $\Delta(B') \leq \Delta(G_t) \leq 3t/n$  **whp** (here we use  $t \gg n \log n$ ). Thus, by Lemma 4.5,  $B'$  has, **whp**, at most  $2b \cdot 6^{k+1} \cdot (3t/n)^k \ll n^2/t$  such paths. Thus, **whp**,  $X \ll n^2/t$ . Therefore, **whp**, the probability of hitting a trap in  $t$  steps is at most  $tX / (\binom{n}{2} - t) \ll 1$ .  $\square$

## 5 Open problems and conjectures

We have proposed a new model for a controlled random graph process that falls into the broader category of online decision-making under uncertainty. The question we considered is the following: given a monotone graph property  $\mathcal{P}$ , is there an online algorithm that decides whether to take or leave any arriving edge that obtains, **whp**, the desired property within given time and budget constraints. We analysed the model for several natural graph properties: connectivity, minimum degree, Hamiltonicity, perfect matchings and the containment of fixed-size trees and cycles. Our focal point was the investigation of the inevitable trade-off between the total number of edges Builder sees (maximum time) and the number of edges he must purchase (minimum budget). In some cases, the trade-off is substantial (for example, to construct a triangle in a close-to-optimal time, namely, when these just appear in the random graph process, Builder must purchase an almost-linear number of edges). For containment of fixed subgraphs, we have quantified that trade-off precisely.

Although we have made some sizable progress, there are a few challenging open problems to consider. Returning to our first result, we have shown (Theorem 1) that the budget needed to obtain minimum degree  $k$  at the hitting time is at most  $o_k n$ . We believe that this result is (asymptotically) optimal in the following sense:

**Conjecture 7.** *For every  $\varepsilon > 0$ , if  $b \leq (o_k - \varepsilon)n$  then for any  $(\tau_k, b)$ -strategy  $B$  of Builder,*

$$\lim_{n \rightarrow \infty} \mathbb{P}(\delta(B_{\tau_k}) \geq k) = 0.$$

We continued by showing (Theorem 2) that if we allow the time to be only asymptotically optimal, then an asymptotically optimal budget suffice. We believe that the same holds for  $k$ -connectivity. Namely, we conjecture the following:

**Conjecture 8.** *Let  $k$  be a positive integer and let  $\varepsilon > 0$ . If  $t \geq (1 + \varepsilon)n \log n/2$  and  $b \geq (1 + \varepsilon)kn/2$  then there exists a  $(t, b)$ -strategy  $B$  of Builder such that*

$$\lim_{n \rightarrow \infty} \mathbb{P}(B_t \text{ is } k\text{-connected}) = 1.$$

The above is (trivially) true for  $k = 1$ , as demonstrated in the introduction.

For Hamilton cycles, we showed (in Theorem 3) that if the time is (asymptotically) optimal, and the budget is inflated (by a constant factor), or if the budget is (asymptotically) optimal, and the time is inflated, then Builder has a strategy that constructs a Hamilton cycle **whp**. We believe that if the time and the budget are both asymptotically optimal, this is not doable. Namely, we conjecture the following:

**Conjecture 9.** *There exists  $\varepsilon > 0$  such that if  $t \leq (1 + \varepsilon)n \log n/2$  and  $b \leq (1 + \varepsilon)n$  then for any  $(t, b)$ -strategy  $B$  of Builder,*

$$\lim_{n \rightarrow \infty} \mathbb{P}(B_t \text{ is Hamiltonian}) = 0.$$

A weaker form of the above conjecture, according to which for any  $k = k(n) \geq 1$ , if  $t \ll n^2/k^2$  and  $b \leq n + k$ , then any  $(t, b)$ -strategy of Builder fails **whp**, follows almost directly from [23] (see Section 5 there). This implies that if  $t$  is asymptotically optimal, then, in particular, a budget of  $n + o(\sqrt{n}/\log n)$  does not suffice. The analogous conjecture for perfect matchings (see Theorem 4) also appears to be challenging.

Theorems 5 and 6 considered the budget thresholds for trees and cycles. It is not hard to extend our result on cycles to any fixed unicyclic graph: after obtaining the cycle, the remaining

forest can be constructed quickly and with a constant budget. The smallest graph not covered by our results is therefore the diamond (a  $K_4$  with one edge removed). One of the difficulties in this particular problem (or, possibly, in the case of any non-unicyclic graph) is that several quite different natural strategies, some are naive and some more sophisticated, turn out to give different upper bounds; each is superior in a different time regime. We handled a similar difficulty when discussing cycles but with lesser apparent complexity. It would be interesting to develop tools and approaches allowing to tackle the case of a generic fixed graph.

Finally, while *the* random graph process might be the most natural underlying graph process, other processes may be considered. One compelling example is the so-called *semi-random graph process* [6] mentioned in the introduction. The semi-random process has already exhibited some intriguing phenomena that significantly distinguish it from the (standard) random graph process. In our context, instead of observing a flow of random edges, Builder sees a flow of random vertices and, at each round, decides whether to connect the observed vertex to any other vertex. Evidently, Builder can achieve connectivity with  $t = b = n - 1$ ; answering questions concerning, e.g., Hamilton cycles or fixed subgraphs would be interesting.

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