

# Counting and packing Hamilton $\ell$ -cycles in dense hypergraphs

Asaf Ferber <sup>\*</sup>      Michael Krivelevich <sup>†</sup>      Benny Sudakov <sup>‡</sup>

## Abstract

A  $k$ -uniform hypergraph  $\mathcal{H}$  contains a Hamilton  $\ell$ -cycle, if there is a cyclic ordering of the vertices of  $\mathcal{H}$  such that the edges of the cycle are segments of length  $k$  in this ordering and any two consecutive edges  $f_i, f_{i+1}$  share exactly  $\ell$  vertices. We consider problems about packing and counting Hamilton  $\ell$ -cycles in hypergraphs of large minimum degree. Given a hypergraph  $\mathcal{H}$ , for a  $d$ -subset  $A \subseteq V(\mathcal{H})$ , we denote by  $d_{\mathcal{H}}(A)$  the number of distinct edges  $f \in E(\mathcal{H})$  for which  $A \subseteq f$ , and set  $\delta_d(\mathcal{H})$  to be the minimum  $d_{\mathcal{H}}(A)$  over all  $A \subseteq V(\mathcal{H})$  of size  $d$ . We show that if a  $k$ -uniform hypergraph on  $n$  vertices  $\mathcal{H}$  satisfies  $\delta_{k-1}(\mathcal{H}) \geq \alpha n$  for some  $\alpha > 1/2$ , then for every  $\ell < k/2$   $\mathcal{H}$  contains  $(1 - o(1))^n \cdot n! \cdot \left(\frac{\alpha}{\ell!(k-2\ell)!}\right)^{\frac{n}{k-\ell}}$  Hamilton  $\ell$ -cycles. The exponent above is easily seen to be optimal. In addition, we show that if  $\delta_{k-1}(\mathcal{H}) \geq \alpha n$  for  $\alpha > 1/2$ , then  $\mathcal{H}$  contains  $f(\alpha)n$  edge-disjoint Hamilton  $\ell$ -cycles for an explicit function  $f(\alpha) > 0$ . For the case where every  $(k-1)$ -tuple  $X \subset V(\mathcal{H})$  satisfies  $d_{\mathcal{H}}(X) \in (\alpha \pm o(1))n$ , we show that  $\mathcal{H}$  contains edge-disjoint Hamilton  $\ell$ -cycles which cover all but  $o(|E(\mathcal{H})|)$  edges of  $\mathcal{H}$ . As a tool we prove the following result which might be of independent interest: For a bipartite graph  $G$  with both parts of size  $n$ , with minimum degree at least  $\delta n$ , where  $\delta > 1/2$ , and for  $p = \omega(\log n/n)$  the following holds. If  $G$  contains an  $r$ -factor for  $r = \Theta(n)$ , then by retaining edges of  $G$  with probability  $p$  independently at random, w.h.p the resulting graph contains a  $(1 - o(1))rp$ -factor.

## 1 Introduction

Hamiltonicity is definitely one of the most studied properties of graphs in the last few decades, and many deep and interesting results have been obtained about it. In his seminal paper [6], Dirac proved that every graph on  $n$  vertices,  $n \geq 3$ , with minimum degree at least  $n/2$  is Hamiltonian. The complete bipartite graph  $K_{m,m+1}$  shows that this theorem is best possible, i.e., the minimum degree condition cannot be improved. Moreover, this extremal example hints that the addition of one more edge creates many Hamilton cycles. It thus natural to ask the following questions:

- (1) How many *edge-disjoint* Hamilton cycles does a *Dirac graph* (that is, a graph  $G$  on  $n$  vertices with minimum degree  $\delta(G) \geq n/2$ ) have?
- (2) How many *distinct* Hamilton cycles does a Dirac graph have?

---

<sup>\*</sup>Institute of Theoretical Computer Science ETH, 8092 Zürich, Switzerland. Email: asaf.ferber@inf.ethz.ch.

<sup>†</sup>School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, 69978, Israel. Email: krivelev@post.tau.ac.il. Research supported in part by USA-Israel BSF Grant 2010115 and by grant 912/12 from the Israel Science Foundation.

<sup>‡</sup>Department of Mathematics, ETH, 8092 Zürich, Switzerland. Email: benjamin.sudakov@math.ethz.ch. Research supported in part by SNSF grant 200021-149111.

These questions have been examined by various researchers and many results are known. Among them are Christofides, Kühn and Osthus [2], Cuckler and Kahn [5], Kühn, Lapinskas and Osthus [15], Nash-Williams [21, 22, 23], Sárközy, Selkow and Szemerédi [27], and the authors of this paper [7]. In particular, it is worth mentioning very recent remarkable results due to Csaba, Kühn, Lo, Osthus and Treglown [4] who settled the long standing conjectures made by Nash-Williams [21, 22, 23]. They showed that every  $d$ -regular Dirac graph contains  $\lfloor \frac{d}{2} \rfloor$  edge-disjoint Hamilton cycles, and that every graph  $G$  on  $n$  vertices with minimum degree  $\delta \geq n/2$  contains at least  $reg_{even}(n, \delta)/2$  edge-disjoint Hamilton cycles, where  $reg_{even}(n, \delta)$  denotes the largest degree of an even-regular spanning subgraph one can guarantee in a graph on  $n$  vertices with minimum degree  $\delta$ . These results are clearly optimal.

Note that if a graph  $G$  contains  $r$  edge-disjoint Hamilton cycles, then in particular  $G$  contains a  $2r$ -factor, that is, a spanning  $2r$ -regular subgraph. Therefore, the following question is also related to the two mentioned above:

- (3) Given a graph  $G$  with minimum degree  $\delta(G)$ , what is the maximal  $r$  for which  $G$  contains an  $r$ -factor?

As in Dirac's Theorem, the complete bipartite graph  $K_{m, m+1}$  with unbalanced parts demonstrates that for  $\delta(G) < n/2$  one can not expect to obtain even a 1-factor. The question about finding the maximal  $r := r(\delta, n)$  such that any graph  $G$  on  $n$  vertices with minimum degree  $\delta$  must contain an  $r$ -factor has also been investigated by various researchers. Among them are Katerinis [11] and Hartke, Martin and Seacrest [9]. The former showed that any Dirac graph contains an  $r$ -factor for  $r \geq \frac{n+5}{4}$  (he also gave an example of a Dirac graph  $G$  on  $n$  vertices that does not contain an  $\frac{n+6}{4}$ -factor), and the latter generalized the result to graphs with minimum degree  $\delta$ , with  $\delta \geq n/2$ .

In this paper we investigate analogous questions in hypergraphs. First we need to define the notion of a Hamilton cycle in a hypergraph. For two positive integers  $0 \leq \ell < k$ , a  $(k, \ell)$ -cycle is a  $k$ -uniform hypergraph whose vertices can be ordered cyclically such that the edges are segments of that order and such that every two consecutive edges share exactly  $\ell$  vertices. In case that  $0 \leq \ell \leq k/2$ , we refer to  $(k, \ell)$ -cycles as *loose cycles*. Now, let  $\mathcal{H}$  be a  $k$ -uniform hypergraph and let  $0 \leq \ell < k$ . We say that  $\mathcal{H}$  contains a *Hamilton  $\ell$ -cycle* if  $\mathcal{H}$  contains a  $(k, \ell)$ -cycle using all the vertices of  $\mathcal{H}$ . Note that in the case  $\ell = 0$  a Hamilton  $\ell$ -cycle corresponds to a *perfect matching*.

Analogously to graphs, the connection between the degrees in hypergraphs and the appearance of Hamilton  $\ell$ -cycles is well studied, and many results have been derived. Of course, an obvious necessary condition for a  $k$ -uniform hypergraph on  $n$  vertices to contain a Hamilton  $\ell$ -cycle is for  $(k - \ell)$  to divide  $n$ . Before we proceed to describe the previous work and to state our results, let us introduce some notation. Given a hypergraph  $\mathcal{H}$ , for a  $d$ -subset  $A \in \binom{V(\mathcal{H})}{d}$ , we denote by  $d_{\mathcal{H}}(A)$  the number of distinct edges  $f \in E(\mathcal{H})$  for which  $A \subseteq f$ , and set

$$\delta_d(\mathcal{H}) = \min d_{\mathcal{H}}(A), \text{ and } \Delta_d(\mathcal{H}) = \max d_{\mathcal{H}}(A),$$

where the minimum and the maximum are taken over all subsets  $A \subseteq V(\mathcal{H})$  of size exactly  $d$ . In a similar way, for two subsets  $X, Y \subseteq V(\mathcal{H})$ , we denote by  $d_{\mathcal{H}}(X, Y)$  the size of the *neighborhood of  $X$  in  $Y$* . That is,  $d_{\mathcal{H}}(X, Y) := |\{Z \subseteq Y : X \cup Z \in E(\mathcal{H})\}|$ . For a fixed set  $Y$  and an integer  $d < k$ , we set

$$\delta_d(Y) = \min d_{\mathcal{H}}(X, Y), \text{ and } \Delta_d(Y) = \max d_{\mathcal{H}}(X, Y),$$

where the minimum and maximum are taken over all subsets  $X \subseteq V(\mathcal{H})$  of size  $d$ .

Katona and Kierstead were the first to obtain a Dirac-type result for hypergraphs. They proved in [12] that if  $\delta_{k-1}(\mathcal{H}) \geq (1 - \frac{1}{2k})n + O_k(1)$ , then  $\mathcal{H}$  contains a Hamilton  $(k-1)$ -cycle. They also gave an example for a hypergraph  $\mathcal{H}$  with  $\delta_{k-1}(\mathcal{H}) = \lfloor \frac{n-k+3}{2} \rfloor$  which does not contain a Hamilton  $(k-1)$ -cycle, and implicitly conjectured that this is the correct bound. For  $k=3$ , this conjecture has been confirmed by Rödl, Ruciński and Szemerédi in [26]. For  $k \geq 4$ , it is proved in [25] that  $\delta_{k-1}(\mathcal{H}) \approx \frac{n}{2}$  is asymptotically the correct bound for the existence of a Hamilton  $(k-1)$ -cycle in  $\mathcal{H}$ . A construction of Markström and Ruciński from [19] demonstrates that  $\delta_{k-1}(\mathcal{H}) \approx \frac{n}{2}$  is necessary for having a perfect matching in  $\mathcal{H}$ , and since whenever  $(k-\ell)|k$ , a Hamilton  $\ell$ -cycle contains a perfect matching, one obtains that indeed  $\delta_{k-1}(\mathcal{H}) \approx \frac{n}{2}$  is the correct (asymptotic) bound for enforcing the existence of a Hamilton  $\ell$ -cycle for each such  $\ell$ . For values of  $\ell$  for which  $(k-\ell) \nmid k$ , Kühn, Mycroft and Osthus showed in [16] that  $\delta_{k-1}(\mathcal{H}) \approx \frac{n}{\lceil \frac{k}{k-\ell} \rceil (k-\ell)}$  is the correct asymptotic bound for enforcing the existence of a Hamilton  $\ell$ -cycle. There are many other important and interesting results regarding the connection between the minimum degree of a hypergraph and the existence of Hamilton  $\ell$ -cycles which we did not mention, and for a more complete list we refer the reader to the excellent survey of Rödl and Ruciński [24].

Now we are ready to state our main results. As far as we know, this paper is the first attempt to deal with Questions (1)–(3) in the hypergraph setting. In our first theorem we show that a dense  $k$ -uniform hypergraph contains the “correct” number of loose Hamilton cycles. That is, we show that given a  $k$ -uniform hypergraph  $\mathcal{H}$  on  $n$  vertices with  $\delta_{k-1}(\mathcal{H}) \geq \alpha n$ , the number of Hamilton  $\ell$ -cycles in  $\mathcal{H}$  is at least (up to a sub-exponential factor) the expected number of Hamilton  $\ell$ -cycles in a random  $k$ -uniform hypergraph with edge probability  $p = \alpha$  (that is, a hypergraph obtained by choosing every  $k$ -subset of  $[n]$  with probability  $p$ , independently at random). The expected number of such cycles is

$$(n-1)! \cdot \frac{k-\ell}{2} \cdot \left( \frac{\alpha}{\ell!(k-2\ell)!} \right)^{\frac{n}{k-\ell}}.$$

Indeed, first enumerate the vertices and define the edges of the  $(k, \ell)$ -cycle accordingly. Then, in each of the  $\frac{n}{k-\ell}$  edges, divide by the number of ways to order the first  $\ell$  vertices and the next  $k-2\ell$  vertices. Finally, divide by  $\frac{2n}{k-\ell}$ , which is the number of different ways to obtain the same cycle.

**Theorem 1.1** *Let  $\ell$  and  $k$  be integers satisfying  $0 \leq \ell < k/2$ , and let  $1/2 < \alpha \leq 1$ . Then, for sufficiently large integer  $n$  the following holds. Suppose that*

- (i)  $(k-\ell)|n$ , and
- (ii)  $\mathcal{H}$  is a  $k$ -uniform hypergraph on  $n$  vertices, and
- (iii)  $\delta_{k-1}(\mathcal{H}) \geq \alpha n$ .

*Then, the number of Hamilton  $\ell$ -cycles in  $\mathcal{H}$  is at least*

$$(1 - o(1))^n \cdot n! \cdot \left( \frac{\alpha}{\ell!(k-2\ell)!} \right)^{\frac{n}{k-\ell}}.$$

This is an extension to hypergraphs of the result obtained by Cuckler and Kahn [5] for the case of graphs. We remark that their bound is more accurate and is phrased in terms of certain entropy function over edge weighting of the graph. We will use their result in our proof.

Since in a  $k$ -uniform hypergraph  $\mathcal{H}$  on  $n$  vertices, a Hamilton  $\ell$ -cycle contains  $\frac{n}{k-\ell}$  edges, one cannot hope to find more than  $|E(\mathcal{H})|/\frac{n}{k-\ell}$  edge-disjoint such cycles. In the following theorem we show that indeed, up to a multiplicative factor, any dense  $k$ -uniform hypergraph  $\mathcal{H}$  contains the correct number of edge-disjoint loose Hamilton cycles.

**Theorem 1.2** *Let  $k$  and  $\ell$  be integers satisfying  $0 \leq \ell < k/2$ , and let  $1/2 < \alpha' < \alpha \leq 1$ . Then for all sufficiently large  $n$  the following holds. Suppose that*

- (i)  $(k - \ell)|n$ , and
- (ii)  $\mathcal{H}$  is a  $k$ -uniform hypergraph on  $n$  vertices, and
- (iii)  $\delta_{k-1}(\mathcal{H}) \geq \alpha n$ .

Then  $\mathcal{H}$  contains at least

$$(1 - o(1)) \cdot \frac{f(\alpha')|E(\mathcal{H})|}{\frac{n}{k-\ell}}$$

edge-disjoint Hamilton  $\ell$ -cycles, where  $f(x) = \frac{x + \sqrt{2x-1}}{2}$ .

We remark that we prove Theorem 1.2 by translating the problem into a problem of graphs. One of the ingredients of our proof is the ability to find a spanning and regular subgraph of a dense bipartite graph. In order to achieve this goal we use a result of Csaba [3] (which is tight for bipartite graphs), and this is where the function  $f$  in Theorem 1.2 comes from.

In the special case where the difference between  $\Delta_{k-1}(\mathcal{H})$  and  $\delta_{k-1}(\mathcal{H})$  is small, we obtain the following asymptotically optimal result.

**Theorem 1.3** *Let  $k$  and  $\ell$  be integers satisfying  $0 \leq \ell < k/2$ , and let  $1/2 < \alpha \leq 1$  be a constant. For every  $\delta > 0$  there exists  $\varepsilon > 0$  such that the following holds. For all sufficiently large  $n$ , if:*

- (i)  $(k - \ell)|n$ , and
- (ii)  $\mathcal{H}$  is a  $k$ -uniform hypergraph on  $n$  vertices, and
- (iii)  $\delta_{k-1}(\mathcal{H}) \geq \alpha n$ , and
- (iv)  $\Delta_{k-1}(\mathcal{H}) \leq (\alpha + \varepsilon)n$ .

Then all but at most  $\delta \binom{n}{k}$  edges of  $\mathcal{H}$  can be packed into Hamilton  $\ell$ -cycles.

Note that Theorem 1.3 is more general than the main result of [8] in the sense that we do not require any ‘‘pseudo-random’’ properties of the hypergraph (except, of course, the assumption that the degrees are large). On the other hand, Theorem 1.3 works only for hypergraphs which

are very dense, but it is known (see e.g. [16]) that below the densities we consider, there are constructions of hypergraphs without Hamilton cycles.

In the proofs of Theorems 1.2 and 1.3 we use (as a tool) the following theorem which is also of independent interest and is related to the concept of robustness of graph properties (see for example [14]). Before discussing and stating the theorem, let us introduce the following notation. Let  $G$  be a graph. Given a positive constant  $0 < p \leq 1$ , we say that a graph  $G'$  is distributed according to  $G_p$ , or  $G' \sim G_p$  for brevity, if  $G'$  is a subgraph of  $G$  obtained by retaining every edge of  $G$  with probability  $p$ , independently at random. In the following theorem we show that, given a bipartite graph  $G$  with both parts of size  $n$  and with  $\delta(G) \geq \alpha n$ , where  $\alpha > 1/2$ , if  $G$  contains an  $r$ -factor for  $r = \Theta(n)$ , then for  $p = \omega\left(\frac{\log n}{n}\right)$ , a random subgraph  $G' \sim G_p$  typically contains a  $(1 - o(1))rp$ -factor. The proof of the theorem appears in Section 2.3.

**Theorem 1.4** *Let  $1/2 < \alpha \leq 1$ ,  $\varepsilon > 0$  and  $0 < \rho \leq \alpha$  be positive constants. Then for sufficiently large integer  $n$ , the following holds. Suppose that:*

- (i)  $G$  is a bipartite graph with parts  $A$  and  $B$ , both of size  $n$ , and
- (ii)  $\delta(G) \geq \alpha n$ , and
- (iii)  $G$  contains a  $\rho n$ -factor.

Then, for  $p = \omega\left(\frac{\ln n}{n}\right)$ , with probability  $1 - n^{-\omega(1)}$  a graph  $G' \sim G_p$  has a  $k$ -factor for  $k = (1 - \varepsilon)\rho np$ .

**Remark 1.5** *We remark that the proof of Theorem 1.4 is still valid even if we choose each edge  $e \in E(G)$  with probability  $p_e \geq p$ . This follows from the monotonicity of the random model  $G_p$ .*

Let  $\mathcal{H}$  be a  $k$ -uniform hypergraph on  $n$  vertices with  $\delta_{k-1}(\mathcal{H}) \geq \alpha n$  for some  $\alpha > 1/2$ . Assume further that  $k \mid n$ . Now, by applying Theorem 1.2 with  $\ell = 0$  to  $\mathcal{H}$  one can obtain that  $\mathcal{H}$  contains an  $r$ -factor for every  $r \leq (1 - o(1))\frac{f(\alpha)|E(\mathcal{H})|}{k}$ . In the following proposition, by slightly extending a known construction, we show that there are hypergraphs  $\mathcal{H}$  with  $\delta_{k-1}(\mathcal{H}) \geq n/2 - O(1)$  which do not contain  $r$ -factors for many values of  $r$ .

**Proposition 1.6** *Let  $k \leq n$  be positive integers. Then there exists a  $k$ -uniform hypergraph  $\mathcal{H}$  on  $n$  vertices with  $\delta_{k-1}(\mathcal{H}) \geq n/2 - k - 1$ , which does not contain an  $r$ -factor for any odd integer  $r$ .*

## 2 Tools

In this section we introduce the main tools to be used in the proofs of our results.

## 2.1 Probabilistic tools

We need to employ standard bounds on large deviations of random variables. We mostly use the following well-known bound on the lower and the upper tails of the Binomial distribution due to Chernoff (see [1], [10]).

**Lemma 2.1** *Let  $X \sim \text{Bin}(n, p)$  and let  $\mu = \mathbb{E}(X)$ . Then*

- $\Pr[X < (1 - a)\mu] < e^{-a^2\mu/2}$  for every  $a > 0$ ;
- $\Pr[X > (1 + a)\mu] < e^{-a^2\mu/3}$  for every  $0 < a < 3/2$ .

*Remark: The conclusions of Lemma 2.1 remain the same when  $X$  has the hypergeometric distribution (see [10], Theorem 2.10).*

The following is a trivial yet useful bound.

**Lemma 2.2** *Let  $X \sim \text{Bin}(n, p)$  and  $k \in \mathbb{N}$ . Then the following holds:*

$$\Pr(X \geq k) \leq \left(\frac{enp}{k}\right)^k.$$

**Proof**  $\Pr(X \geq k) \leq \binom{n}{k} p^k \leq \left(\frac{enp}{k}\right)^k$ . □

We also make an extensive use of the following inequality, whose proof can be found at [20], Section 3.2.

**Theorem 2.3** *Let  $S_n$  denote the set of permutations of  $[n]$  and let  $f : S_n \rightarrow \mathbb{R}$  be such that  $|f(\pi) - f(\pi')| \leq u$  whenever  $\pi'$  is obtained from  $\pi$  by transposing two elements. Then if  $\pi$  is chosen randomly from  $S_n$  then*

$$\Pr[|f(\pi) - \mathbb{E}(f)| \geq t] \leq 2 \exp\left(-\frac{2t^2}{nu^2}\right).$$

## 2.2 Factors in graphs

In the proofs of our main results we translate the problem from hypergraphs to graphs by introducing some auxiliary graphs and then by trying to find large factors in each such graph. For this goal we will make use of the following theorem due to Csaba [3].

**Theorem 2.4** *Let  $G = (A \cup B, E)$  be a bipartite graph with parts of size  $n$  and with minimum degree  $\delta(G) \geq n/2$ . Then  $G$  contains a  $\lfloor \rho n \rfloor$ -factor for  $\rho = \frac{\delta + \sqrt{2\delta - 1}}{2}$ , where  $\delta := \delta(G)/n$ .*

In case the graph is almost regular, a better bound can be obtained as stated in the following theorem:

**Theorem 2.5** *For every  $\alpha > 1/2$  there exist  $\varepsilon_0 > 0$  and an integer  $n_0 = n(\alpha)$  such that for every  $n \geq n_0$  and  $\varepsilon \leq \varepsilon_0$  the following holds. Suppose that:*

- (i)  *$G$  is a bipartite graph with two parts  $A$  and  $B$  of size  $n$ , and*
- (ii)  *$d_G(v) \in (\alpha \pm \varepsilon)n$  for every  $v \in V(G)$ .*

*Then, for every  $r \leq (\alpha - 10\sqrt{\varepsilon})n$ ,  $G$  contains an  $r$ -factor.*

**Proof** [Sketch] Before we sketch the proof, note that there exists a standard bijection between bipartite graphs with parts of size  $n$  and digraphs (self loops are allowed!) on  $n$  vertices. For showing it, assume that  $G = (A \cup B, E)$  is a bipartite graph with  $|A| = |B| = n$ , and define a digraph  $D = (A, E')$  as follows (we may assume that  $A = B$ ): the arc  $ab \in E'$  if and only if the corresponding edge appears in  $G$ . For the other direction, assume that  $D = (V, E)$  is a digraph. Define  $G = (A \cup B, E')$  as follows: the parts  $A$  and  $B$  are two copies of  $V$ . An edge  $ab \in E'$  if and only if the arc  $ab \in E$ . Now, note that by deleting at most one edge adjacent to each vertex one can delete all loops and the proof of Theorem 2.5 follows immediately by combining Lemmas 13.2 and 5.2 of [17].  $\square$

In addition, we make use of the following theorem due to Cuckler and Kahn, which provides a good lower bound on the number of perfect matchings in a bipartite graph with respect to the minimum degree (see [5], p.3).

**Theorem 2.6** *Let  $G$  be a bipartite graph with both parts of size  $m$ , and let  $\delta(G) = \delta m \geq m/2$  be its minimum degree. Then the number of perfect matchings in  $G$  is at least*

$$\delta^m \cdot m! (1 - o(1))^m.$$

### 2.3 Factors in random subgraphs of dense graphs

In this subsection we prove Theorem 1.4. In the proof we make use of the following condition for having a  $k$ -factor in a bipartite graph due to Gale and Ryser [13] (a proof can also be found at [18], Problem 7.16).

**Proposition 2.7** *A bipartite graph  $G = (A \cup B, E)$  with  $|A| = |B|$  contains an  $r$ -factor if and only if for all  $X \subseteq A$  and  $Y \subseteq B$  the following holds:*

$$r|X| \leq e_G(X, Y) + r(|B| - |Y|).$$

Now we are ready to prove Theorem 1.4.

**Proof** Let  $G$  be a graph as described in the theorem. We wish to show that a graph  $G' \sim G_p$  is w.h.p such that

$$(*) \quad k|X| \leq e_{G_p}(X, Y) + k(n - |Y|),$$

for all  $X \subseteq A$  and  $Y \subseteq B$ , where  $k = (1 - \varepsilon)\rho np$  (and then by Proposition 2.7 we are done). We distinguish between several cases and consider each of them separately:

**Case 1:**  $|X| + |Y| \leq n$ . In this case, since  $n - |Y| \geq |X|$ , it follows that (\*) is trivial.

**Case 2:**  $|X| + |Y| > n$  (that is,  $|Y| \geq n - |X| + 1$ ) and  $|X| \leq f(n)$ , where  $f(n) = n/\ln \ln n$ . Here, since  $|Y| > n - |X| = (1 - o(1))n$ ,  $\delta(G) = \alpha n$  and  $\alpha > 1/2$ , it follows that  $e_G(X, Y) \geq (1 - o(1))\alpha n|X|$ . Using the fact that  $e_{G_p}(X, Y)$  is binomially distributed, applying Chernoff and the union bound we obtain that

$$\begin{aligned}
\Pr(\exists \text{ such } X, Y \text{ with } e_{G_p}(X, Y) \leq (1 - \varepsilon/2)e_G(X, Y)p) &\leq \sum_{x=1}^{f(n)} \sum_{y=n-x+1}^n \binom{n}{x} \binom{n}{y} e^{-\Theta(np x)} \\
&= \sum_{x=1}^{f(n)} \binom{n}{x} \left( \sum_{y=n-x+1}^n \binom{n}{y} \right) e^{-\Theta(np x)} \\
&\leq \sum_{x=1}^{f(n)} x \binom{n}{x} \binom{n}{x-1} e^{-\Theta(np x)} \\
&= \sum_{x=1}^{f(n)} \frac{x^2}{n-x+1} \binom{n}{x}^2 e^{-\Theta(np x)} \\
&\leq (f(n))^2 \cdot \sum_{x=1}^{f(n)} \binom{n}{x}^2 e^{-\Theta(np x)},
\end{aligned}$$

which is (recall that  $np = \omega(\ln n)$ ) at most

$$(f(n))^2 \cdot \sum_{x=1}^{f(n)} \left( \frac{e^2 n^2}{x^2} e^{-\omega(\ln n)} \right)^x = n^{-\omega(1)}.$$

Hence, since  $(1 - \varepsilon)\rho < (1 - 2\varepsilon/3)\alpha$ , it follows that with probability  $1 - n^{-\omega(1)}$  we have

$$\begin{aligned}
e_{G_p}(X, Y) + (1 - \varepsilon)\rho np(n - |Y|) &\geq e_{G_p}(X, Y) \\
&\geq (1 - 2\varepsilon/3)\alpha np|X| \geq (1 - \varepsilon)\rho np|X|,
\end{aligned}$$

for each such  $X$  and  $Y$ , and (\*) holds.

**Case 3:**  $|X| + |Y| > n$  and  $|X| > f(n)$ . Let

$$\eta_G(x, y) = \min\{e_G(X, Y) : X \subseteq A, Y \subseteq B, |X| = x, \text{ and } |Y| = y\}.$$

Note that by our assumptions we only consider  $x$  and  $y$  for which  $x \geq f(n)$  and  $x + y \geq n + 1$ .

Clearly,  $e_G(X, Y) \geq x(\alpha n + y - n)$  and  $e_G(X, Y) \geq y(\alpha n + x - n)$  for arbitrary sets  $X \subseteq A$  and  $Y \subseteq B$  of sizes  $x$  and  $y$ , respectively. Therefore we have that

$$\eta_G(x, y) \geq \max\{x(\alpha n + y - n), y(\alpha n + x - n)\}.$$

Assume first that  $x \leq y$  (and therefore, the maximum in the right hand side of the above inequality is  $x(\alpha n + y - n)$ ). Since  $\eta_G(x, y) \geq x(\alpha n + y - n)$ , it follows that for each such  $X$



and  $Y$  we have that  $e_G(X, Y) \geq x(\alpha n + y - n)$ . Applying Chernoff and the union bound, using the fact that  $\alpha n + y - n = \Theta(n)$  (here we use that  $\alpha - 1/2 \geq c > 0$  for some constant  $c$ ) we obtain that

$$\begin{aligned} \Pr(\exists \text{ such } X, Y \text{ with } e_{G_p}(X, Y) \leq (1 - \varepsilon)e_G(X, Y)p) &\leq 4^n \cdot \sum_{x=f(n)}^n e^{-\Theta(xnp)} \\ &= e^{-\omega(n)}. \end{aligned}$$

By symmetry, the above estimate is valid for  $y \leq x$  as well.

Now, recall that  $G$  contains a  $\rho n$ -factor and hence by Proposition 2.7 satisfies  $\rho n x \leq e_G(X, Y) + \rho n(n - y)$  for all  $X \subseteq A$  and  $Y \subseteq B$ . Multiply both sides by  $(1 - \varepsilon)p$ . Since if  $X$  and  $Y$  satisfy the assumptions of Case 3, we have with probability  $1 - n^{-\omega(1)}$  that  $e_{G_p}(X, Y) \geq (1 - \varepsilon)e_G(X, Y)p$ , and it follows that

$$\begin{aligned} (1 - \varepsilon)\rho n p |X| &\leq (1 - \varepsilon)e_G(X, Y)p + (1 - \varepsilon)\rho n p(n - |Y|) \\ &\leq e_{G_p}(X, Y) + (1 - \varepsilon)\rho n p(n - |Y|) \end{aligned}$$

holds for each  $X \subseteq A$  and  $Y \subseteq B$  covered by Case 3. Therefore, by Proposition 2.7 we conclude that with probability  $1 - n^{-\omega(1)}$  the random subgraph  $G_p$  contains a  $(1 - \varepsilon)\rho n p$ -factor as desired.  $\square$

## 2.4 Properties of random partitions of vertices

In this subsection we introduce several lemmas about properties of random partitions of vertices of dense hypergraphs. The following lemma shows that the vertex set of a dense  $k$ -uniform hypergraph can be partitioned in such a way that the proportion of the degrees to each part remains about the same as in the hypergraph.

**Lemma 2.8** *Let  $k$  be a positive integer and let  $\delta > 0$  and  $\varepsilon > 0$  be real numbers. Then, for every  $c > 0$  and a sufficiently large integer  $n$ , the following holds. Suppose that*

- (i)  $\mathcal{H}$  is a  $k$ -uniform hypergraph with  $n$  vertices, and
- (ii)  $\delta_{k-1}(\mathcal{H}) \geq \delta n + \varepsilon n$ , and
- (iii)  $m_1, \dots, m_t$  are integers such that  $m_i \geq cn$  for  $1 \leq i \leq t$ , and  $m_1 + \dots + m_t = n$ , and
- (iv)  $V(\mathcal{H}) = V_1 \cup \dots \cup V_t$  is a partition of  $V(\mathcal{H})$ , chosen uniformly at random among all partitions into  $t$  parts, with part  $V_i$  of size exactly  $m_i$  for every  $1 \leq i \leq t$ .

Then, with probability  $1 - e^{-\Theta(n)}$  the following holds:

$$\delta_{k-1}(V_i) \geq (\delta + 2\varepsilon/3)m_i \text{ for every } 1 \leq i \leq t.$$

**Proof** Let  $V(\mathcal{H}) = V_1 \cup \dots \cup V_t$  be a random partition of  $V(\mathcal{H})$  into  $t$  parts, each of size exactly  $m_i$ , and set

$$a_i = (\delta + 2\varepsilon/3)m_i.$$

Now, note that for each  $X \in \binom{V(\mathcal{H})}{k-1}$  and for each  $1 \leq i \leq t$ , the parameter  $d_{\mathcal{H}}(X, V_i)$  has a hypergeometric distribution with mean  $\mu \geq (\delta + \varepsilon)m_i$ . Therefore, by Lemma 2.1 it follows that

$$\Pr[d_{\mathcal{H}}(X, V_i) < a_i] \leq e^{-\Theta(m_i)} = e^{-\Theta(n)}.$$

Applying the union bound we obtain that

$$\Pr \left[ \exists X \in \binom{V(\mathcal{H})}{k-1} \text{ and } 1 \leq i \leq t \text{ such that } d_{\mathcal{H}}(X, V_i) < a_i \right] \leq \Theta(n^{k-1})e^{-\Theta(n)} = e^{-\Theta(n)}.$$

This completes the proof.  $\square$

Let  $\mathcal{H}$  be a  $k$ -uniform hypergraph on  $n$  vertices and let  $0 \leq \ell \leq k/2$  be an integer. Assume in addition that  $n$  is divisible by  $k - \ell$  and that our goal is to find Hamilton  $\ell$ -cycles in  $\mathcal{H}$ . We distinguish between two cases and for each case, in a similar way as in [8], we define an auxiliary graph that will serve us throughout the paper.

(1) Case  $1 \leq \ell < k/2$ . Let  $V(\mathcal{H}) = A \cup B$  be a partition of  $V(\mathcal{H})$  for which  $|A| = \ell \cdot \frac{n}{k-\ell}$ . Let  $\mathcal{M}_A = (F_0, \dots, F_{m-1})$  be a sequence of  $m := \frac{n}{k-\ell}$  disjoint  $\ell$ -subsets of  $A$  and let  $\mathcal{M}_B$  be a (non-ordered) collection of  $\frac{|B|}{k-2\ell} = (n - \ell \cdot \frac{n}{k-\ell}) / (k - 2\ell) = m$  disjoint  $(k - 2\ell)$ -subsets of  $B$ . Note that  $\mathcal{M}_A$  can be considered as a spanning  $(2\ell, \ell)$ -cycle of  $A$  and  $\mathcal{M}_B$  as a perfect matching of the complete  $(k - 2\ell)$ -uniform hypergraph on the vertex set  $B$ . Define an auxiliary bipartite graph  $G_{\mathcal{H}} := G(\mathcal{M}_A, \mathcal{M}_B, \mathcal{H}) = (S \cup T, E)$ , with both parts of size  $|S| = |T| = m$ , as follows:

- (i)  $S := \{F_i F_{i+1} : 0 \leq i \leq m - 1\}$  (we refer to  $m$  as 0), and
- (ii)  $T := \mathcal{M}_B$ , and
- (iii) for  $s \in S$  and  $t \in T$ ,  $st \in E$  if and only if  $t \cup F_i \cup F_{i+1} \in E(\mathcal{H})$ , where  $i$  is the unique integer for which  $s = F_i F_{i+1}$ .

A moment's thought now reveals that there is an injection between perfect matchings of  $G_{\mathcal{H}}$  and Hamilton  $\ell$ -cycles of  $\mathcal{H}$ . This fact is used extensively throughout the paper.

(2) Case  $\ell = 0$  (note that a Hamilton 0-cycle is a perfect matching). Here we take a partition  $V(\mathcal{H}) = A \cup B$  into two sets  $A$  and  $B$  such that  $|A| = \frac{\lfloor k/2 \rfloor \cdot n}{k}$ . Let  $\mathcal{M}_A$  be a collection of  $\frac{n}{k}$  disjoint subsets of  $A$ , each of size exactly  $\lfloor k/2 \rfloor$ , and let  $\mathcal{M}_B$  be a collection of  $\frac{n}{k}$  disjoint subsets of  $B$ , each of size exactly  $\lceil k/2 \rceil$ . Define an auxiliary bipartite graph  $G_{\mathcal{H}} := G(\mathcal{M}_A, \mathcal{M}_B, \mathcal{H}) = (S \cup T, E)$ , with parts  $S$  and  $T$  as follows:

- (i)  $S = \mathcal{M}_A$  and  $T = \mathcal{M}_B$ , and
- (ii) for  $s \in S$  and  $t \in T$ ,  $st \in E$  if and only if  $s \cup t \in E(\mathcal{H})$ .

Note that in this case every perfect matching in  $G_{\mathcal{H}}$  corresponds to a perfect matching (a Hamilton  $(k, 0)$ -cycle) of  $\mathcal{H}$ .

The following lemma shows that by picking  $V(\mathcal{H}) = A \cup B$ ,  $\mathcal{M}_A$  and  $\mathcal{M}_B$  at random, the auxiliary graph  $G_{\mathcal{H}}$  typically possesses some desirable properties.

**Lemma 2.9** *Let  $\ell$  and  $k$  be integers for which  $0 \leq \ell < k/2$ . Let  $\delta > 0$  and  $\varepsilon > 0$  be real numbers. Then, for sufficiently large integers  $n$  the following holds. Suppose that*

- (i)  $(k - \ell) | n$ , and
- (ii)  $\mathcal{H}$  is a  $k$ -uniform hypergraph on  $n$  vertices, and
- (iii)  $\delta_{k-1}(\mathcal{H}) \geq \delta n + \varepsilon n$ .

*Then, for a random uniform choice of  $A$ ,  $B$ ,  $\mathcal{M}_A$  and  $\mathcal{M}_B$  as described above, with probability  $1 - e^{-\Theta(n)}$  we get that  $\delta(G_{\mathcal{H}}) \geq (\delta + \varepsilon/2)m$ , where  $m = |\mathcal{M}_A|$ .*

**Proof** First, consider the case where  $1 \leq \ell < k/2$ . Let  $V(\mathcal{H}) = A \cup B$  be a typical partition as obtained by Lemma 2.8 with  $m_1 = \frac{\ell}{k-\ell} \cdot n$  and  $m_2 = n - m_1$ . The conclusion of Lemma 2.9 for this case is an immediate consequence of the following two claims:

**Claim 2.10** *With probability  $1 - e^{-\Theta(n)}$  a random collection  $\mathcal{M}_B$  as described above is such that*

$$|\{Y \in \mathcal{M}_B : X \cup Y \in E(\mathcal{H})\}| \geq (\delta + \varepsilon/2)|\mathcal{M}_B|$$

*holds for each  $X \in \binom{V(\mathcal{H})}{2\ell}$ . In particular,  $d_{G_{\mathcal{H}}}(s) \geq (\delta + \varepsilon/2)m$  for every  $s \in S$ .*

**Proof** We pick  $\mathcal{M}_B$  as follows: Let  $\{v_0, \dots, v_{|B|-1}\}$  be a random enumeration of the elements of  $B$  and define

$$\mathcal{M}_B := \{\{v_j, \dots, v_{j+k-2\ell-1}\} : j = (k - 2\ell)i, 0 \leq i \leq m - 1\}.$$

Now, for a subset  $X \in \binom{V(\mathcal{H})}{2\ell}$ , define  $d_B(X) = |\{Y \in \mathcal{M}_B : X \cup Y \in E(\mathcal{H})\}|$ . We wish to show that

$$\Pr \left[ \exists X \in \binom{V(\mathcal{H})}{2\ell} \text{ such that } d_B(X) < (\delta + \varepsilon/2)m \right] = e^{-\Theta(n)}.$$

Indeed, fix  $X \in \binom{V(\mathcal{H})}{2\ell}$ , and for each  $0 \leq i \leq m - 1$ , let  $Y_i$  be the indicator random variable for the event “ $X \cup \{v_j, \dots, v_{j+k-2\ell-1}\} \in E(\mathcal{H})$ ”, where  $j = (k - 2\ell) \cdot i$ . Starting the enumeration of the elements of  $B$  from the  $j^{\text{th}}$  place and using the fact that

$$d_{\mathcal{H}}(X \cup \{v_j, \dots, v_{j+k-2\ell-2}\}, B) \geq (\delta + 2\varepsilon/3)|B|,$$

we obtain that  $\mathbb{E}(Y_i) \geq \delta + 2\varepsilon/3$ , for every  $0 \leq i \leq m - 1$ . Hence,

$$\mathbb{E}(d_B(X)) = \sum_{i=0}^{m-1} \mathbb{E}(Y_i) \geq (\delta + 2\varepsilon/3)m.$$

Now, given an enumeration of  $B = \{v_0, \dots, v_{|B|-1}\}$ , by switching between two elements  $v_i$  and  $v_j$ ,  $d_B(X)$  can change by at most 2. Therefore, using Theorem 2.3 it follows that

$$\begin{aligned} \Pr[d_B(X) < (\delta + \varepsilon/2)|\mathcal{M}_B|] &\leq \Pr[d_B(X) < \mathbb{E}(d_B(X)) - \varepsilon m/6] \\ &\leq 2 \exp\left(\frac{-\varepsilon^2 m^2}{|B|72}\right) = e^{-\Theta(n)}. \end{aligned}$$

Applying the union bound we obtain that

$$\Pr\left[\exists X \in \binom{V(\mathcal{H})}{2\ell} \text{ such that } d_B(X) < (\delta + \varepsilon/2)|\mathcal{M}_B|\right] \leq \Theta(n^{2\ell})e^{-\Theta(n)} = e^{-\Theta(n)}$$

as desired.  $\square$

**Claim 2.11** *With probability  $1 - e^{-\Theta(n)}$ , a random (enumerated) collection  $\mathcal{M}_A = \{F_0, \dots, F_{m-1}\}$  as described above is such that*

$$|\{i : 0 \leq i \leq m-1 \text{ and } X \cup F_i \cup F_{i+1} \in E(\mathcal{H})\}| \geq (\delta + \varepsilon/2)m$$

*holds for each  $X \in \binom{V(\mathcal{H})}{k-2\ell}$ . In particular,  $d_{G_{\mathcal{H}}}(t) \geq (\delta + \varepsilon/2)m$  for every  $t \in T$ .*

**Proof** We pick  $\mathcal{M}_A$  as follows: Let  $\{u_0, \dots, u_{|A|-1}\}$  be a random enumeration of the elements of  $A$ , and for each  $0 \leq i \leq m-1$ , define

$$F_i = \{u_{\ell \cdot i}, \dots, u_{\ell \cdot (i+1) - 1}\}$$

and set

$$\mathcal{M}_A = \{F_0, \dots, F_{m-1}\}.$$

Now, for a subset  $X \in \binom{V(\mathcal{H})}{k-2\ell}$ , define  $d_A(X) = |\{i : 0 \leq i \leq m-1 \text{ and } X \cup F_i \cup F_{i+1} \in E(\mathcal{H})\}|$ , we wish to show that

$$\Pr\left[\exists X \in \binom{V(\mathcal{H})}{k-2\ell} \text{ such that } d_A(X) < (\delta + \varepsilon/2)m\right] = e^{-\Theta(n)}.$$

Indeed, fix  $X \in \binom{V(\mathcal{H})}{k-2\ell}$ , and for each  $0 \leq i \leq m-1$ , let  $X_i$  be the indicator random variable for the event “ $X \cup F_i \cup F_{i+1} \in E(\mathcal{H})$ ”. From here, the proof is similar to the proof of Claim 2.10 so we omit the details (the only difference is that here, switching two elements can change  $d_A(X)$  by at most 4 and not 2, which does not cause any problem).  $\square$

Next, consider the case where  $\ell = 0$ . In this case, let  $V(\mathcal{H}) = A \cup B$  be a typical partition as obtained by Lemma 2.8 with  $t = 2$  and  $m_1 = \frac{\lfloor k/2 \rfloor n}{k}$ . Now, randomly define  $\mathcal{M}_A$  and  $\mathcal{M}_B$  as described above. Finally, Claim 2.10 shows that with high probability we obtain  $\delta(G_{\mathcal{H}}) \geq (\delta + \varepsilon/2)m$  as desired.

This completes the proof.  $\square$

**Remark 2.12** *If we change Condition (iii) of Lemma 2.9 to  $(\delta - \varepsilon)n \leq \delta_{k-1}(\mathcal{H}) \leq \Delta_{k-1}(\mathcal{H}) \leq (\delta + \varepsilon)n$ , then the same proof (more or less line by line) shows that  $d_{G_{\mathcal{H}}}(v) \in (\delta \pm 2\varepsilon)m$  for every  $v \in V(G_{\mathcal{H}})$ . We will make use of this fact in the proof of Theorem 1.3.*

### 3 Proofs of the main results

#### 3.1 Proof of Theorem 1.1

In this subsection we prove Theorem 1.1.

**Proof** In order to prove Theorem 1.1 we show that for every  $\varepsilon > 0$ , the number of Hamilton  $\ell$ -cycles in  $\mathcal{H}$  is at least

$$(1 - o(1))^n \cdot n! \cdot \left( \frac{\alpha - \varepsilon/2}{\ell!(k - 2\ell)!} \right)^{\frac{n}{k-\ell}}.$$

Let  $\varepsilon > 0$  be a positive constant. Denote  $\delta = \alpha - \varepsilon$  and observe that  $\delta_{k-1}(\mathcal{H}) \geq (\delta + \varepsilon)n$ . First, consider the case where  $1 \leq \ell < k/2$ . Assume that  $V(\mathcal{H}) = A \cup B$  is a partition of  $V(\mathcal{H})$  into two sets  $A$  and  $B$  with  $|A| = \ell \cdot \frac{n}{k-\ell}$ , equipped with  $\mathcal{M}_A$  and  $\mathcal{M}_B$  as described in Subsection 2.4. By applying Lemma 2.9 to  $\mathcal{H}$ , it follows that a  $(1 - o(1))$ -fraction of these partitions are such that  $\delta(G_{\mathcal{H}}) \geq (\delta + \varepsilon/2)m$  (where  $m = \frac{n}{k-\ell}$  and  $G_{\mathcal{H}}$  is the auxiliary graph as defined in Subsection 2.4). Now, using Theorem 2.6 we obtain that the number of perfect matchings in each such  $G_{\mathcal{H}}$  is at least

$$(1 - o(1))^n (\delta + \varepsilon/2)^m \cdot m! = (1 - o(1))^n (\delta + \varepsilon/2)^{\frac{n}{k-\ell}} \left( \frac{n}{k-\ell} \right)!.$$

Next, note that each perfect matching of  $G_{\mathcal{H}}$  corresponds to a Hamilton  $\ell$ -cycle. Moreover, note that given two such partitions  $A \cup B$  and  $A' \cup B'$  of  $V(\mathcal{H})$ , if  $A \neq A'$ , then clearly the obtained Hamilton  $\ell$ -cycles coming from these partitions are distinct. In case  $A = A'$ , note that as long as  $\mathcal{M}_A$  and  $\mathcal{M}_{A'}$  do not define the same cyclic ordering for the same sets  $F_0, \dots, F_{m-1}$  (for a fixed  $\mathcal{M}_A$  there are at most  $2m$  such  $\mathcal{M}'_A$ ), the Hamilton  $\ell$ -cycles obtained from such distinct structures are all distinct. All in all, combining the above mentioned, each Hamilton  $\ell$ -cycle is being counted at most  $2m$  times and we obtain that the number of Hamilton  $\ell$ -cycles in  $\mathcal{H}$  is at least

$$\begin{aligned} & \frac{1}{2m} \cdot (1 - o(1))^n \cdot n! \cdot \left( \frac{\delta + \varepsilon/2}{\ell!(k - 2\ell)!} \right)^{\frac{n}{k-\ell}} \\ &= (1 - o(1))^n \cdot n! \cdot \left( \frac{\alpha - \varepsilon/2}{\ell!(k - 2\ell)!} \right)^{\frac{n}{k-\ell}}. \end{aligned}$$

Indeed, we need to multiply the above estimate by the number of auxiliary graphs  $G_{\mathcal{H}}$ . For this, take a permutation of  $V(\mathcal{H})$ , define  $A$  to be its first  $\ell \cdot \frac{n}{k-\ell}$  vertices,  $\mathcal{M}_A$  to be the first  $\frac{n}{k-\ell}$  consecutive (and disjoint)  $\ell$ -tuples, and  $\mathcal{M}_B$  to be the last  $\frac{n}{k-\ell}$  consecutive  $(k - 2\ell)$ -tuples. Then, divide by the ordering inside the tuples and the ordering between the tuples in  $\mathcal{M}_B$ .

Next, for the case where  $\ell = 0$  the proof is more or less the same. Here, the partitions we consider are of the form  $(A, B)$  where  $|A| = \frac{\lfloor k/2 \rfloor \cdot n}{k}$ , and  $\mathcal{M}_A$  in the definition of the auxiliary graph  $G_{\mathcal{H}}$  is just a collection of sets, not enumerated. In addition, every perfect matching can be obtained by  $\binom{k}{\lfloor k/2 \rfloor}^{\frac{n}{k}}$  partitions (from each edge, choose  $\lfloor k/2 \rfloor$  elements to be in  $A$ ). All in all, there are at least

$$\begin{aligned}
& \frac{n!}{(\lfloor k/2 \rfloor! \lceil k/2 \rceil!)^{\frac{n}{k}}} \cdot \frac{((1 - o(1))(\delta + \varepsilon/2))^{\frac{n}{k}} (n/k)!}{\binom{k}{\lfloor k/2 \rfloor}^{\frac{n}{k}} (n/k)! (n/k)!} \\
&= \frac{n!}{(k!)^{\frac{n}{k}} (n/k)!} \cdot ((1 - o(1))(\delta + \varepsilon/2))^{\frac{n}{k}} \\
&= (1 - o(1))^n \cdot n! \cdot \left( \frac{\alpha - \varepsilon/2}{k!} \right)^{\frac{n}{k}}
\end{aligned}$$

perfect matchings in  $\mathcal{H}$ . This completes the proof of Theorem 1.1.  $\square$

### 3.2 Proofs of Theorems 1.2 and 1.3

In this subsection we prove Theorems 1.2 and 1.3. We start with Theorem 1.2.

**Proof** The proof is rather similar to the proof of the main results in [8]. The main difference is that here we use Theorem 1.4 in order to find many edge-disjoint perfect matchings in random subgraphs of a graph which is not necessarily the complete bipartite graph. Let  $\varepsilon = \alpha - \alpha'$ , and note that  $\delta_{k-1}(\mathcal{H}) \geq (\alpha' + \varepsilon)n$ . We distinguish between two cases:

**Case I:**  $1 \leq \ell < k/2$ . In this case the general scheme goes as follows:

First, choose  $r := |E(\mathcal{H})| \cdot \left( \frac{(k-\ell)\ln n}{n} \right)^2$  random partitions of  $V(\mathcal{H})$ ,  $\{(A_i, B_i) : 1 \leq i \leq r\}$ , such that  $|A_i| = \ell \cdot \frac{n}{k-\ell}$  for each  $i$ , equipped with  $\mathcal{M}_{A_i}$  and  $\mathcal{M}_{B_i}$  as described in Section 2.4. For each such partition  $(A_i, B_i)$ , denote the corresponding auxiliary graph  $G_{\mathcal{H}}$  by  $G^{(i)}$ , and use the notation  $\mathcal{M}_{A_i} = \{F_{i,0}, \dots, F_{i,m-1}\}$ , where  $m = \frac{n}{k-\ell}$ . Note that by Lemma 2.9 we have that w.h.p  $\delta(G^{(i)}) \geq (\alpha' + \varepsilon/2)m$  for every  $1 \leq i \leq r$ .

Second, for each edge  $f \in E(\mathcal{H})$ , we say that  $i$  is a *candidate* for  $f$  if there exist  $j$  and  $B \in \mathcal{M}_{B_i}$  such that  $f = F_{i,j} \cup B \cup F_{i,j+1}$ . For each edge  $f \in E(\mathcal{H})$  let  $\psi(f)$  denote the number of candidates it has. Each  $f \in E(\mathcal{H})$  with  $\psi(f) > 0$  picks one candidate  $i$  at random among the  $\psi(f)$  candidates. For each  $1 \leq i \leq r$ , consider the subhypergraph  $\mathcal{H}_i$  obtained from the partition  $(A_i, B_i)$  together with the edges that chose  $i$ , and denote the corresponding auxiliary subgraph of  $G^{(i)}$  by  $H_i$ . Observe that the hypergraphs  $\mathcal{H}_i$  are edge-disjoint.

Finally, we wish to show that w.h.p every auxiliary graph  $H_i$  contains  $(1 - o(1)) \frac{f(\alpha')m}{\ln^2 n}$  edge-disjoint perfect matchings, where  $f(\alpha') = \frac{\alpha' + \sqrt{2\alpha' - 1}}{2}$ . We then conclude that every subhypergraph  $\mathcal{H}_i$  contains  $(1 - o(1)) \frac{f(\alpha')m}{\ln^2 n}$  edge-disjoint Hamilton  $\ell$ -cycles for each  $i$ , and therefore  $\mathcal{H}$  contains at least

$$(1 - o(1))r \cdot \frac{f(\alpha')m}{\ln^2 n} = (1 - o(1)) \frac{|E(\mathcal{H})| \cdot f(\alpha')}{\frac{n}{k-\ell}}$$

edge-disjoint Hamilton  $\ell$ -cycles as required. To this end we need the following claim:

**Claim 3.1** *With high probability the following holds: every  $H_i$  contains at least  $(1 - o(1)) \frac{f(\alpha')n}{\ln^2 n}$  edge-disjoint perfect matchings.*

**Proof** Let  $f \in E(\mathcal{H})$  be an edge and recall that the random variable  $\psi(f)$  counts the number of partitions  $(A_i, B_i)$  which are candidates for  $f$ . Observe that for every edge  $f$  and index  $i$ , the  $i^{\text{th}}$  partition is a candidate for  $f$  with the same probability

$$q \leq \frac{|\mathcal{M}_{A_i}| \cdot |\mathcal{M}_{B_i}|}{|E(\mathcal{H})|} = \frac{m^2}{|E(\mathcal{H})|}.$$

(Indeed, the partition  $(A_i, B_i)$  is a candidate for at most  $|\mathcal{M}_{A_i}| \cdot |\mathcal{M}_{B_i}| = m^2$  edges, among all the  $|E(\mathcal{H})| = \Theta(n^k)$  edges of the hypergraph. Due to symmetry this bound is obtained.)

Therefore, since  $\psi(f) \sim \text{Bin}(r, q)$ , applying Chernoff and the union bound we obtain that w.h.p  $\psi(f) \leq (1+o(1))r \cdot q \leq (1+o(1)) \ln^2 n$  for each  $f \in E(\mathcal{H})$ . Hence, we conclude that the edges of  $G^{(i)}$  remain in  $H_i$  with probability  $p \geq \frac{1-o(1)}{\ln^2 n}$ . Now, combining Theorem 2.4, Remark 1.5, and applying the union bound we conclude that w.h.p. every  $H_i$  contains a  $(1-o(1)) \frac{f(\alpha')m}{\ln^2 n}$ -factor. In order to complete the proof, recall that  $H_i$  is bipartite and thus each such factor can be decomposed to  $(1-o(1)) \frac{f(\alpha')m}{\ln^2 n}$  edge-disjoint perfect matchings.  $\square$

**Case II:**  $\ell = 0$ . The proof for this case is similar to previous case so we omit it. The only difference is that here we use a slightly different auxiliary graph, so for this case we need to take  $r = |E(\mathcal{H})| \cdot \left(\frac{k \ln n}{n}\right)^2$  partitions  $(A_i, B_i)$  with  $|A_i| = \frac{\lfloor k/2 \rfloor \cdot n}{k}$ . All the other calculations remain the same.

This completes the proof of Theorem 1.2.  $\square$

Now we prove Theorem 1.3.

**Proof** The proof of Theorem 1.3 is quite similar to the previous proof, so we might omit few details. Let  $\delta > 0$  be a constant, let  $\varepsilon > 0$  be a sufficiently small constant (to be determined later), and let  $\mathcal{H}$  be a  $k$ -uniform hypergraph which satisfies the assumptions of the theorem. Throughout the proof we use similar notation as in the proof of Theorem 1.2.

First, let  $(A, B)$  be a random partition of  $V(\mathcal{H})$  into two sets with  $|A| = \ell \cdot \frac{n}{k-\ell}$ , equipped with  $\mathcal{M}_A$  and  $\mathcal{M}_B$  as described in Section 2.4. Using Remark 2.12 we conclude that with probability  $1 - e^{-\Theta(n)}$  we have  $(\alpha - \varepsilon)m \leq \delta(G_{\mathcal{H}}) \leq \Delta(G_{\mathcal{H}}) \leq (\alpha + 2\varepsilon)m$ . Conditioning on that, similarly to the calculation in Claim 3.1, we conclude that for such a partition  $(A, B)$  and an edge  $f \in E(\mathcal{H})$ , the probability that  $(A, B)$  is a candidate for  $f$  is bounded between

$$\frac{|\mathcal{M}_A| \cdot (\alpha - \varepsilon)m}{|E(\mathcal{H})|} = \frac{(\alpha - \varepsilon)m^2}{|E(\mathcal{H})|}$$

and

$$\frac{|\mathcal{M}_A| \cdot (\alpha + 2\varepsilon)m}{|E(\mathcal{H})|} = \frac{(\alpha + 2\varepsilon)m^2}{|E(\mathcal{H})|}.$$

Second, let  $q = \frac{(\alpha - \varepsilon)m^2}{|E(\mathcal{H})|}$  (clearly,  $q$  is a lower bound for that probability), and choose  $r := |E(\mathcal{H})| \cdot \left(\frac{k-\ell}{n}\right)^2 \cdot \frac{1}{q}$  random partitions of  $V(\mathcal{H})$ ,  $\{(A_i, B_i) : 1 \leq i \leq r\}$ , such that  $|A_i| = \ell \cdot \frac{n}{k-\ell}$  for each  $i$ , equipped with  $\mathcal{M}_{A_i}$  and  $\mathcal{M}_{B_i}$  as described in Section 2.4 (and in the proof of Theorem 1.2).

Third, since  $\psi(f)$  is binomially distributed with probability  $q \leq q_f \leq (1 + 7\varepsilon)q$ , by Chernoff's inequality and the union bound we obtain that  $\psi(f) \in (1 \pm 8\varepsilon)rq$  holds for each  $f \in E(\mathcal{H})$ .

Next, using the fact that all the  $G^{(i)}$ 's are almost regular (all the degrees lie in the interval  $(\alpha \pm 2\varepsilon)m$ ), combining Theorem 2.5 with Theorem 1.4, using the fact that  $\varepsilon$  is sufficiently small, we obtain that with probability  $1 - n^{-\omega(1)}$  each  $H_i$  contains at least  $(1 - o(1))(\alpha - 20\sqrt{2\varepsilon})\frac{m}{rq}$  edge-disjoint perfect matchings. Therefore, for each  $i$ , by taking all the edge-disjoint Hamilton  $\ell$ -cycles in  $\mathcal{H}_i$ , there is at most a  $40\sqrt{2\varepsilon}$ -fraction of edges in  $\mathcal{H}_i$  which are unused. All in all, there is at most  $40\sqrt{2\varepsilon}$ -fraction of edges in  $\mathcal{H}$  which are not covered by any of the Hamilton  $\ell$ -cycles. Finally, by taking  $\varepsilon$  to be small enough such that  $40\sqrt{2\varepsilon} \leq \delta$  we complete the proof.  $\square$

### 3.3 Proof of Proposition 1.6

**Proof** Let  $k \leq n$  be positive integers. Define a  $k$ -uniform hypergraph  $\mathcal{H}$  on  $n$  vertices as follows: Let  $V(\mathcal{H}) = [n]$ , and partition  $V(\mathcal{H}) = A \cup B$  into two sets  $A$  and  $B$  such that  $n/2 - 1 \leq |A| \leq n/2 + 1$  is an odd integer. Let  $E(\mathcal{H})$  consists of all the  $k$ -tuples  $f \in \binom{[n]}{k}$  for which  $|A \cap f|$  is even, and observe that  $\delta_{k-1}(\mathcal{H}) \geq n/2 - k$ . Now, let  $r$  be an odd integer and assume towards a contradiction that  $\mathcal{H}$  contains an  $r$ -factor  $\mathcal{H}' \subseteq \mathcal{H}$ . Let  $\mathcal{H}''$  be the multi-hypergraph on the vertex set  $A$  which consists of the (multi-)set of edges  $\{A \cap f : f \in E(\mathcal{H}')\}$ . Since all the edges of  $\mathcal{H}''$  are of even size, the size of  $A$  is odd, and since all the vertex degrees are  $r$  (which is odd), we derive a contradiction.  $\square$

## 4 Concluding remarks and open problems

To the best of our knowledge, this paper is the first to deal with problems of counting and packing in general dense hypergraphs. Here we have obtained some preliminary results, which suggest many interesting and challenging problems for further study.

In Theorem 1.1 we showed that, for every  $\ell < k/2$ , the number of Hamilton  $\ell$ -cycles in  $k$ -uniform hypergraphs with large minimum degree is lower bounded (up to sub-exponential factor) with the expected number of such cycles in a random hypergraph with the same density. It would be interesting to generalize it to every  $\ell < k$ .

In Theorems 1.2 and 1.3 we dealt with the question of packing Hamilton  $\ell$ -cycles into dense  $k$ -uniform hypergraph. We showed that for  $\ell < k/2$ , if  $\delta_{k-1}(\mathcal{H}) \geq \alpha n$ , for some  $\alpha > 1/2$ , then one can find  $\frac{f(\alpha)|E(\mathcal{H})|}{\binom{n}{k-\ell}}$  edge-disjoint Hamilton  $\ell$ -cycles. It is natural to try to obtain the best possible  $f(\alpha)$ , and to try to generalize our results for every  $\ell \leq k - 1$ .

As was mentioned in the introduction, Kühn, Mycroft and Osthus showed in [16] that  $\delta_{k-1} \approx \frac{n}{\lceil \frac{k}{k-\ell} \rceil (k-\ell)}$  is the correct asymptotic bound for the existence of a Hamilton  $\ell$ -cycle. Note that for certain choices of  $k$  and  $\ell$  (for example,  $k = 3$  and  $\ell = 1$ ), this bound is much smaller than the bound of  $n/2$  that we considered. It would be nice to extend our results to hypergraphs with minimum degrees starting at  $\frac{n}{\lceil \frac{k}{k-\ell} \rceil (k-\ell)}$ , for every  $\ell < k$ .

**Acknowledgement.** A major part of this work was carried out when Benny Sudakov was visiting Tel Aviv University, Israel. He would like to thank the School of Mathematical Sciences of Tel Aviv University for hospitality and for creating a stimulating research environment.



## References

- [1] N. Alon and J. H. Spencer, **The Probabilistic Method**, third ed., Wiley, New York, 2008.
- [2] D. Christofides, D. Kühn and D. Osthus, *Edge-disjoint Hamilton cycles in graphs*, J. Combinatorial Theory Series B (2012) 102, 1035–1060.
- [3] B. Csaba, *Regular spanning subgraphs of bipartite graphs of high minimum degree*, Electronic Journal of Combinatorics, (2007) 14 (1), publication 21.
- [4] B. Csaba, D. Kühn, A. Lo, D. Osthus and A. Treglown, *Proof of the 1-factorization and Hamilton decomposition conjectures*, Memoirs of the American Mathematical Society, to appear.
- [5] B. Cuckler and J. Kahn, *Hamiltonian cycles in Dirac graphs*, Combinatorica (2009) 29 (3), 299–326.
- [6] G. A. Dirac, *Some theorems on abstract graphs*, Proceedings of the London Mathematical Society, (1952) 2, 69–81.
- [7] A. Ferber, M. Krivelevich and B. Sudakov, *Counting and packing Hamilton cycles in dense graphs and oriented graph*, to appear in J. Combinatorial Theory Series B.
- [8] A. Frieze and M. Krivelevich, *Packing Hamilton cycles in random and pseudo-random hypergraphs*, Random Structures and Algorithms 41 (2012), 1–22.
- [9] S. G. Hartke, R. Martin, and T. Seacrest, *Relating minimum degree and the existence of a  $k$ -factor*, Research manuscript (2010).
- [10] S. Janson, T. Łuczak and A. Ruciński, **Random graphs**, Wiley, New York, 2000.
- [11] P. Katerinis, *Minimum degree of a graph and the existence of  $k$ -factors*, In Proceedings of the Indian Academy of Sciences-Mathematical Sciences, vol. 94, no. 2-3, pp. 123–127. Springer India, 1985.
- [12] G. Y. Katona and H. A. Kierstead, *Hamiltonian chains in hypergraphs*, J. Graph Theory, (1999) 30, 205–212.
- [13] M. Krause, *A simple proof of the Gale-Ryser theorem*, The American Mathematical Monthly 4 (1996), 335–337.
- [14] M. Krivelevich, C. Lee and B. Sudakov, *Robust Hamiltonicity of Dirac graphs*, Transactions of the American Mathematical Society 366 (2014), 3095–3130.
- [15] D. Kühn, J. Lapinskas and D. Osthus, *Optimal packings of Hamilton cycles in graphs of high minimum degree*, Combinatorics, Probability and Computing, (2013) 22 (03), 394–416.
- [16] D. Kühn, R. Mycroft and D. Osthus, *Hamilton  $\ell$ -cycles in uniform hypergraphs*, J. Combinatorial Theory Series A (2010) 117, 910–927.

- [17] D. Kühn and D. Osthus, *Hamilton decompositions of regular expanders: a proof of Kelly's conjecture for large tournaments*, *Advances in Mathematics* (2013) 237, 62–146.
- [18] L. Lovász, **Combinatorial problems and exercises**, Vol. 361 American Mathematical Society, 2007.
- [19] K. Markström and A. Ruciński, *Perfect matchings (and Hamilton cycles) in hypergraphs with large degrees*, *European Journal of Combinatorics*, (2011) 32 (5), 677–687.
- [20] C. McDiarmid, Concentration, in: *Probabilistic Methods for Algorithmic Discrete Mathematics*, *Algorithms Combin.*, 16, Springer, Berlin, 1998, 195–248.
- [21] C. Nash-Williams, *Hamiltonian lines in graphs whose vertices have sufficiently large valencies*, *Combinatorial Theory and Its Applications*, III, North-Holland, 1970, 813–819.
- [22] C. Nash-Williams, *Edge-disjoint Hamiltonian circuits in graphs with vertices of large valency*, *Studies in Pure Mathematics* (L. Mirsky, ed.), 157–183, Academic Press, London, 1971.
- [23] C. Nash-Williams, *Hamiltonian arcs and circuits*, *Recent Trends in Graph Theory*, Springer, 1971, 197–210.
- [24] V. Rödl and A. Ruciński, *Dirac-type questions for hypergraphs – a survey (or more problems for Endre to solve)*, in **An Irregular Mind**, Springer Berlin Heidelberg (2010), 561–590.
- [25] V. Rödl, A. Ruciński and E. Szemerédi, *An approximate Dirac-type theorem for  $k$ -uniform hypergraphs*, *Combinatorica*, (2008) 28(2), 229–260.
- [26] V. Rödl, A. Ruciński and E. Szemerédi, *Dirac-type conditions for hamiltonian paths and cycles in 3-uniform hypergraphs*, *Advances in Mathematics*, (2011) 227 (3), 1225–1299.
- [27] G. Sárközy, S. Selkow and E. Szemerédi, *On the number of Hamiltonian cycles in Dirac graphs*, *Discrete Math.* (2003) 265, 237–250.