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Turán-type problems for long cycles in random and pseudo-random graphs

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Abstract

We study the Turán number of long cycles in random and pseudo-random graphs. Denote by $\mathrm{ex}(G(n,p),H)$ the random variable counting the number of edges in a largest subgraph of $G(n,p)$ without a copy of H. We determine the asymptotic value of $\mathrm{ex}(G(n,p),C_t)$ where C_t is a cycle of length t, for $p \geq \frac{C}{n}$ and $A\log n \leq t \leq (1-\varepsilon)n$. The typical behaviour of $\mathrm{ex}(G(n,p),C_t)$ depends substantially on the parity of t. In particular, our results match the classical result of Woodall on the Turán number of long cycles, and can be seen as its random version, showing that the transference principle holds here as well. In fact, our techniques apply in a more general sparse pseudo-random setting. We also prove a robustness-type result, showing the likely existence of cycles of prescribed lengths in a random subgraph of a graph with a nearly optimal density. Finally, we also present further applications of our main tool (the Key Lemma) for proving results on Ramsey-type problems about cycles in sparse random graphs.

1. *Introduction*

One of the most central topics in extremal graph theory is the so-called Turán-type problems. Recall that $\mathrm{ex}(n,H)$ denotes the maximum possible number of edges in a graph on n vertices without having H as a subgraph. Determining the value of $\mathrm{ex}(n,H)$ for a fixed graph H has become one of the most central problems in extremal combinatorics and there is a rich literature investigating it. Mantel [**45**] proved in 1907 that $\mathrm{ex}(n,K_3) = \lfloor \frac{n^2}{4} \rfloor$; Turán [**56**] found the value of $\mathrm{ex}(n,K_t)$ for $t \geq 3$ in 1941. In 1968, Simonovits [**52**] showed that the result of Mantel can be extended for an odd cycle of a fixed length, that is, $\mathrm{ex}(n,C_{2t+1}) = \lfloor \frac{n^2}{4} \rfloor$, where the extremal example is the complete bipartite graph†. For the general case, it was proved in 1946 by Erdős and Stone [**15**] that $\mathrm{ex}(n,H) = \left(1 - \frac{1}{\chi(H)-1} + o(1)\right)\binom{n}{2}$, where $\chi(H)$ is the chromatic number of the fixed graph H. Note that when H is a graph with chromatic number 2, an even cycle for instance, then from the above result we can only obtain that $\mathrm{ex}(n,H) = o(n^2)$. Bondy and Simonovits [**8**] proved in 1974 that for even cycles we have $\mathrm{ex}(n,C_{2t}) = O(n^{1+1/t})$. Unfortunately, a matching lower bound is known only for the cases where $t = 2,3,5$. For a survey see [**53**, **57**].

In this paper we consider the case where $H = C_t$ and $t \coloneqq t(n)$ tends to infinity with n. In this direction, it was proved by Erdős and Gallai [**13**], among other things, that if $t \coloneqq t(n)$, then $\mathrm{ex}(n,P_t) = \lfloor \frac{1}{2}(t-1)n \rfloor$. For long cycles, it was shown by Woodall [**59**] that if $t \geq \frac{1}{2}(n+3)$ then $\mathrm{ex}(n,C_t) = \binom{t-1}{2} + \binom{n-t+2}{2}$, where the extremal example is given by two cliques intersecting in exactly one vertex. In the same paper, Woodall also showed that for odd cycles C_t shorter than $\frac{1}{2}(n+3)$, the trivial bound $\mathrm{ex}(n,C_t) \geq \lfloor \frac{n^2}{4} \rfloor$ is still tight.

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†Throughout the paper we denote by P_t and C_t the path and the cycle of length t (i.e., the path and the cycle with t edges), respectively.

In the past few decades several generalisations of the classical Turán number $\mathrm{ex}(n, H)$ were suggested and many results have been established in this area. Denote by $\mathrm{ex}(G, H)$ the number of edges in a largest subgraph of a graph G containing no copy of H. Note that the value of $\mathrm{ex}(G, H)$ is bounded from below by the number of edges in G that are not contained in any copy of H. As a consequence, if the number of copies of H in G is much smaller than the number of edges in G, then we obtain that $\mathrm{ex}(G, H) \geq (1 - o(1))e(G)$. Thus, it makes sense to restrict our attention to graphs G for which the number of copies of H is at least proportional to the number of edges.

We focus on the case where the host graph G is either a random graph or pseudo-random graph. Given a positive integer n and a real number $p \in [0, 1]$, we let $G(n, p)$ be the *binomial random graph*, that is, a graph sampled from the family of all labeled graphs on the vertex set $[n] \coloneqq \{1, \ldots, n\}$, where each pair of elements of $[n]$ forms an edge with probability $p \coloneqq p(n)$, independently. We denote by $\mathrm{ex}(G(n, p), H)$ the number of edges in a largest subgraph of $G(n, p)$ without a copy of H (note that $\mathrm{ex}(G(n, p), H)$ is a random variable). Clearly, in this case we want to consider only the values of p for which $G(n, p)$ contains a copy of H with high probability (w.h.p., i.e., with probability tending to 1 as $n \to \infty$), and in fact, the number of copies of H in G is typically "large enough".

For fixed-size graphs H, this parameter has already been considered by various researchers. It is known that the threshold probability for a random graph to have the property that a typical edge is contained in a copy of H, for a fixed graph H, is $n^{-1/m_2(H)}$, where $m_2(H)$ is the *maximum 2-density* and defined to be $m_2(H) = \max\left\{\frac{e(H')-1}{v(H')-2} \mid H' \subseteq H,\ v(H') \geq 3\right\}$ (see [**22**] for more details). Therefore, it makes sense to consider graphs $G(n, p)$ for the regime $p = \Omega(n^{-1/m_2(H)})$. The cases $H = K_3$, $H = C_4$, and $H = K_4$ were solved by Frankl and Rödl [**18**], Füredi [**23**], and by Kohayakawa, Łuczak, and Rödl [**36**], respectively. For fixed odd cycles, it was shown by Haxell, Kohayakawa, and Łuczak [**29**] that for $p \geq Cn^{-1/m_2(C_{2t+1})}$ we have that $\frac{1}{2}e(G(n, p)) \leq \mathrm{ex}(G(n, p), C_{2t+1}) \leq \left(\frac{1}{2} + \varepsilon\right) e(G(n, p))$, for any $\varepsilon > 0$ and large enough n. For fixed even cycles, the same group of authors showed [**28**] that for $p = \omega(n^{-1/m_2(C_{2t})})$ we have $\mathrm{ex}(G(n, p), C_{2t}) = o(e(G(n, p)))$ (for more precise bounds on the fixed even cycle case, see Kohayakawa, Kreuter, and Steger [**35**], and Morris and Saxton [**48**]). The authors of [**29**, **28**, **36**] conjectured that a similar behaviour should also hold for any fixed-size graph H, that is, that the value of $\mathrm{ex}(G(n, p), H)$ should be asymptomatically equal to $\frac{\mathrm{ex}(n,H)}{\binom{n}{2}} \cdot e(G(n, p))$, for suitable values of p. This conjecture was proved independently by Conlon and Gowers [**10**] (with certain constraints on H) and by Schacht [**51**]. More precisely, they proved that for $p \geq Cn^{-1/m_2(H)}$, and for a fixed graph H, w.h.p. $\mathrm{ex}(G(n, p), H) \leq (1 - \frac{1}{\chi(H)-1} + \varepsilon)e(G(n, p))$. A matching lower bound can be obtained by a random placement of the extremal example of $\mathrm{ex}(n, H)$ on top of a random graph. The phenomenon that we observe here is frequently called *the transference principle*, which in this context can be interpreted as a random graph "inheriting" its (relative) extremal properties from the classical deterministic case, i.e., the complete graph. In their papers, Conlon and Gowers [**10**] and Schacht [**51**] discussed this principle and showed transference of several extremal results from the classical deterministic setting to the probabilistic setting.

In this paper we aim to study the transference principle in the context of long cycles. The first step is to understand what should be the relevant regime of p. It is easy to observe that if $p = o(\frac{1}{n})$ then a typical $G(n, p)$ is a forest, that is, does not contain any cycle. Thus, when looking at the appearance of a cycle in $G(n, p)$, it is natural to restrict ourselves to the regime $p = \Omega(\frac{1}{n})$. Furthermore, it is well known that cycles start to appear in $G(n, p)$ at probability $p = \Theta\left(\frac{1}{n}\right)$. We shall further recall what are the typical lengths of cycles one can expect to have in this regime. Note that for $p = \frac{C}{n}$ w.h.p. there are linearly many isolated vertices. Therefore, in this regime of p, we can hope to find in $G(n, p)$ cycles of length at most $(1 - \varepsilon)n$ for some constant $\varepsilon = \varepsilon(C) > 0$. Indeed, the typical appearance of nearly spanning cycles was shown in a series of papers by Ajtai, Komlós, and Szemerédi [**1**], Fernandez de la Vega [**16**], Bollobás [**6**], Bollobás, Fenner and Frieze [**7**], Frieze [**21**], Anastos and Frieze [**3**]), and Łuczak showed [**43**]. Observe that when looking at cycles of length $o(\log n)$ in the context of Turán-type problems, the regime $p = \Theta(\frac{1}{n})$ is not quite relevant. It is easy to verify that for $p = \Theta(\frac{1}{n})$ w.h.p. one expects $o(e(G(n,p)))$ cycles of such lengths, and hence they can be destroyed by deleting a negligible proportion of edges. Therefore, when requiring that the number of copies of C_t will be w.h.p. at least proportional to the number of edges, combining it with the fact that $p = \Omega\left(\frac{1}{n}\right)$, we get that $t = \Omega(\log n)$.

Moving back to the extremal problem, it was shown by Dellamonica, Kohayakawa, Marciniszyn, and Steger [**11**] that if $p = \omega(\frac{1}{n})$, then for all $\alpha > 0$, if G' is a subgraph of $G(n,p)$ with $e(G') \geq (1-(1-w(\alpha))(\alpha + w(\alpha)) + o(1))\, e(G(n,p))$, then w.h.p. G' contains a cycle of length at least $(1-\alpha)n$, where $w(\alpha) = 1-(1-\alpha)\lfloor(1-\alpha)^{-1}\rfloor$. This result is asymptotically tight by the classical result of Woodall [**59**] that guarantees a cycle of length at least $(1-\alpha)n$ in any graph G with $e(G) \geq (1-(1-w(\alpha))(\alpha+w(\alpha)) + o(1))\binom{n}{2}$.

Very recently, Balogh, Dudek and Li [**4**] studied the asymptotic behaviour of $\mathrm{ex}(G(n,p), P_\ell)$ for various ranges of $\ell = \ell(n)$.

In this paper we investigate the appearance of long cycles of a **given length** in subgraphs of $G(n,p)$ and of pseudo-random graphs. More precisely, we determine the asymptotic value $\mathrm{ex}(G(n,p), C_t)$, where $p = \Omega(\frac{1}{n})$ and t is between $\Theta(\log n)$ and $(1-\varepsilon)n$. Here and later, $\log n$ refers to the natural logarithm.

We now present the main result of this paper.

THEOREM 1.1. *For every $0 < \beta < \frac{1}{4}$, there exists $C > 0$ such that if $p \geq \frac{C}{n}$ and $G = G(n,p)$, then with probability $1 - e^{-\Omega(n)}$ the following holds. For any $\frac{C_1}{\log(1/\beta)}\log n \leq t \leq (1-C_2\beta)n$,*

$$\mathrm{ex}(G(n,p), C_t) \leq \left(\frac{\mathrm{ex}(n, C_t)}{\binom{n}{2}} + \beta\right) e(G(n,p)),$$

where $C_1, C_2 > 0$ are absolute constants.

Thus, also here, we observe a manifestation of the transference principle, that is, the random graph $G(n,p)$ preserves the relative behaviour of the Turán number of long cycles observed in the classical case, i.e., in the complete graph K_n. The exact value of $\mathrm{ex}(n, C_t)$, which is a crucial quantity in this paper, is known due to Woodall [**59**] to be

$$\mathrm{ex}(n, C_t) = \begin{cases} \binom{t-1}{2} + \binom{n-t+2}{2} + 1, & \text{if } t \geq \frac{1}{2}(n+3), \\ \lfloor \frac{1}{4}n^2 \rfloor + 1, & \text{if } t < \frac{1}{2}(n+3) \text{ and odd.} \end{cases}$$

When $t < \frac{1}{2}(n+3)$ and even, we note that $\mathrm{ex}(n, C_t) \leq \mathrm{ex}(n, P_t) = \lfloor \frac{1}{2}n(t-1) \rfloor$, as given by Erdős and Gallai in [**13**], and is asymptotically equal to this upper bound. This will be further discussed in detail in Definition 2 and after Theorem 1.2.

REMARK 1. In Theorem 1.1 we obtain in fact, for a given t, all cycles of length q, where $\frac{C_1}{\log(1/\beta)}\log n \leq q \leq t$, with the same parity as t.

Theorem 1.1 is asymptotically optimal in a stronger form; a matching lower bound is true for any graph G on n vertices, not only for random graphs. That is, for any graph G on the vertex set $[n]$ there exists a subgraph G_0 with at least $\frac{\mathrm{ex}(n,C_t)}{\binom{n}{2}} e(G)$ edges containing no cycle of length t. Indeed, let W_t be a graph on n vertices with $\mathrm{ex}(n, C_t)$ edges containing no cycle of length t. By averaging, there exists a placement σ of the vertices of W_t onto $[n]$ such that when intersecting with G, we have $e(G\cap W_t^\sigma) \geq \frac{\mathrm{ex}(n,C_t)}{\binom{n}{2}} e(G)$. Clearly, the resulting graph $G\cap W_t^\sigma$ contains no cycles of length t. This gives the following.

FACT 1. For every graph G on $n \geq 3$ vertices and every integer $t \in [3,n]$ we have

$$\mathrm{ex}(G,C_t) \geq \frac{\mathrm{ex}(n,C_t)}{\binom{n}{2}} e(G).$$

Theorem 1.1 follows from a more general statement that deals with a class of graphs which is larger than the random graphs class.

DEFINITION 1. Let G be a graph on n vertices. Suppose $0 < \eta \leq 1$ and $0 < p \leq 1$. We say that G is *(p,η)-upper-uniform* if for every $U, W \subseteq V(G)$ with $U \cap W = \emptyset$ and $|U|, |W| \geq \eta n$, we have $e_G(U,W) \leq (1+\eta)p|U||W|$.

It is not hard to verify that for $p = \Omega_\eta(1/n)$, $G(n,p)$ is w.h.p. (p,η)-upper-uniform (more details can be found in, e.g., [**25**]). We prove the following result, from which Theorem 1.1 follows as a corollary.

The following notation is based on results by Erdős-Gallai [**13**] and by Woodall [**59**] (see Theorem 2.1 and Theorem 2.2 for more information).

DEFINITION 2. The functions g_o, g_e are given as follows.
If t is odd, then

$$g_o(t,n)\cdot\binom{n}{2} := \mathrm{ex}(n,C_t)+1 = \begin{cases} \binom{t-1}{2}+\binom{n-t+2}{2}+1, & \text{if } t \geq \frac{1}{2}(n+3),\\ \left\lfloor \frac{1}{4}n^2 \right\rfloor + 1, & \text{if } t < \frac{1}{2}(n+3).\end{cases}$$

If t is even and $\gamma > 0$ is a parameter,

$$g_e^\gamma(t,n)\cdot\binom{n}{2} := \begin{cases} \mathrm{ex}(n,C_t)+1 = \binom{t-1}{2}+\binom{n-t+2}{2}+1, & \text{if } t \geq \frac{1}{2}(n+3),\\ \mathrm{ex}(n,P_t)+1 = \left\lfloor \frac{1}{2}n(t-1) \right\rfloor + 1, & \text{if } \gamma n \leq t < \frac{1}{2}(n+3),\\ 0, & \text{if } t < \gamma n,\end{cases}$$

Furthermore, the function $g^\gamma : [0,1] \to [0,1]$ is defined as follows.

$$g^\gamma(t,n) = \begin{cases} g_o(t,n), & \text{if } t \text{ is odd}\\ g_e^\gamma(t,n), & \text{if } t \text{ is even.}\end{cases}$$

We are now ready to state our general result.

THEOREM 1.2. *For every $0 < \beta < \frac{1}{4}$, there exist $\eta, n_0, \gamma > 0$ such that for every $n \geq n_0$ the following holds. Let G be a (p,η)-upper-uniform graph on n vertices with $e(G) \geq (1-\beta/2)p\binom{n}{2}$ for some $0 < p := p(n) \leq 1$. Then for any $\frac{C_1}{\log(1/\beta)}\log n \leq t \leq (1-C_2\beta)n$, where $C_1, C_2 > 0$ are absolute constants, if G' is a subgraph of G with*

$$e(G') \geq (g^\gamma(t,n)+\beta)\, e(G)$$

edges, then G' contains a cycle of length t.

Since $\mathrm{ex}(n, P_t) = \mathrm{ex}(n, P_{t-1}) + O(n)$, we have $\mathrm{ex}(n, P_t) - O(n) \le \mathrm{ex}(n, P_{t-1}) \le \mathrm{ex}(n, C_t)$, and we can deduce from Theorem 1.2 the following corollary.

COROLLARY 1.3. *For every $0 < \beta < \frac{1}{5}$, there exist $\eta, n_0 > 0$ such that for every $n \ge n_0$ the following holds. Let G be a (p, η)-upper-uniform graph on n vertices with $e(G) \ge (1 - \beta/2)p\binom{n}{2}$ for some $0 < p := p(n) \le 1$. Then for any $\frac{C_1}{\log(1/\beta)} \log n \le t \le (1 - C_2\beta)n$,*

$$\mathrm{ex}(G, C_t) \le \left(\frac{\mathrm{ex}(n, C_t)}{\binom{n}{2}} + \beta \right) e(G),$$

where $C_1, C_2 > 0$ are some absolute constants.

Since random graphs are upper-uniform (with an appropriate choice of parameters), Theorem 1.1 now follows immediately from Corollary 1.3.

As mentioned in Remark 1, given t, the statement holds for every $\frac{C_1}{\log(1/\beta)} \log n \le q \le t$ with the same parity as t.

REMARK 2. Theorem 1.2 does not assume any positive lower bound on the value of p. However, graphs G satisfying the conditions of the statement exist only for a restricted spectrum of values for p. More specifically, for $0 < p = o(\frac{1}{n})$ there are no (p, η)-upper-uniform graphs G with $e(G) \ge (1 - \beta/2)p\binom{n}{2}$, which makes the statement relevant only for $p \ge \frac{C}{n}$ where $C > 0$ is some constant. To see this, take G to be a (p, η)-upper-uniform graph with $p = o(\frac{1}{n})$. By a standard double-counting argument one can see that $e(G) \le (1 + \eta)p\binom{n}{2} = o(n)$. Thus, there is a subset I of isolated vertices in G of size $\frac{n}{2}$. By the assumption $e(G) \ge (1 - \beta)p\binom{n}{2}$ we obtain that $e(G[V \setminus I]) \ge (1 - \beta)p\binom{n}{2}$. This contradicts the upper uniformity of G since $e(G[V \setminus I]) \le (1 + \eta)p\binom{n/2}{2}$.

The random graph case is probably the most natural application of Theorem 1.2. However, another natural application of Theorem 1.2 is for (n, d, λ)-graphs, which can be shown to be (p, η)-upper-uniform for suitable values of d, λ.

DEFINITION 3. A graph G is an (n, d, λ)*-graph* if G has n vertices, is d-regular, and the second largest (in absolute value) eigenvalue of its adjacency matrix is bounded from above by λ.

(n, d, λ)-graphs have been studied extensively, mainly due to their good pseudo-random properties. For a detailed background see [**40**]. Recently, it was shown in [**20**] that for a given $\beta > 0$, if $\frac{d}{\lambda} \ge C(\beta)$, then (n, d, λ)-graphs contain cycles of all lengths between $\frac{C_1}{\log(1/\beta)} \log n$ and $(1 - C_2\beta)n$ (for some absolute constants $C_1, C_2 > 0$), improving the result in [**30**].

Using the Expander Mixing Lemma due to Alon and Chung [**2**], we can easily show that for suitable values of d and λ, an (n, d, λ)-graph is also upper-uniform. Hence we obtain the following corollary.

COROLLARY 1.4. *For every $0 < \beta < \frac{1}{4}$ there exist $n_0, \gamma, \delta > 0$ such that for every $n \ge n_0$ and for every $d, \lambda > 0$ satisfying $\lambda \le \delta d$ the following holds. If G is an (n, d, λ)-graph, then for any $\frac{C_1}{\log(1/\beta)} \log n \le t \le (1 - C_2\beta)n$, where $C_1, C_2 > 0$ are absolute constants, if G' is a* *subgraph of G with*

$$e(G') \geq (g^{\gamma}(t,n) + \beta)e(G),$$

edges, then G' contains a cycle of length t.

Note that the lower bound on t is tight due to the existence of (n,d,λ)-graphs with large girth. More explicitly, it was shown in [**42**, **46**] that there exist infinitely many (n,d,λ)-graphs with girth $\Omega(\log n)$, such that $\frac{d}{\lambda}$ is larger than a given constant. Details of the proof and further discussion on this application can be found in Section 6.1.

Using very similar techniques, we can also obtain a *robustness*-type result (for a detailed survey on robustness problems see [**54**]). In this type of results, we consider a graph G satisfying some *extremal conditions* that guarantee a graph property $\mathcal{P}$ (in our case, containment of long cycles). The aim is to measure quantitatively the strength of these specific conditions. For this, we let G be a graph satisfying these conditions, and let $G(p)$ be the random graph obtained by keeping each edge of G independently with probability $p \in [0,1]$. Note that if $G = K_n$ then $G(p) = G(n,p)$. In the next theorem we show that if G has (slightly more than) the minimum number of edges that guarantees a long cycle of a given length, then with high probability $G(p)$ also contains such a cycle for $p = \Omega\left(\frac{1}{n}\right)$. This value of p is best possible due to the threshold for the existence of cycles in $G(n,p)$.

THEOREM 1.5. *For every $\beta > 0$ there exists $C > 0$ such that for $\frac{C_1}{\log(1/\beta)} \log n \leq t \leq (1 - C_2\beta)n$ (where $C_1, C_2 > 0$ are absolute constants), and for any $p \geq \frac{C}{n}$, the following holds. If G is a graph on n vertices satisfying*

$$e(G) \geq \mathrm{ex}(n, C_t) + \beta\binom{n}{2},$$

then w.h.p. $G(p)$ contains a copy of C_t.

Note that starting with a graph with exactly $\mathrm{ex}(n,C_t)+1$ edges is not enough. Indeed, let G be an extremal example for $\mathrm{ex}(n,C_t)$ with an arbitrary edge e added to it. Then when taking $G(p)$ with $p = o(1)$ w.h.p. e is deleted. The above theorem shows that adding $\beta\binom{n}{2}$ edges to the extremal number will be enough, and, in fact, for many values of t this order of magnitude of edges is tight. However, there are values of t for which only $\omega(1/p)$ extra edges to the extremal number suffice. More precisely, this happens when $t < \frac{1}{2}n$ is odd, and recall that in this case we have $\mathrm{ex}(n,C_t) = \lfloor \frac{n^2}{4} \rfloor$. This demonstrated in the following theorem (for more discussion see Section 5).

THEOREM 1.6. *For every $\beta > 0$ there exists $C > 0$ such that for an odd t with $\frac{C_1}{\log(1/\beta)} \log n \leq t \leq \left(\frac{1}{2} - \beta\right) n$ (where $C_1 > 0$ is an absolute constant), and for any $p \geq \frac{C}{n}$, the following holds. If G is a graph on n vertices satisfying*

$$e(G) \geq \mathrm{ex}(n, C_t) + \omega\left(\tfrac{1}{p}\right),$$

then w.h.p. $G(p)$ contains a copy of C_t.

Finally, in Section 6.2 we show some applications to Ramsey-type problems about cycles in random graphs. In particular, we use the power of our Key Lemma (see Lemma 3.1) in order to argue about a typical appearance of a monochromatic cycle of a prescribed length in multicoloured sparse random graphs.

1.1. *Organisation*

In Section 2.2 we discuss the Sparse Regularity Lemma on which we rely heavily in the proof of Theorem 1.2. In Section 3 we present the Key Lemma used in the paper to convert a cycle (or a path) in the reduced graph to a cycle of an appropriate length in the original graph. In the same section we give the proof of Theorem 1.2 using the Key Lemma. Section 4 is devoted for the proof of the Key Lemma. In Section 6 we give some further related results.

2. *Notation and preliminaries*

Our graph-theoretic notation is standard, in particular we use the following. For a graph $G = (V, E)$ and a set $U \subset V$, let $G[U]$ denote the corresponding induced subgraph of G. We also denote $e(G) = |E(G)|$ and $v(G) = |V(G)|$. For $U \subset V$ we let $\Gamma_G(U) = \{v \in V \setminus U \mid \exists u \in U \text{ s.t. } \{u, v\} \in E\}$ be the (external) neighbourhood of U in G. For an integer k and $V_i \subseteq V$, $i \in [k]$, we say that $\Pi = (V_1, \ldots, V_k)$ is a *partition* of V if $V = \bigcup_{i\in[k]} V_i$ and $V_i \cap V_j = \emptyset$ for every $i \neq j$.

Let $[n] := \{1, \ldots, n\}$. For a positive real number ℓ, we denote by $\lfloor \ell \rfloor_{odd}$ (respectively, $\lfloor \ell \rfloor_{even}$) the largest odd (respectively, even) integer m with $m \leq \ell$.

2.1. *Known extremal results*

To prove our result, we use two classic theorems, one by Woodall [**59**] about cycles, and the other one by Erdős and Gallai [**13**] regarding paths.

THEOREM 2.1 ([**13**], Theorem 2.6). *Let G be an n-vertex graph with more than $\lfloor \frac{1}{2} n(t-1) \rfloor$ edges. Then G contains a path of length at least t.*

THEOREM 2.2 ([**59**], Corollary 11). *Let G be a graph on $n \geq 3$ vertices and let $3 \leq t \leq n$. Assume that $e(G) \geq w(t, n) \cdot \binom{n}{2}$ where*

$$w(t, n) \cdot \binom{n}{2} := \begin{cases} \binom{t-1}{2} + \binom{n-t+2}{2} + 1, & \text{if } t \geq \frac{1}{2}(n+3), \\ \lfloor \frac{1}{4} n^2 \rfloor + 1, & \text{if } t < \frac{1}{2}(n+3). \end{cases}$$

Then G contains a cycle of length d for any $3 \leq d \leq t$.

Both Theorems 2.1 and 2.2 are tight, as can be seen in the constructions given in [**13**] and in [**59**], respectively.

Note that the function $w(t, n)$ of Woodall is strongly related to the function $g^\gamma(t, n)$ given in Definition 2. In particular, for odd values of t we have $w(t, n) = g_o(t, n)$, and furthermore, $w(t, n) \geq \frac{1}{2}$ for any t and n. In addition, $w(t, n)$ is monotone increasing in t for this case. So for any odd t, $w(t, n)\binom{n}{2} = g_o(t, n)\binom{n}{2} = \text{ex}(n, C_t) - 1$. As for even values of t, we get $w(t, n) = g_e(t, n)$ only when $t \geq \frac{1}{2}(n + 3)$. For an even $t < \frac{1}{2}(n + 3)$, note that we **do not** necessarily have $w(t, n)\binom{n}{2} = \text{ex}(n, C_t) + 1$ (although, as mentioned, Theorem 2.2 is still tight because of the requirement of having all cycles, also the odd ones, of length at most t.) For this reason, we also make use of $\text{ex}(n, P_t)$ in Definition 2 for even values of $t < \frac{1}{2}(n + 3)$.

REMARK 3. Note that if $0 < \varphi < \frac{1}{2}$ is constant and $t = (1 - \varphi + o_n(1))n$ then $w(t,n) = 1 - 2\varphi + 2\varphi^2 + o_n(1)$. In particular, if $e(G) \geq (1-\varphi)\binom{n}{2}$, then G contains a cycle of length d for any $3 \leq d \leq (1-\varphi)n$.

Another result to be used in this paper in a significant way is by Friedman and Pippenger [**19**], regarding the existence of large trees in expanding graphs.

THEOREM 2.3 ([**19**], Theorem 1). *Let T be a tree on k vertices of maximum degree at most d. Let H be a non-empty graph such that, for every $X \subset V(H)$ with $|X| \leq 2k-2$ we have $|\Gamma_H(X)| \geq (d+1)|X|$. Let further $v \in V(H)$ be an arbitrary vertex of H. Then H contains a copy of T, rooted at v.*

2.2. *Sparse Regularity Lemma*

In order to prove Theorem 1.2, we make use of a variant of Szemerédi's Regularity Lemma [**55**] for sparse graphs, the so-called Sparse Regularity Lemma due to Kohayakawa [**34**] and Rödl (see [**9**, **25**, **37**]). The sparse version of the Regularity Lemma is based on the following definition.

DEFINITION 4. Let a graph $G = (V, E)$ and a real number $p \in (0, 1]$ be given. We define the *p-density* of a pair of non-empty, disjoint sets $U, W \subseteq V$ in G by

$$d_{G,p}(U,W) = \frac{e_G(U,W)}{p|U||W|}.$$

For any $0 < \varepsilon \leq 1$, the pair (U, W) is said to be *(ε, G, p)-regular*, or just *(ε, p)-regular* for short, if, for all $U' \subseteq U$ with $|U'| \geq \varepsilon|U|$ and all $W' \subseteq W$ with $|W'| \geq \varepsilon|W|$, we have

$$|d_{G,p}(U,W) - d_{G,p}(U',W')| \leq \varepsilon. \tag{2.1}$$

We say that a partition $\Pi = (V_1, \dots, V_k)$ of V is *(ε, p)-regular* if $||V_i| - |V_j|| \leq 1$ for all $i, j \in [k]$, and, furthermore, at least $(1-\varepsilon)\binom{k}{2}$ pairs (V_i, V_j) with $1 \leq i < j \leq k$ are (ε, p)-regular.

In the case $p = 1$ we say that the pair (or the partition) is ε-regular.

THEOREM 2.4 (Sparse Regularity Lemma [**34**]). *For any given $\varepsilon > 0$ and $k_0 \geq 1$, there are constants $\eta = \eta(\varepsilon, k_0) > 0$ and $K_0 = K_0(\varepsilon, k_0) \geq k_0$ such that any (p, η)-upper-uniform graph G on n vertices, for large enough n, with $0 < p \leq 1$ admits an (ε, p)-regular partition of its vertex set into k parts, where $k_0 \leq k \leq K_0$.*

2.2.1. *The Reduced Graph*

DEFINITION 5 (Reduced Graph). Let $\varepsilon > 0$, $k \geq 1$ an integer, $0 \leq p \leq 1$, and $0 < \rho \leq 1$. Let G_0 be a graph on n vertices, and $\Pi = (V_1, \dots, V_k)$ a partition of its vertices. We define the *reduced graph* $R(G_0, \Pi, \rho, \varepsilon, p)$ to be the graph on the vertex set $\{1, \dots, k\}$, where vertices i and j are connected by an edge if and only if (V_i, V_j) is (ε, p)-regular and $d_{G_0,p}(V_i, V_j) \geq \rho$. If we consider the reduced graph where $p = 1$, we omit this parameter from the notation.

The following lemma is a standard use of the Regularity Lemma, showing that the reduced graph "inherits" the proportion of edges from the underlying graph. Its proof follows the same lines, up to the choice of constants, as proofs of similar results. We refer the reader to [**25**] for more details. We state it here in a version that suits our proof of Theorem 1.2. However, we use slightly different versions of it in proof of other results (see Section 5 and Section 6).

LEMMA 2.5. *Let $0 < \beta < \frac{1}{4}$, $x \in [0,1)$ such that $x + \beta < 1$. Let $\varepsilon \leq \frac{\beta}{1000}$, $k \geq \frac{100}{\beta}$, and $\eta \leq \frac{1}{3k}$ be positive. Assume that G is an (p,η)-upper uniform graph, and $e(G) \geq (1-\beta/2)p\binom{n}{2}$, for some $0 < p := p(n) \leq 1$. Let G' be the graph obtained from G by keeping at least $(x+\beta)\,e(G)$ edges, and assume that $\Pi = (V_1, \ldots, V_k)$ is an (ε, p)-regular partition of G'. Let $R := R(G', \Pi, \rho, \varepsilon, p)$ be the reduced graph as in Definition 5 for $\rho = 10\varepsilon$. Then*

$$e(R) \geq (x + \beta/32)\binom{k}{2}.$$

3. *Key Lemma and proof of Theorem 1.2*

In this section we state the Key Lemma and then use it to prove Theorem 1.2.

DEFINITION 6. Let G be a graph and let $V_1, V_2 \subseteq V(G)$ be two disjoint subsets of vertices with $|V_1|, |V_2| \in \{\lfloor m \rfloor, \lceil m \rceil\}$ for some positive number m. Let $\varepsilon > 0$. We say that the pair (V_1, V_2) satisfies the *ε-property* in G if for every two subsets $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$ with $|U_1|, |U_2| \geq \varepsilon m$, G contains at least one edge between them, i.e., $e(G[U_1, U_2]) > 0$.

DEFINITION 7. Let $\varepsilon > 0$ and let k be a positive integer. Let G_0 be a graph on n vertices and let $\Pi = (V_1, \ldots, V_k)$ be a partition of its vertices into k parts satisfying $||V_i| - |V_j|| \leq 1$ for all $i, j \in [k]$. We define the ε-graph $S := S(G_0, \Pi, \varepsilon)$ to be the graph with vertex set $[k]$ where $\{i, j\} \in E(S)$ if the pair (V_i, V_j) satisfies the ε-property in G_0 with $m := \frac{n}{k}$.

LEMMA 3.1 (Key Lemma). *Let $0 < \varepsilon < \frac{1}{85}$. Let G_0 be a graph on n vertices, for large enough n, and let $\Pi = (V_1, \ldots, V_k)$ be a partition of its vertices satisfying $||V_i| - |V_j|| \leq 1$ for all $i, j \in [k]$, where $k \geq \frac{2}{\varepsilon^2}$ is a constant. Let $S := S(G_0, \Pi, \varepsilon)$ be the corresponding ε-graph as in Definition 7. Then for any absolute constant $C_1 > 2.1$ we have the following.*

(i) *If S contains a path of an odd length b, $1 \leq b < k$, then G_0 contains cycles of all even lengths in $\left[\frac{C_1}{\log(1/\varepsilon)} \log n, (1-48\varepsilon)an\right]$, with $a := \frac{b+1}{k}$.*
(ii) *If S contains a cycle of an odd length b, $3 \leq b < k$, then G_0 contains cycles of all odd lengths in $\left[\frac{(b-1)C_1}{2\log(1/\varepsilon)} \log n, (1-48\varepsilon)an\right]$, with $a := \frac{b-1}{k}$.*

The assumption in the first item that b is odd is of technical nature and is in fact an artifact of our proof strategy. The proof of the Key Lemma can be found in Section 4.3.

Using this Key Lemma, we can deduce the existence of long cycles in a graph in cases where there are enough edges in a corresponding ε-graph.

COROLLARY 3.2. *Let $0 < \beta < 1/3$, $\varepsilon = \frac{\beta}{10000}$, $k \geq \frac{2}{\varepsilon^2}$, and $\gamma \leq \frac{2(1-48\varepsilon)}{k}$. Let G_0 be a graph on n vertices, for large enough n, and let $\frac{C_1}{\log(1/\beta)} \log n \leq t \leq (1-C_2\beta)n$, for any absolute constants $C_2 \geq \frac{48}{10000}$ and $C_1 > 2.1$. Assume that there exists a partition $\Pi = (V_1, \ldots, V_k)$ of the vertices of G_0, satisfying $||V_i| - |V_j|| \leq 1$ for all $i, j \in [k]$, such that for the corresponding ε-graph $S := S(G_0, \Pi, \varepsilon)$ we have $e(S) \geq (g^\gamma(t,n) + \beta/32)\binom{k}{2}$, where $g^\gamma(t,n)$ is defined in Definition 2. Then G_0 contains a cycle of length t.*

Proof. We split the proof into four cases by the parity and the value of t. Throughout all following cases we use the facts that $1 - C_2\beta \le 1 - 48\varepsilon$ and that $\varepsilon < \beta$.

Case 1: t is even and $t < \gamma n$. In this case we have $g_e^\gamma(t,n) = 0$, implying $e(S) \ge \frac{\beta}{32}\binom{k}{2}$. By Theorem 2.1 we get that S contains a path of an odd length at least $\lfloor \frac{\beta}{32} \cdot k \rfloor_{odd} > 1$ and hence, by Lemma 3.1, G_0 contains a cycle of length t.

Case 2: t is even and $\gamma n \le t < \frac{1}{2}(n+3)$. In this case we have $g_e^\gamma(t,n) = \frac{\lfloor \frac{1}{2}n(t-1)\rfloor + 1}{\binom{n}{2}}$, and in particular $e(S) \ge \left(\frac{\lfloor \frac{1}{2}n(t-1)\rfloor+1}{\binom{n}{2}} + \frac{\beta}{32}\right)\binom{k}{2} \ge \left(\frac{t}{n} + \frac{\beta}{50}\right)\binom{k}{2}$. By Theorem 2.1 we get that S contains a path of an odd length at least $\lfloor \left(\frac{t}{n} + \frac{\beta}{50}\right) k \rfloor_{odd}$ and hence, by Lemma 3.1, and since $t < \left(\frac{t}{n} + \frac{\beta}{50}\right)(1-48\varepsilon)n$, G_0 contains a cycle of length t.

Case 3: t is odd and $t < \frac{1}{2}(n+3)$. In this case we have $g_o(t,n) = \frac{\lfloor \frac{1}{4}n^2\rfloor+1}{\binom{n}{2}}$, and in particular $e(S) \ge \left(\frac{\lfloor \frac{1}{4}n^2\rfloor+1}{\binom{n}{2}} + \frac{\beta}{32}\right)\binom{k}{2} > \left(\frac{1}{2} + \frac{\beta}{32}\right)\binom{k}{2}$. By Remark 3 and Theorem 2.2 we get that S contains cycles of all lengths up to $\left(\frac{1}{2} + \frac{\beta}{32}\right)k > \left(\frac{n+3}{2n} + \frac{\beta}{40}\right)k > \left(\frac{t}{n} + \frac{\beta}{50}\right)k + 1$ and hence, by Lemma 3.1, G_0 contains a cycle of length t.

Case 4: $\frac{1}{2}(n+3) \le t \le (1 - C_2\beta)n$. In this case we have $g^\gamma(t,n) = \frac{\binom{t-1}{2}+\binom{n-t+2}{2}+1}{\binom{n}{2}}$, and thus $e(S) \ge \left(\frac{\binom{t-1}{2}+\binom{n-t+2}{2}+1}{\binom{n}{2}} + \frac{\beta}{32}\right)\binom{k}{2} \ge \frac{t}{n}(1 + \frac{144\varepsilon}{1-48\varepsilon})\binom{k}{2}$. By Remark 3 and Theorem 2.2 we get that S contains cycles of all lengths up to $\frac{t}{n}\left(1 + \frac{144\varepsilon}{1-48\varepsilon}\right)k$ and in particular a path and a cycle of lengths $\left\lfloor \frac{t}{n}\left(1 + \frac{96\varepsilon}{1-48\varepsilon}\right)k \right\rfloor_{odd}$. By Lemma 3.1, since $t \le \left(\frac{t}{n}\left(1 + \frac{96\varepsilon}{1-48\varepsilon}\right)k - 2\right)(1-48\varepsilon)\frac{n}{k}$, we get that G_0 contains a cycle of length t, where if t is odd then we look at the cycle in S and if t is even then we look at the path. □

REMARK 4. Given γ, note that the function $g^\gamma(t,n)$ is monotone in the following sense. For any $0 < t < \frac{1}{4}(n+3)$ we have $g^\gamma(2t+1,n) \ge g^\gamma(2t,n)$, $g^\gamma(2t+1,n) \ge g^\gamma(2t-1,n)$, and $g^\gamma(2t+2,n) \ge g^\gamma(2t,n)$. In addition, if $t \ge \frac{1}{2}(n+3)$ then $g^\gamma(t+1,n) \ge g^\gamma(t,n)$. Consequently, under the assumptions of Corollary 3.2, if t is odd then G contains **all** cycles of lengths between $\frac{C_1}{\log(1/\beta)}\log n$ and t, and if t is even then G contains **all even** cycles of lengths between $\frac{C_1}{\log(1/\beta)}\log n$ and t. In addition, if $t \ge \frac{1}{2}(n+3)$ then G contains **all** cycles of lengths between $\frac{C_1}{\log(1/\beta)}\log n$ and t (regardless of the parity of t).

The following is a well-known (and easy to show) fact.

FACT 2. Let n be an integer, $\varepsilon > 0$, $\varepsilon < \rho < \frac{1}{2}$. Let G_0 be a graph on n vertices and let $V_1, V_2 \subseteq V(G_0)$ be two subsets of vertices satisfying: $V_1 \cap V_2 = \emptyset$, $|V_1|, |V_2| \in \{\lfloor m \rfloor, \lceil m \rceil\}$ for some m, and the pair (V_1, V_2) is (ε, p)-regular in G_0 with $d_{G_0,p}(V_1,V_2) \ge \rho$, for some $0 < p := p(n) \le 1$. Then the pair (V_1, V_2) satisfies the ε-property in G_0.

Using Corollary 3.2 we can immediately prove our Theorem 1.2.

Proof of Theorem 1.2. Let $\varepsilon = \frac{\beta}{10000}$, $\rho = 10\varepsilon$, and $k_0 = \frac{2}{\varepsilon^2}$. Let $\eta_0 := \eta_0(\varepsilon, k_0) > 0$ and $K_0 := K_0(\varepsilon, k_0) \ge k_0$. Let $\eta := \min\{\eta_0, \frac{1}{3k_0}\}$ and let $\gamma = \frac{2(1-48\varepsilon)}{k_0}$. Recall that G is a (p,η)-upper-uniform graph for some $0 < p := p(n) \le 1$, with $e(G) \ge (1 - \beta/2)p\binom{n}{2}$. Let G' be a graph obtained from G by keeping at least $(g^\gamma(t,n)+\beta)e(G)$ edges, and note that G' is also (p,η)-upper-uniform. Let $\Pi=(V_1,\dots,V_k)$ be an (ε,p)-regular partition of G' guaranteed by the Sparse Regularity Lemma (Theorem 2.4) for the relevant parameters, for some $k_0 \le k \le K_0$. Let $R \coloneqq R(G',\Pi,\rho,\varepsilon,p)$ be the reduced graph on k vertices with parameters ρ,ε,p and k, as in Definition 5. By Fact 2, if $\{i,j\}$ is an edge in R, then the pair (V_i,V_j) satisfies the ε-property in G'. Hence, the reduced graph R is a subgraph of the ε-graph $S \coloneqq S(G',\Pi,\varepsilon)$, as defined in Definition 7. In particular $e(S)\ge e(R)$, and every path or cycle contained in R is also contained in S.

Let $C_1>0$ be the constant from Lemma 3.1, and let $C_2>0$ be the constant from Corollary 3.2. Let $\frac{C_1}{\log(1/\beta)}\log n \le t \le (1-C_2\beta)n$. By Lemma 2.5 we have that $e(R)\ge (g^\gamma(t,n)+\beta/32)\binom{k}{2}$, and thus $e(S)\ge (g^\gamma(t,n)+\beta/32)\binom{k}{2}$. Applying Corollary 3.2, we get a cycle of length t in G. □

Applying Remark 4 to Theorem 1.2, note that if t is odd then G contains **all** cycles of lengths between $\frac{C_1\log n}{\log(1/\beta)}$ and t, and if t is even then G contains **all even** cycles of lengths between $\frac{C_1\log n}{\log(1/\beta)}$ and t. In addition, if $t>\frac{1}{2}(n+3)$ then G contains **all** cycles of lengths between $\frac{C_1\log n}{\log(1/\beta)}$ and t (regardless of the parity of t).

4. *Proof of the Key Lemma*

In this section we prove the Key Lemma (Lemma 3.1) using several claims and results regarding tree embeddings in expander graphs. The main idea is to show that every two vertices connected by an edge in the reduced graph represent a pair of clusters in the original graph that has "good expansion" properties (Section 4.1). Then, we show that the graph induced by any pair of such clusters contains a very specific tree (Section 4.2), which will later be used to embed the desired cycle (Section 4.3).

4.1. *Expander graphs*

Definition 8. A graph $G=(V,E)$ is called a (B,ℓ)-*expander* if for every $X\subseteq V$ with $|X|\le B$ we have $|\Gamma_G(X)|\ge \ell|X|$.

For the proofs in this section we also need a somewhat more specific definition of expander graphs for the special case of bipartite graphs.

Definition 9. A bipartite graph $G=(V_1\cup V_2,E)$ is called a (B,ℓ)-*bipartite-expander* if for every $X\subseteq V_i$, $1\le |X|\le B$, we have $|\Gamma_G(X)|\ge \ell|X|$.

Remark 5. If a bipartite graph G is an $(A,\ell+1)$-bipartite-expander, then it is a $\left(2A,\frac{1}{2}\ell\right)$-expander.

Proposition 4.1. *Let $\varepsilon>0$ and let $a,b>0$ satisfy $(2b+2)(1-\varepsilon-ab)>1$ and $(2b+2)\varepsilon\ge 1$. Let G be a bipartite graph with parts V_1,V_2 with $|V_1|,|V_2|\ge(2b+2)\varepsilon m$ for some integer m, and assume that every two subsets $V_1',\subseteq V_1$, $V_2'\subseteq V_2$ with $|V_1'|,|V_2'|\ge \varepsilon m$ span at least one edge in G, i.e., $e(G[V_1',V_2'])>0$. Then there exist $U_1\subseteq V_1$ and $U_2\subseteq V_2$ with $|U_1|\ge(1-\varepsilon)|V_1|$ and $|U_2|\ge(1-\varepsilon)|V_2|$ such that the bipartite graph $G[U_1,U_2]$ is an (ax,b)-bipartite-expander, where $x=\min(|V_1|,|V_2|)$.*

Proof. If every subset of $X_i \subseteq V_i$ of size at most ax satisfies $|\Gamma_G(X_i)| \geq b|X_i|$ then we are done by setting $U_1 = V_1$ and $U_2 = V_2$. Otherwise, there are subsets violating the expansion condition. We iteratively remove such subsets of size at most εm, one by one, to create an $(\varepsilon m, b)$-bipartite-expander. We then show that the expander we have created is, in fact, an (ax, b)-bipartite-expander. More formally, we define $V_1^0 = V_1$, $V_2^0 = V_2$ and $W_1^0 = W_2^0 = \emptyset$. Let $r \in \mathbb{N} \cup \{0\}$. If for $1 \leq i \neq j \leq 2$, there exists $W \subset V_i^r$ with $|W| \leq \varepsilon m$ and $|\Gamma(W) \cap V_j^r| < b|W|$, then we define $V_i^{r+1} = V_i^r \setminus W$, $V_j^{r+1} = V_j$, and $W_i^{r+1} = W_i^r \cup W$, $W_j^{r+1} = W_j^r$. If at some point r_0 there are no more subsets violating the $(\varepsilon m, b)$-expansion condition in $V_1^{r_0}, V_2^{r_0}$, and we have $|W_1^{r_0}|, |W_2^{r_0}| < \varepsilon m$, then we define $U_1 = V_1^{r_0}$, $U_2 = V_2^{r_0}$, which means that the graph $G[U_1, U_2]$ is an $(\varepsilon m, b)$-bipartite-expander. Otherwise, for some r_0 we have, for the first time in this process, $|W_i^{r_0}| \geq \varepsilon m$ for some $i \in \{1, 2\}$. Since in each step r of the process we add to one of W_1^{r-1}, W_2^{r-1} at most εm vertices, it follows that $\varepsilon m \leq |W_i^{r_0}| \leq 2\varepsilon m$. By the definition of $W_i^{r_0}$ we get $|\Gamma(W_i^{r_0}) \cap V_j^{r_0}| < b|W_i^{r_0}|$, where $i \neq j \in \{1, 2\}$. By the choice of r_0 we know that $|W_j^{r_0}| < \varepsilon m$ $(j \neq i)$, and thus

$$|V_j^{r_0} \setminus \Gamma(W_i^{r_0})| > |V_j| - |W_j^{r_0}| - b|W_i^{r_0}| \geq (2b+2)\varepsilon m - \varepsilon m - 2b\varepsilon m \geq \varepsilon m.$$

Since we know that every two large enough sets span an edge, it follows that $e_G(W_i^{r_0}, V_j \setminus \Gamma(W_i^{r_0})) > 0$, which is a contradiction. Hence in the end of this vertex-removal process we are left with $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$ of sizes $|U_1| \geq (1-\varepsilon)|V_1|$ and $|U_2| \geq (1-\varepsilon)|V_2|$ such that the bipartite graph $G[U_1, U_2]$ is an $(\varepsilon m, b)$-bipartite-expander.

We conclude by proving that $G[U_1, U_2]$ is in fact an (ax, b)-bipartite-expander. Assume, for contradiction, that for $1 \leq i \neq j \leq 2$ there exists $W \subseteq U_i$ with $\varepsilon m < |W| \leq ax$ and such that $|\Gamma(W) \cap U_j| < b|W|$. Recall that $x = \min\{|V_1|, |V_2|\} \geq (2b+2)\varepsilon m$, so it follows that

$$|U_j \setminus \Gamma(W)| > (1-\varepsilon)x - abx = (1 - \varepsilon - ab)x \geq (2b+2)(1-\varepsilon-ab)\varepsilon m > \varepsilon m.$$

By the assumption that every two subsets of size at least εm span an edge we get that $e(W, U_j \setminus \Gamma(W)) > 0$, which is a contradiction. □

COROLLARY 4.2. *Let G be a bipartite graph with parts V_1, V_2, let m be some integer, let $0 < \varepsilon < \frac{1}{85}$, and denote $x = \min(|V_1|, |V_2|)$. Assume that every two subsets $V_1' \subseteq V_1$, $V_2' \subseteq V_2$ with $|V_1'|, |V_2'| \geq \varepsilon m$ span at least one edge in G, i.e., $e(G[V_1', V_2']) > 0$. Then there exist $U_1, W_1 \subseteq V_1$ and $U_2, W_2 \subseteq V_2$ with $|U_1|, |W_1| \geq (1-\varepsilon)|V_1|$ and $|U_2|, |W_2| \geq (1-\varepsilon)|V_2|$ such that*
(i) *If $|V_1|, |V_2| \geq \frac{1}{2}m - 1$ then the bipartite graph $G[W_1, W_2]$ is a $(6\varepsilon x, \frac{1}{8\varepsilon} + 1)$-bipartite-expander, and hence a $(12\varepsilon x, \frac{1}{16\varepsilon})$-expander.*
(ii) *If $|V_1|, |V_2| \geq 20\varepsilon m$ then the bipartite graph $G[U_1, U_2]$ is a $(\frac{1}{10}x, 9)$-bipartite-expander, and hence a $(\frac{1}{5}x, 4)$-expander.*

4.2. *Tree embeddings*

We start by defining the following trees, playing a key role in our proofs.

DEFINITION 10. Let $T^{(r,h)}$ be the r-ary tree of depth h (that is, the tree where each vertex, but a leaf, has r children, and the distance, in edges, between the root and every leaf is exactly h). Let $T_\ell^{(r,h)}$ be the tree consisting of two disjoint copies of $T^{(r,h)}$ and a path of length ℓ connecting their roots.

REMARK 6. Note that a longest path in $T_\ell^{(r,h)}$ is of length $\ell + 2h$. Furthermore, the tree $T_\ell^{(r,h)}$ has exactly $\ell - 1 + 2 \cdot \frac{r^{h+1}-1}{r-1}$ vertices.

The main ingredients in the proof of Lemma 3.1 are the following claims regarding tree embeddings in bipartite-expander graphs.

PROPOSITION 4.3. *Let G be a bipartite graph with parts V_1, V_2 with $|V_1|, |V_2| \in \{\lfloor m \rfloor, \lceil m \rceil\}$ for some positive number m. Let $0 < \varepsilon < \frac{1}{85}$ and assume that the pair (V_1, V_2) satisfies the ε-property in G. Then G contains every tree on at most $6\varepsilon m$ vertices with maximum degree at most $\frac{1}{16\varepsilon} - 1$. In particular, G contains a copy of $T_\ell^{(r,h)}$ where $r = \lfloor \frac{1}{16\varepsilon} \rfloor - 2$, $h = \lceil \frac{\log(\varepsilon m)}{\log r} \rceil$, and any integer $\ell \in [1, 2\varepsilon m]$.*

Proof. By Corollary 4.2 there are subsets $U_1, \subseteq V_1$, $U_2 \subseteq V_2$ for which the graph $G[U_1, U_2]$ is an $(12\varepsilon m, \frac{1}{16\varepsilon})$-expander. By Theorem 2.3 we get that $G[U_1, U_2]$ contains a copy of any tree on at most $6\varepsilon m$ vertices with maximum degree at most $\frac{1}{16\varepsilon} - 1$. Set $r = \lfloor \frac{1}{16\varepsilon} \rfloor - 2$, $h = \lceil \frac{\log(\varepsilon m)}{\log r} \rceil$, and $\ell \in [1, 2\varepsilon m]$. By Remark 6 the tree $T_\ell^{(r,h)}$ has at most $6\varepsilon m$ vertices and maximum degree at most $\frac{1}{16\varepsilon} - 1$, so in particular $G[U_1, U_2]$ contains a copy of it. □

PROPOSITION 4.4. *Let G be a bipartite graph on parts V_1, V_2 with $|V_1|, |V_2| \in \{\lfloor m \rfloor, \lceil m \rceil\}$ for some positive number m. Let $0 < \varepsilon < \frac{1}{85}$ and assume that the pair (V_1, V_2) satisfies the ε-property in G. Then $G[V_1, V_2]$ contains a copy of $T_\ell^{(2,h)}$ for $h = \lceil \frac{\log(\varepsilon m)}{\log 2} \rceil$, and any integer $\ell \in [1, 2(1 - 48\varepsilon)m]$. Moreover, if ℓ is even then we can embed a copy of $T_\ell^{(2,h)}$ with all leaves in V_i for any $i \in \{1, 2\}$.*

Proof. We start with the case where we consider odd values of ℓ. Let $U_{11}, U_{12} \subseteq V_1$ be disjoint, and $U_{21}, U_{22} \subseteq V_2$ be also disjoint, such that $|U_{ij}| = \lceil 21\varepsilon m \rceil$ for any $i, j \in \{1, 2\}$. By Corollary 4.2 (item 2) applied separately on $G[U_{11}, U_{21}]$ and on $G[U_{12}, U_{22}]$ we get four subsets $W_{ij} \subseteq U_{ij}$, $i, j \in \{1, 2\}$, all of size at least $20\varepsilon m$, such that each of the graphs $G[W_{11}, W_{21}]$ and $G[W_{12}, W_{22}]$ is a $(\frac{1}{2}\varepsilon m, 4)$-expander.

Let $X_1 \subseteq V_1 \setminus (W_{11} \cup W_{12})$ and let $X_2 \subseteq V_2 \setminus (W_{21} \cup W_{22})$ be such that $|X_1| = |X_2| = \lfloor (1 - 43\varepsilon)m \rfloor$. Let $\ell \in [1, 2(1 - 48\varepsilon)m]$ be odd, and let $q = 4\lceil \varepsilon m \rceil$. We now find a path of length exactly $\ell - 4 + q$. We do this using the following claim, implied by a standard DFS-based argument, stated implicitly in [**5**] and more explicitly in, e.g., [**49**]. For a more extensive discussion about the DFS (Depth First Search) algorithm in finding paths in expander graphs we refer the reader to [**39**].

CLAIM 4.5. *For every graph G there exists a partition of its vertices $V = S \cup T \cup U$ such that $|S| = |T|$, G has no edges between S and T, and U spans a path in G.*

Apply Claim 4.5 to the graph $G[X_1, X_2]$. Notice that $|U| = |X_1 \cup X_2| - |S| - |T| = 2|X_1| - 2|S|$ and in particular $|U|$ is even. U spans a path in $G[X_1, X_2]$, which is a bipartite graph, so we get $|U \cap X_1| = |U \cap X_2|$. Assume w.l.o.g. that $|S \cap X_1| \geq |S \cap X_2|$, then $|T \cap X_2| \geq |T \cap X_1|$. If $|S| = |T| \geq 2\lceil \varepsilon m \rceil - 1$ then we get $|S \cap X_1|, |T \cap X_2| \geq \varepsilon m$. However, we know that $e(S \cap X_1, T \cap X_2) \leq e(S, T) = 0$, contradicting the ε-property of the pair (V_1, V_2) in G. Hence we get that $|S| = |T| \leq 2\lceil \varepsilon m \rceil - 2$, which means that $|U| \geq 2\lfloor (1 - 43\varepsilon)m \rfloor - 4\lceil \varepsilon m \rceil + 4 \geq 2(1 - 45\varepsilon)m - 2$, and in particular $G[X_1, X_2]$ contains a path of length at least $2(1 - 45\varepsilon)m - 3$. Thus, let P_0 be a path of length $\ell - 4 + q \leq 2(1 - 45\varepsilon)m - 3$ and denote its endpoints by $u^* \in X_1$ and $v^* \in X_2$. Let $u_1, \ldots, u_q$ be the first q vertices of P_0 when moving from u^*, that is $u^* = u_1$, and let $v_1, \ldots, v_q$ be the first q vertices of P_0 when moving from v^*, that is $v^* = v_1$. Note that the vertices $\{u_1, \ldots, u_q\}$ are distributed equally between X_1 and X_2, having exactly

$2\lceil \varepsilon m\rceil$ vertices in each set, and similarly the vertices $\{v_1,\dots,v_q\}$. Consider now only the $2\lceil \varepsilon m\rceil$ vertices with odd indices, i.e., $\{u_1,u_3,\dots,u_{q-1}\}$ and $\{v_1,v_3,\dots,v_{q-1}\}$, and note that we have $\{u_1,u_3,\dots,u_{q-1}\}\in X_1$ and $\{v_1,v_3,\dots,v_{q-1}\}\in X_2$. Hence, by the ε-property of the pair (V_1,V_2) in G, at least $\lceil \varepsilon m\rceil+1$ of the vertices in $\{u_1,u_3,\dots,u_{q-1}\}$ have some neighbour in W_{21}, and similarly, at least $\varepsilon m+1$ of the vertices $\{v_1,v_3,\dots,v_{q-1}\}$ have some neighbour in W_{12}. By the pigeonhole principle, there exists (an odd) $s\in\{1,\dots,q-1\}$ such that u_s is connected to some vertex in W_{21} and v_{q-s} is connected to some vertex in W_{12}. Denote by P the subpath of P_0 with endpoints u_s and v_{q-s}, denoted by u,v, respectively, and note that it is of length exactly $\ell-2$.

Now, let w_1 be a neighbour of u in W_{21} and w_2 be a neighbour of v in W_{12}. Recall that $G[W_{11},W_{21}]$ is an $(\frac{1}{2}\varepsilon m,4)$-expander, so by Theorem 2.3 there exists a copy of $T^{(2,h)}$ in $G[W_{11},W_{21}]$, for $h=\lceil\frac{\log(\varepsilon m)}{\log 2}\rceil$, rooted in any predetermined vertex of W_{21}. Similarly, there exists a copy of $T^{(2,h)}$ in $G[W_{12},W_{22}]$ rooted in any predetermined vertex of W_{12}. Let T_{w_1},T_{w_2} be these copies of $T^{(2,h)}$ in $G[W_{11},W_{21}]$ and in $G[W_{12},W_{22}]$, respectively, rooted in $w_1\in W_{21}$ and in $w_2\in W_{12}$, respectively. Joining T_{w_1} and T_{w_2} to P, we get a copy of $T_\ell^{(2,h)}$, as required.

For even values of ℓ we repeat the same argument, with a minor change. Note first that if ℓ is even then any embedded copy of $T_\ell^{(2,h)}$ in $G[V_1,V_2]$ has all leaves in either V_1 or V_2. Assume that we wish to embed a copy of $T_\ell^{(2,h)}$ with all leaves in V_i for some $i\in\{1,2\}$. Note further that if ℓ is even then P_0 is of an even length $\ell-4+q$, and hence both of its endpoints u^* and v^* are in X_j for some $j\in\{1,2\}$. Now, we look at $\{u_1,\dots,u_q\}$ and $\{v_1,\dots,v_q\}$ and split into two possible cases by the parity of h and by the part in which the endpoints of P_0 are contained. If h is even and $i\neq j$, or if h is odd and $i=j$, then we consider only vertices of odd indices, i.e., $\{u_1,u_3,\dots,u_{q-1}\}$ and $\{v_1,v_3,\dots,v_{q-1}\}$. If h is even and $i=j$, or if h is odd and $i\neq j$, then we consider only vertices of even indices, i.e., $\{u_2,u_4,\dots,u_q\}$ and $\{v_2,v_2,\dots,v_q\}$. For simplicity we assume now that h is even and $j=1$, $i=2$ (in particular $i\neq j$), where all other cases are handled similarly. This means that by the pigeonhole principle there exists (an odd) $s\in\{1,\dots,q-1\}$ such that u_s is connected to some vertex in W_{21} and v_{q-s} is connected to some vertex in W_{22}, and equivalently to the odd ℓ case, we embed trees T_{w_1} and T_{w_2}, having $w_1\in W_{21}$ and $w_2\in W_{22}$. □

4.3. *Proof of the Key Lemma*

We are now ready to prove Lemma 3.1 using Proposition 4.3 and Proposition 4.4.

Proof of Lemma 3.1. Throughout the proof we denote $m:=\frac{n}{k}$. Note that k is constant, so $m=\Theta(n)$. Recall that S is the ε-graph obtained from G_0 with respect to the partition $\Pi=(V_1,\dots,V_k)$, that is, every edge $\{i,j\}\in E(S)$ represents a pair (V_i,V_j) which satisfies the ε-property in G_0.

The general idea is to convert a cycle (or a path) from the graph S to a cycle in G_0 of the desired length, by using tree embeddings between clusters of G_0. Assume that $(1,\dots,b)$ is a cycle in S and that b is odd. Roughly speaking, we divide the cycle in S into pairs of vertices that are connected with an edge $(2i,2i+1)$, for $i\in[(b-1)/2]$. We then embed in each pair of corresponding clusters (V_{2i},V_{2i+1}) a tree $T_\ell^{(r,h)}$ with appropriate parameters, such that the leaf sets of the two copies of $T^{(r,h)}$ in $T_\ell^{(r,h)}$ are in V_{2i} and V_{2i+1}, respectively. Since each of these leaf sets contains at least εm vertices, we can use the ε-property to connect some leaf from the leaves in V_{2i+1} and some leaf from the leaves in V_{2i+2} by an edge. This way, we are able to connect different copies of $T_\ell^{(r,h)}$ to a very large tree, containing a copy of $T_{\ell^*}^{(r,h)}$ for an appropriate ℓ^*, where its leaf sets are in V_2 and V_b. We then use one vertex v from V_1 and connect it to both leaf sets. This creates a cycle in G_0 of length exactly $t=\ell^*+2h+2$. For converting a path in S to an even cycle in G_0 we use a similar argument, only this time we split each cluster into two clusters and use both endpoints of the path in S to "close" the cycle in G_0. We give the full details below.

We start with the first item. Suppose that S contains a path of an odd length b, where $1 \le b < k$, and let $t \in [\frac{C_1}{\log(1/\varepsilon)} \log n, (1-48\varepsilon)an]$ be even, $a \coloneqq \frac{b+1}{k}$. Assume w.l.o.g. that this path is $(1,\dots,b+1)$, and consider the sequence of corresponding clusters $V_1,\dots,V_{b+1}$. We separate the case where t is even into three parts. The first part deals with the case where $t \in [\frac{C_1}{\log(1/\varepsilon)} \log n, 2\varepsilon m]$, the second part deals with the case where $t \in [2\varepsilon m, (1-48\varepsilon)an]$ and $b = 1$, and the third part deals with all other cases, i.e., $t \in [2\varepsilon m, (1-48\varepsilon)an]$ and $b \ge 3$ (and is further separated into two subcases by the value of b). In each part we divide the vertices of the path into pairs, and embed a certain tree in the bipartite subgraph of the original graph induced by each pair. This is where we use the assumption of b being odd, i.e., the path has an even number of vertices. A similar cluster pairing strategy was presented and used by Dellamonica et al. [**11**, Theorem 7].

If $t \in [\frac{C_1}{\log(1/\varepsilon)} \log n, 2\varepsilon m]$ is even, then we look at a single edge in the path, say, $\{1,2\}$. The graph $G_0[V_1,V_2]$ is bipartite and the pair (V_1,V_2) satisfies the ε-property in G_0. By Proposition 4.3 we know that $G_0[V_1,V_2]$ contains a copy of every tree with at most $6\varepsilon m$ vertices and maximum degree at most $\frac{1}{16\varepsilon} - 1$. In particular, $G_0[V_1,V_2]$ contains a copy of $T_\ell^{(r,h)}$ (as in Definition 10) for $r = \lfloor \frac{1}{16\varepsilon} \rfloor - 2$, $h = \lceil \frac{\log(\varepsilon m)}{\log r} \rceil$ and any odd $\ell \in [1, 2\varepsilon m]$ (as $T^{(r,h)}$ has at most $2\varepsilon m$ vertices for these values of r and h, and thus $T_\ell^{(r,h)}$ has at most $6\varepsilon m$). Note that a maximal path in $T_\ell^{(r,h)}$ is of length $2h + \ell$. Set $\ell = t - 2h - 1$ (note that it satisfies the constraints, as $1 \le t - 2h - 1 \le 2\varepsilon m$) and we get that a maximal path in a $T_\ell^{(r,h)}$-copy is of length exactly $t-1$. Now, note that this copy of $T_\ell^{(r,h)}$ has at least εm leaves in V_1 and εm leaves in V_2, due to parity considerations. By the ε-property of the pair (V_1,V_2) in G_0 there is an edge between these two sets of leaves, closing a cycle of length $\ell + 2h + 1 = t$, as required.

If $b = 1$ and $t \in [2\varepsilon m, (1-48\varepsilon)an]$ is even, for $a \coloneqq \frac{b+1}{k}$, then once again the graph $G_0[V_1,V_2]$ is bipartite and the pair (V_1,V_2) satisfies the ε-property in G_0. We repeat the previous argument but with the only change of embedding a different tree in $G_0[V_1,V_2]$. By Proposition 4.4 we know that $G_0[V_1,V_2]$ contains a copy of $T_\ell^{(2,h)}$ for $h = \lceil \frac{\log(\varepsilon m)}{\log 2} \rceil$ and $\ell = t - 1 - 2h$. Also here, note that this copy of $T_\ell^{(2,h)}$ has at least εm leaves in V_1 and εm leaves in V_2, due to parity considerations. Again, by the ε-property of the pair (V_1,V_2) in G_0 there is an edge between these two sets of leaves, closing a cycle of length t, as required.

If $b \ge 3$ and $t \in [2\varepsilon m, (1-48\varepsilon)an]$ is even, for $a \coloneqq \frac{b+1}{k}$, then we look at the full path $(1,\dots,b+1)$ and the set of corresponding clusters $V_1,\dots,V_{b+1}$. Informally, we embed two copies of $T_\ell^{(2,h)}$ for some carefully chosen values h,ℓ, one in $G_0[V_1,V_2]$, and one in $G_0[V_b,V_{b+1}]$. Then, if we have used all the clusters already for tree embedding (i.e., $b = 3$), then we connect these two trees by two edges to create a cycle of the desired length. Otherwise, we keep embedding trees in all clusters we have not touched yet. Formally, we further separate this case into two subcases and argue as follows.

Assume first that $b = 3$. For following the arguments of this subcase Figure 1 can be helpful. Note that each of the pairs (V_1,V_2) and (V_3,V_4) satisfies the ε-property in G_0, and that we have $|V_j| \in \{\lfloor m \rfloor, \lceil m \rceil\}$ for any $j \in [4]$. Now let $j \in \{1,3\}$. By Proposition 4.4 we know that $G_0[V_j,V_{j+1}]$ contains a copy of $T_{\ell_j}^{2,h}$ where $h = \lceil \frac{\log(\varepsilon m)}{\log 2} \rceil$ and ℓ_j is such that $\ell_1 + \ell_3 = t - 4h - 2$, $|\ell_1 - \ell_3| \le 2$, and both are even. Note here that $\ell_j \le \frac{1}{2}t - 2h \le 2(1-48\varepsilon)m$. We embed (two trees) $T_{\ell_1}^{(2,h)}$ and $T_{\ell_3}^{(2,h)}$, such that the leaf sets L_2, L_2' and L_3, L_3' are in V_2 and V_3, respectively (which is possible as ℓ_1,ℓ_3 are even). Having $|L_2|, |L_2'|, |L_3|, |L_3'| \ge \varepsilon m$, by the ε-property of the pair (V_2,V_3) in G_0, there exist two edges, one between L_2 and L_3, and the other between L_2' and L_3'. These two edges close a cycle of length exactly t.

Assume now that $b \ge 5$. When following the arguments of this subcase Figure 2 can be helpful. In this subcase too we embed two $T_\ell^{(2,h)}$-copies, for a suitable choice of h,ℓ, in $G_0[V_1,V_2]$

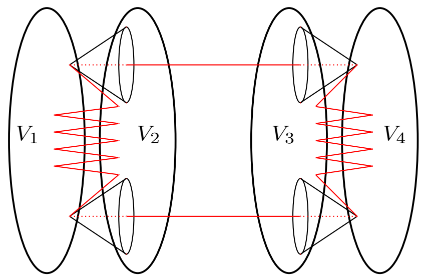

FIGURE 1. *Embedding trees to create an even cycle (in red), proof of Lemma 3.1, $b = 3$. (For the simplicity of the figure the roots of the $T_\ell^{(r,h)}$-copies are contained in V_1 and V_4, but they can rather be contained in V_2 and V_3, respectively, as well).*

and in $G_0[V_b, V_{b+1}]$. However, we do not connect them directly by two edges, but through other $T_\ell^{(2,h)}$-copies we embed in the rest of clusters. More precisely, for each $j \in \{3, \dots, b-1\}$, arbitrarily split the vertex set V_j into two equally sized subsets (up to possibly one vertex), denoted by U_j, U'_j. Set some $i \in \{2, \frac{1}{2}(b-1)\}$ and look at the pair (V_{2i-1}, V_{2i}). Since (V_{2i-1}, V_{2i}) satisfies the ε-property in G_0, it follows that each of the pairs (U_{2i-1}, U_{2i}) and (U'_{2i-1}, U'_{2i}) satisfies the ε'-property in G_0, for ε' satisfying $\varepsilon'(\frac{1}{2}m - 1) = \varepsilon m$. We have $|U_{2i-1}|, |U_{2i}| \geq \frac{1}{2}m - 1$, so by Proposition 4.4 (taking $\frac{1}{2}m - 1$ instead of m and ε' instead of ε) we get that $G_0[U_{2i-1}, U_{2i}]$ contains a copy of $T_{\ell_0}^{(2,h)}$ for $h = \lceil \frac{\log(\varepsilon m)}{\log 2} \rceil$ and $\ell_0 = \lfloor \frac{t-b+1}{b+1} \rfloor_{odd} - 2h \leq 2(1 - 48\varepsilon)(\frac{1}{2}m - 1)$. Denote this copy by $T_{2i-1,2i}$ and its leaf sets in U_{2i-1}, U_{2i} by L_{2i-1}, L_{2i}, respectively. We do the same for $G_0[U'_{2i-1}, U'_{2i}]$ where we denote the embedded copy of $T_{\ell_0}^{(2,h)}$ by $T'_{2i-1,2i}$, and its leaf sets in U'_{2i-1}, U'_{2i} by L'_{2i-1}, L'_{2i}, respectively. Do this for every $i \in \{2, \frac{1}{2}(b-1)\}$ with the same notation. Recall that for every $i \in \{2, \frac{1}{2}(b-3)\}$ the pair (V_{2i}, V_{2i+1}) satisfies the ε-property in G_0, and moreover, note that we have $|L_{2i}|, |L_{2i+1}|, |L'_{2i}|, |L'_{2i+1}| \geq \varepsilon m$. Thus, for every $i \in \{2, \frac{1}{2}(b-3)\}$ we have $e_{G_0}(L_{2i}, L_{2i+1}), e_{G_0}(L'_{2i}, L'_{2i+1}) > 0$, so we add an edge between every such two leaf sets, summing up to total of $b-5$ new edges (note that for $b = 5$ this part is redundent so we do not add any edges). This creates two disjoint copies of $T_{\ell^*}^{(2,h)}$ in G_0, where $h = \lceil \frac{\log(\varepsilon m)}{\log 2} \rceil$, $\ell^* = \frac{1}{2}(\ell_0 + 2h)(b-3) + \frac{1}{2}(b-5) - 2h$, one contained in $U \coloneqq \bigcup_{j=3}^{b-1} U_j$ and the other in $U' \coloneqq \bigcup_{j=3}^{b-1} U'_j$. Moreover, the first $T_{\ell^*}^{(2,h)}$-copy, embedded in U, has at least εm leaves in U_3 and at least εm leaves in U_{b-1}. Similarly, the other $T_{\ell^*}^{(2,h)}$-copy, embedded in U', has at least εm leaves in U'_3 and at least εm leaves in U'_{b-1} (see Figure 2). Now, we treat the pairs (V_j, V_{j+1}) where $j \in \{1, b\}$ almost similarly to how we treated them in the subcase $b = 3$. More formally, we note that (V_j, V_{j+1}) also has the ε-property in G_0 and that $|V_j|, |V_{j+1}| \in \{\lfloor m \rfloor, \lceil m \rceil\}$. So by Proposition 4.4 we get that $G_0[V_j, V_{j+1}]$ contains a copy of $T_{\ell_j}^{(2,h)}$ for $h = \lceil \frac{\log(\varepsilon m)}{\log 2} \rceil$ and the ℓ_j's are such that $\ell_1 + \ell_b = (t - 2\ell^* - 4h) - 4h - 4$, $|\ell_1 - \ell_b| \leq 2$, and both are even (which means that $\ell_j \leq \frac{1}{2}(t - 2\ell^* - 4h) - 2h - 1 \leq 2(1 - 48\varepsilon)m$), where the leaf sets L_2, L'_2, and L_b, L'_b are in V_2 and V_b, respectively (as ℓ_j is even). Since $|L_2|, |L'_2|, |L_b|, |L'_b| \geq \varepsilon m$, once again, by the ε-property, we can connect some $v_2 \in L_2$ with $v_3 \in L_3$, some $v'_2 \in L'_2$ with $v'_3 \in L'_3$, some $v_{b-1} \in L_{b-1}$ with $v_b \in L_b$, and some $v'_{b-1} \in L'_{b-1}$ with $v'_b \in L'_b$ (see Figure 2). By doing that we complete a cycle of length exactly $\ell_1 + \ell_b + 2\ell^* + 8h + 4 = t$.

We now prove the second item. Suppose now that S contains an odd cycle of length b, where $3 \leq b < k$, and let $t \in \left[\frac{(b-1) \cdot C_1}{2\log(1/\varepsilon)} \log n, (1 - 48\varepsilon)an\right]$ be odd, $a = \frac{b}{k}$. Assume w.l.o.g. that this cycle is $(1, \dots, b)$ and consider the set of corresponding clusters $V_1, \dots, V_b$. Let $i \in \{1, \dots, \frac{1}{2}(b-1)\}$ and look at the pair (V_{2i-1}, V_{2i}). Using Proposition 4.3 and Proposition 4.4 we embed one of two different possible trees in $G_0[V_{2i-1}, V_{2i}]$, depending on the value of t, to eventually create a cycle of the required length. Recall that the pair (V_{2i-1}, V_{2i}) satisfies

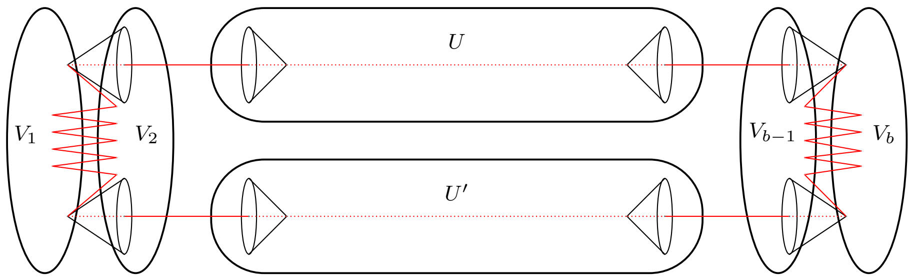

FIGURE 2. *Embedding trees to create an even cycle (in red), proof of Lemma 3.1, $b \geq 5$. (For the simplicity of the figure the roots of the $T_\ell^{(r,h)}$-copies are contained in V_1 and V_b, but they can rather be contained in V_2 and V_{b-1}, respectively, as well).*

the ε-property in G_0, and furthermore, that $|V_{2i-1}|, |V_{2i}| \geq \lfloor m \rfloor$. Hence, by Proposition 4.3 and Proposition 4.4, $G_0[V_{2i-1}, V_{2i}]$ contains a copy of $T_\ell^{(r,h)}$ where $h = \lceil \frac{\log(\varepsilon m)}{\log r} \rceil$ for both $r = \lfloor \frac{1}{16\varepsilon} \rfloor - 2, \ell \in [1, 2\varepsilon m]$ and $r = 2, \ell \in [1, 2(1-48\varepsilon)m]$, respectively. Thus, we embed a copy of $T_{\ell_i}^{(r,h)}$ in $G_0[V_{2i-1}, V_{2i}]$ for $h = \lceil \frac{\log(\varepsilon m)}{\log r} \rceil$, where $r = \lfloor \frac{1}{16\varepsilon} \rfloor - 2$ if $t \in \left[\frac{(b-1)\cdot C_1}{2\log(1/\varepsilon)} \log n, 2\varepsilon m \right]$, and $r = 2$ if $t \in [2\varepsilon m, (1-48\varepsilon)an]$. We choose the value of ℓ_i as follows. For all $i \in \{2, \ldots, \frac{1}{2}(b-1)\}$ we set $\ell_i = \ell_0 := \lfloor \frac{2t-2-(1+2h)(b-1)}{b-1} \rfloor_{odd}$, and $\ell_1 = t - 1 - \frac{b-1}{2} - h(b-1) - \frac{1}{2}(b-3)\ell_0$. Note that ℓ_1 is also odd, and moreover, that $\ell_0, \ell_1 \in [1, 2(1-48\varepsilon)m]$. For every $i \in \{1, \ldots, \frac{1}{2}(b-1)\}$ we denote the embedded $T_{\ell_i}^{(r,h)}$-copy in $G_0[V_{2i-1}, V_{2i}]$ by $T_{2i-1,2i}$, and further denote by L_{2i-1}, L_{2i} its leaf sets in V_{2i-1} and in V_{2i}, respectively. Note that for every $i \in \{1, \ldots, \frac{1}{2}(b-1)\}$, a maximal path in $T_{2i-1,2i}$ is of length $2h + \ell_i$. Recall that for every $i \in \{1, \ldots, \frac{1}{2}(b-3)\}$, also the pair (V_{2i}, V_{2i+1}) satisfies the ε-property in G_0, and note that we have $|L_{2i}|, |L_{2i+1}| \geq \varepsilon m$. Thus we have $e_{G_0}(L_{2i}, L_{2i+1}) > 0$ for every $i \in \{1, \ldots, \frac{1}{2}(b-3)\}$, so we add an edge between every such pair of leaf sets, summing up to $\frac{1}{2}(b-3)$ new edges. Thus we get in G_0 a copy of the tree $T_{\ell^*}^{(r,h)}$, where $\ell^* = \sum_{i=1}^{\frac{1}{2}(b-1)} \ell_i + (b-3)h + \frac{1}{2}(b-3) = t - 2h - 2$, with at least εm leaves in V_1 and at least εm leaves in V_{b-1}. Some maximal path inside this tree (connecting the mentioned two leaf sets) will be used to get a cycle of length t along with extra two edges. Now, we note that there exists a vertex $v_b \in V_b$ which is adjacent both to a vertex in L_1 and a vertex in L_{b-1}. Indeed, otherwise one of L_1, L_{b-1} would have fewer than $(1-\varepsilon)\lfloor m \rfloor$ neighbours in V_b, which contradicts the ε-property of the pairs (V_1, V_b) and (V_{b-1}, V_b) in G_0. Thus we can connect the vertex v_b to a vertex in L_1 and to a vertex in L_{b-1}, adding two more edges and closing a cycle of length exactly t. □

5. *Robustness*

In this section we prove Theorem 1.5, and discuss its tightness. We show that this result is tight for many values of t, and we prove Theorem 1.6, giving a tighter result for the cases in which Theorem 1.5 is not tight enough.

For the proofs in this section we use Szemerédi's celebrated Regularity Lemma [**55**].

THEOREM 5.1 (Szemerédi's Regularity Lemma [**55**]). *For every positive real ε and for every positive integer k_0 there are positive integers n_0 and K_0 with the following property: for every graph G on $n \geq n_0$ vertices there is an ε-regular partition $\Pi = (V_1, \ldots, V_k)$ of $V(G)$ such that $||V_i| - |V_j|| \leq 1$ and $k_0 \leq k \leq K_0$.*

The following lemma connects the reduced graph of G and the ε-graph of $G(p)$, with respect to the same partition Π.

Lemma 5.2. *Let $0 < x < 1$, $0 < \beta < 1 - x$ and let G be a graph on n vertices with $e(G) \geq (x+\beta)\binom{n}{2}$ and an ε-regular partition $\Pi = (V_1, \dots, V_k)$ of its vertices with $\varepsilon \leq \frac{\beta}{100}$ and $k \geq \frac{2}{\varepsilon^2}$. Let $R := R(G, \Pi, \rho, \varepsilon)$ be the reduced graph as in Definition 5, with $\rho = 10\varepsilon$. Then there exists $C := C(\varepsilon, k)$ such that for $p \geq \frac{C}{n}$ the following holds. Let $S := S(G(p), \Pi, \varepsilon)$ be the ε-graph corresponding to $G(p)$, as in Definition 7. Then w.h.p. $R \subseteq S$, and therefore w.h.p. $e(S) \geq (x + \beta/2)\binom{k}{2}$.*

Proof. Denote $\lfloor m \rfloor \leq |V_i| \leq \lceil m \rceil$, where $m = \frac{n}{k}$. It is sufficient to prove that for every i, j where (V_i, V_j) is an ε-regular pair with $d(V_i, V_j) \geq \rho = 10\varepsilon$, we have that w.h.p. (V_i, V_j) satisfies the ε-property in the random graph $G(p)$. Let $U_i \subseteq V_i$ and $U_j \subseteq V_j$ be such that $|U_i|, |U_j| \geq \varepsilon m$. By ε-regularity we have $|d(V_i, V_j) - d(U_i, U_j)| \leq \varepsilon$. Combining it with the assumption $d(V_i, V_j) \geq \rho$, we have that

$$e_G(U_i, U_j) \geq (\rho - \varepsilon)|U_i||U_j| = 9\varepsilon |U_i||U_j|.$$

For two disjoint subsets U_i, U_j of $V(G)$, denote by $e_p(U_i, U_j)$ the random variable counting the number of edges between these sets in $G(p)$. Then $e_p(U_i, U_j)$ is distributed binomially with parameters $e_G(U_i, U_j)$ and p. Hence, the probability that there exist two such sets that do not satisfy the ε-property in $G(p)$ is at most $\binom{n}{\varepsilon m}^2 \Pr[e_p(U_i, U_j) = 0] \leq e^{-\Omega(n)}$, for, say, $p \geq \frac{\log k}{\varepsilon^2 m}$. □

We can now prove Theorem 1.5.

Proof of Theorem 1.5. We can assume $0 < \beta < 1/4$. Set $\varepsilon = \frac{\beta}{10000}$ and $k_0 = \frac{2}{\varepsilon^2}$. Take n_0, K as given in the Regularity Lemma (Theorem 5.1), and also set $\gamma = \frac{2(1-48\varepsilon)}{k}$. Let $\frac{C_1}{\log(1/\beta)} \log n \leq t \leq (1 - C_2\beta)n$, where C_1, C_2 are the absolute constants from Corollary 3.2. Let G be a graph on $n \geq n_0$ vertices with $e(G) \geq \mathrm{ex}(n, C_t) + \beta\binom{n}{2} \geq (g^\gamma(t,n) + \beta/2)\binom{n}{2}$ (recall that $\mathrm{ex}(n, C_t) \geq g^\gamma(t,n)\binom{n}{2} - 1$). Then by Theorem 5.1 there exists an ε-regular partition $\Pi = (V_1, \dots, V_k)$ of $V(G)$ such that $||V_i| - |V_j|| \leq 1$, with $k_0 \leq k \leq K$.
We next look at the graph $G(p)$ with the same partition and consider the ε-graph $S := S(G(p), \Pi, \varepsilon)$. By Corollary 5.2 we have that w.h.p. $e(S) \geq (g^\gamma(t,n) + \beta/4)\binom{k}{2}$. Using Corollary 3.2 we get that w.h.p. $G(p)$ contains a cycle of length t. □

Remark 7. As mentioned in the introduction and in the beginning of this section, Theorem 1.5 is tight in the sense that for many values of t taking a graph G with $\Theta(n^2)$ extra edges above the extremal number $\mathrm{ex}(n, C_t)$ is in fact necessary for having w.h.p. a copy of C_t in $G(p)$ where $p \geq \frac{C}{n}$. However, there are values of t for which only $\omega(1/p)$ extra edges suffice.

The following claim gives a description of the cases for which Theorem 1.5 is tight.

Claim 5.3. *If t is even, or is odd with $t \geq \frac{n}{2}$, then adding $\beta\binom{n}{2}$ edges to the extremal amount of edges in Theorem 1.5 is necessary.*

Proof. Assume first that $t = o(n)$ and even, and take a graph $G = G(n, p_0)$ for some $p_0 = o(1)$ and $p_0 = \omega(n^{-1+2/t})$. Note that $\mathbb{E}[e(G)] = \Theta(n^2 p_0) \gg \mathrm{ex}(n, C_t)$ (recall that $\mathrm{ex}(n, C_t) = O\left(n^{1+2/t}\right)$ in this case), and furthermore, taking $G(p)$ with $p = \frac{C}{n}$ for some constant $C > 0$ is equivalent to sampling a graph from $G(n, p_0 p)$. Having $p_0 p = o(\frac{1}{n})$, we get that $G(p)$ is w.h.p. acyclic, and in particular that taking only $o(n^2)$ more edges than the extremal number is not enough in this case.

Assume now that $t = \Theta(n)$ is either even, or odd satisfying $t \geq \frac{n}{2}$. Let a be a constant such that $t \geq an$ (even or odd). It is known (and an easy exercise) that for any constant $C > 0$, there exists some $\alpha \coloneqq \alpha(C) > 0$ such that w.h.p. for any $an \leq t_0 \leq n$ the graph $G(t_0, p)$ has w.h.p. at least αn isolated vertices, where $p = \frac{C}{n}$. Now, let $an \leq t < n$, let $0 < \varepsilon < \alpha$ be some constant, and take G to be the graph on n vertices consisting of two cliques sharing exactly one vertex, one of size $(1+\varepsilon)t$, denoted by K^1, and the other of size $n - (1+\varepsilon)t + 1$, denoted by K^2. Now take $G(p)$ and look at a subgraph of it that is induced by the vertices of K^1. This subgraph is exactly $G((1+\varepsilon)t, p)$ and thus w.h.p. $G(p)[K^1]$ contains at least αn isolated vertices. Therefore, w.h.p. $G(p)[K^1]$ does not contain any cycle of length $(1+\varepsilon)t - \alpha n < t$ or larger, and in particular $G(p)$ does not contain any cycle of length t or larger. On the other hand, $e(G) = \binom{(1+\varepsilon)t}{2} + \binom{n-(1+\varepsilon)t+1}{2} \geq \binom{t-1}{2} + \binom{n-t+2}{2} + \frac{\varepsilon}{4}n^2 = \mathrm{ex}(n, C_t) + \frac{\varepsilon}{4}n^2$. Note that here we look at a graph that can be constructed by taking the extremal example of Woodall (see [**59**]), moving εn vertices from the smaller clique to the larger clique, and adjusting all relevant edges accordingly. □

5.1. *Robustness for odd cycles*

In this subsection we discuss the supplemental part of Claim 5.3, where we prove a tight robustness result for odd cycles shorter than $\frac{n}{2}$.

Proof of Theorem 1.6. Let $0 < \varepsilon \leq \min\left\{\frac{\beta}{1000}, \frac{1}{1105^2}\right\}$, and let $k_0 = \lceil \frac{2}{\varepsilon^2} \rceil$. Let n_0, K_0 be as given in the Regularity Lemma (Theorem 5.1). Let $t \in [\frac{C_1}{\log(1/\beta)} \log n, \left(\frac{1}{2} - \beta\right) n]$ be odd. Let G be a graph on $n \geq n_0$ vertices with $e(G) \geq \mathrm{ex}(n, C_t) + \frac{1}{p} \cdot f(n) = \lfloor \frac{1}{4}n^2 \rfloor + \frac{1}{p} \cdot f(n)$, where $f(n)$ is a monotone increasing function tending to infinity with n, and assume that $f(n) = o(n)$. By Theorem 5.1 there exists an ε-regular partition $\Pi = (V_1, \ldots, V_k)$ of $V(G)$, for some $k_0 \leq k \leq K_0$, such that $||V_i| - |V_j|| \leq 1$ for every $i, j \in [k]$. Let $\rho = 10\varepsilon$ and let $R \coloneqq R(G, \Pi, \rho, \varepsilon)$ be the reduced graph (as in Definition 5). We separate the proof into two cases, by the number of edges in R.

Case 1: Assume that $e(R) > \frac{1}{4}k^2$, then by Theorem 2.2 R contains a cycle of an odd length $b = \lfloor \left(\frac{t}{n} + \beta\right) k \rfloor_{odd}$, and also a triangle. Let $C \coloneqq C(\varepsilon)$ be as given in Lemma 5.2, look at the graph $G(p)$ for $p \geq \frac{C}{n}$, and consider the ε-graph $S \coloneqq S(G(p), \Pi, \varepsilon)$. Recall that, by Corollary 5.2, w.h.p. $R \subseteq S$. Let C_1 be the absolute constant from Lemma 3.1. If we have $t \in [\frac{C_1}{\log(1/\beta)} \log n, \frac{2}{k}n]$, then we look at a triangle in S and by Lemma 3.1 we get that w.h.p. cycles of all lengths in $[\frac{C_1}{\log(1/\beta)} \log n, \frac{2}{k}n]$ in $G(p)$, and in particular a cycle of length t. For larger values of t we consider a cycle of length b in S. Using Lemma 3.1, as $(b-1)(1-48\varepsilon)\frac{n}{k} > t$, we get that w.h.p. $G(p)$ contains a cycle of length ℓ, for any $\ell \in \left[\frac{2}{k}n, (b-1)(1-48\varepsilon)\frac{n}{k}\right]$, and in particular a cycle of length t.

Case 2: Assume now that $e(R) \leq \frac{1}{4}k^2$. By a similar result to Lemma 2.5 (by dropping the assumption of upper-uniformity, taking $G' = G$, and considering $p = 1$), one can get that $e(R) \geq \left(\frac{1}{4} - 6\varepsilon\right) k^2$. In addition, we may assume that $\delta(G) \geq \frac{n}{5}$. Indeed, otherwise we iteratively remove vertices from G in the following way. Let $G_0 = G$. If for $i \geq 0$ we have $\delta(G_i) < \frac{v(G_i)}{5}$ then we define $G_{i+1} = G_i - v_i$ for some $v_i \in V(G_i)$ with $d_{G_i}(v_i) < \frac{v(G_i)}{5}$. Let i_0 be minimal such that $\delta(G_{i_0}) \geq \frac{v(G_{i_0})}{5}$. Let $\varepsilon' = \frac{9}{5}\beta$ and denote $n' = \lceil (1-\varepsilon')n \rceil$. If $v(G_{i_0}) \geq n'$, then denote $G' = G_{i_0}$ and consider G' instead of G, as we still have $e(G') \geq \frac{1}{4}(n')^2 + \frac{1}{p}f(n')$.

Otherwise, let i_1 be such that $v(G_{i_1}) = n'$, and denote $G'' = G_{i_1}$. Note that now we have $e(G'') > \frac{1}{4}n^2 - \frac{n}{5} \cdot \varepsilon' n \geq \frac{1}{4}(n')^2 + \beta\binom{n'}{2}$. By Theorem 1.5 there exists $C' > 0$ such that for $p \geq \frac{C'}{n'}$ w.h.p. the graph $G''(p)$ contains an odd cycle of length t for any $\frac{C_1}{\log(1/\beta)} \log n' \leq t \leq \frac{1}{2}n'$. Taking $C = \frac{C'}{1-\varepsilon'}$ so that $p \geq \frac{C}{n}$, we get that, in particular, w.h.p. the graph $G(p)$ contains an odd cycle of length t for any $\frac{C_1}{\log(1/\beta)} \log n \leq t \leq \left(\frac{1}{2} - \beta\right) n$ (as $\log n > \log n'$ and $(\frac{1}{2} - \beta)n < \frac{1}{2}n'$). Hence, from now on we assume that $\delta(G) \geq \frac{n}{5}$, since otherwise we can consider G' instead of G. We now further separate this case into two sub-cases, by the structure of the reduced graph R. We say that a graph H on h vertices is η-far from being bipartite if at least ηh^2 edges must be removed from H in order to make it bipartite. Otherwise, we say that H is η-close to being bipartite. Take $\eta = 2\varepsilon$.

Subcase 2.1: Assume that R is η-close to being bipartite, and recall that $\eta = 2\varepsilon$. Let $A \subset V(G)$ be such that $[A, A^c]$ is a max-cut in G. Again, by a similar result to Lemma 2.5, one can get that $e_G(A, A^c) \geq \left(\frac{1}{4} - 6\varepsilon - \eta\right) n^2$, and thus $|A|, |A^c| \geq \left(\frac{1}{2} - \sqrt{6\varepsilon + \eta}\right) n = \left(\frac{1}{2} - \sqrt{8\varepsilon}\right) n$. Recall that $e(G) \geq \lfloor \frac{1}{4}n^2 \rfloor + \frac{f(n)}{p}$, so w.l.o.g. we have $e(A) \geq \frac{f(n)}{2p} = \omega\left(\frac{1}{p}\right)$. In fact, this is the only part of the proof where we use the assumption about G having at least $\omega\left(\frac{1}{p}\right)$ extra edges above the Turán number for an odd cycle. To obtain $G(p)$ we first note that $G(p) \supseteq G[A](p) \cup G[A, A^c](p)$. We expose separately the edges of $G[A](p)$ and the edges of $G[A, A^c](p)$. Starting with the edges inside A, we show that w.h.p. $G[A](p)$ contains a matching of size $\omega(1)$. Let m be the size of a maximal matching one can find in $G[A](p)$. Then we have

$$\mathbb{P}[\text{maximal matching in } G[A](p) \text{ is of size at most } m] \leq \sum_{i=0}^{m} \binom{e_G(A)}{i} p^i (1-p)^{e_G(A) - 2in},$$

where each summand bounds the probability of having a maximal matching of size i, by considering the probability of having i edges in $G[A](p)$, where any other edge shares at least one vertex with an edge in this set of i edges (as this is a maximal matching). As there are at least $e_G(A) - |A| \cdot 2i \geq e_G(A) - 2in$ edges that share no vertex with a given matching of size i, we get this bound. Now, considering, say, $m = \sqrt{f(n)}$ we get

$$\mathbb{P}[\text{maximal matching in } G[A](p) \text{ is of size at most } m] = o(1).$$

Hence, w.h.p. $m \geq \sqrt{f(n)}$ holds. Let M be a matching in $G[A]$ of size $\lfloor \sqrt{f(n)} \rfloor$. We now expose the edges of $G[A, A^c]$ in three stages. Let p_1 be such that $(1 - p_1)^3 = 1 - p$, i.e., $p_1 = 1 - (1 - p)^{\frac{1}{3}} \geq \frac{c_1}{n}$ for some constant $c_1 > 0$. Note that $G[A, A^c](p)$ is the same as taking $G_1 \cup G_2 \cup G_3$ where $G_i = G[A, A^c](p_1)$ for each $i = 1, 2, 3$, independently. Recall that $[A, A^c]$ is a max-cut, and that $\delta(G) \geq \frac{n}{5}$, so we have that $d_G(v, A^c) \geq \frac{n}{10}$ for every $v \in A$. Moreover, at most $4\sqrt{\varepsilon}n$ vertices in A^c have less than $\left(\frac{1}{2} - 3\sqrt{\varepsilon}\right) n$ neighbours into A in G. Indeed, if there are x vertices in A^c with less than $\left(\frac{1}{2} - 3\sqrt{\varepsilon}\right) n$ neighbours in A, then

$$\left(\frac{1}{4} - 8\varepsilon\right) n^2 \leq e_G(A, A^c) = \sum_{u \in A^c} d(u, A) < x\left(\frac{1}{2} - 3\sqrt{\varepsilon}\right) n + (|A^c| - x)\,|A|.$$

Recalling that both $|A|$ and $|A^c|$ are of size at least $\left(\frac{1}{2} - \sqrt{8\varepsilon}\right) n$, we get that $x \leq 4\sqrt{\varepsilon}n$. Consider $G_1 = G[A, A^c](p_1)$, and let uv be an edge in the matching M. For each of u, v look at the set of their neighbours in A^c with degree in G at least $\left(\frac{1}{2} - 3\sqrt{\varepsilon}\right) n$ into A. We know that there are at least $\left(\frac{1}{10} - 4\sqrt{\varepsilon}\right) n$ such neighbours for each of u, v and at least $\frac{1}{2}\left(\frac{1}{10} - 4\sqrt{\varepsilon}\right) n$ neighbours for each of u, v such that these sets of neighbours are disjoint. Hence, the probability that, in G_1, both u and v have each at least one neighbour in A^c with degree at least $\left(\frac{1}{2} - 3\sqrt{\varepsilon}\right) n$ into A in G, where these neighbours of u are disjoint from those of v, is at least $1 - 2(1 - p_1)^{\frac{1}{2}\left(\frac{1}{10} - 4\sqrt{\varepsilon}\right)n}$. Hence, with some probability bounded away from 0, we have that uv is contained in a path on three edges in G_1, with endpoints in A^c of high degree into A in G.

We now find, w.h.p., a set of at least $f(n)^{1/4}$ such paths, with distinct endpoints, one by one. Let $M' \subseteq M$ be some proper subset of edges in the matching (might be empty), and assume that for every $e \in M'$ we have found in G_1 a path consisting of three edges such that e is the middle edge, the endpoints of this path are in A^c, and each has at least $\left(\frac{1}{2} - 3\sqrt{\varepsilon}\right) n$ neighbours into A in G. Denote this set of paths by M''. Now take some edge $e' = uv \in M \setminus M'$. Then the probability that both u and v have each, in G_1, at least one neighbour in A^c with degree at least $\left(\frac{1}{2} - 3\sqrt{\varepsilon}\right) n$ into A in G, where these neighbours are distinct and are not contained in the vertices of M'', is at least

$$1 - 2(1-p_1)^{\frac{1}{2}\left(\frac{1}{10} - 4\sqrt{\varepsilon}\right)n - 2|M'|} \geq 1 - 2(1-p_1)^{\frac{1}{2}\left(\frac{1}{10} - 4\sqrt{\varepsilon}\right)n - 2|M|} \geq 1 - e^{-a_1} =: q_1, \qquad (5.1)$$

for some constant $a_1 > 0$. If we can find such a path for an edge $e' = uv \in M \setminus M'$, then we update M' to contain also e', and M'' to contain also this path, and we repeat this with a new edge of $M \setminus M'$. Let B be the event that we succeed only at most m_1 times in G_1 (i.e., that starting with $M' = \emptyset$, we end with $|M'| \leq m_1$). Assuming $m_1 \leq f(n)^{1/4}$ and recalling that $|M| = \lfloor \sqrt{f(n)} \rfloor$, we get

$$\mathbb{P}[B] \leq \sum_{i=0}^{m_1} \binom{|M|}{i} (1-q_1)^{|M|-i} = o(1).$$

Hence, w.h.p. we have $m_1 \geq f(n)^{1/4}$. That is, w.h.p. there are at least $f(n)^{1/4}$ edges of M where, in G_1, each is the middle edge of a P_3-copy with endpoints which have at least $\left(\frac{1}{2} - 3\sqrt{\varepsilon}\right) n$ neighbours into A in G, and all of the endpoints are distinct. Denote by M_1 a subset of $\lfloor (f(n))^{1/4} \rfloor$ many such P_3-copies in G_1 (note that $|G_1|$ is smaller than εn). Let U be the set of remaining vertices, i.e., $U = V(G) \setminus V(M_1)$, and hence $|U| \geq (1-4\varepsilon)n$. Further denote $U_1 = U \cap A$ and $U_2 = U \cap A^c$. We continue to the second exposure. We look at the graph $G_2 = G[A, A^c](p_1)$ and focus on $G_2[U_1, U_2]$. Since $G[A, A^c]$, as a bipartite graph, is missing at most $8\varepsilon n^2$ edges, we get that w.h.p. the pair (A, A^c) has the $(5\sqrt{\varepsilon})$-property in G_2, and in particular the pair (U_1, U_2) has, w.h.p., the ε''-property in G_2, for some $\varepsilon'' > 0$ satisfying $\varepsilon'' \left(\left(\frac{1}{2} - \sqrt{8\varepsilon}\right) n - 2\lfloor f(n)^{1/4} \rfloor\right) \leq 5\sqrt{\varepsilon}\left(\frac{1}{2} - \sqrt{8\varepsilon}\right) n$ (more precisely, $\varepsilon'' \cdot \min(|U|_1, |U_2|) = 5\sqrt{\varepsilon} \cdot \min(|A|, |A^c|)$). Hence, using either Proposition 4.4 or Proposition 4.3, we embed a copy of $T_\ell^{(r,h)}$ in $G_2[U_1, U_2]$, where $\ell = t - 5 - 2h$, and the values of r and h are determined by the value of t in the following way. Take C_1 to be the absolute constant from Lemma 3.1. If $t \in [\frac{C_1}{\log(1/\beta)} \log n, 5\sqrt{\varepsilon}(1 - 2\sqrt{8\varepsilon})n]$, then we set $r = \lfloor \frac{1}{16 \cdot 5\sqrt{\varepsilon}} \rfloor - 2$, $h = \lceil \frac{\log(5\sqrt{\varepsilon}(1/2 - \sqrt{8\varepsilon})n)}{\log r} \rceil$, and use Proposition 4.3 to embed a copy of $T_\ell^{(r,h)}$. If $t \in [5\sqrt{\varepsilon}(1 - 2\sqrt{8\varepsilon})n, \frac{1}{2}n]$ then we use Proposition 4.4 to embed a copy of $T_\ell^{(2,h)}$ where $h = \lceil \frac{\log(5\sqrt{\varepsilon}(1/2 - \sqrt{8\varepsilon})n)}{\log 2} \rceil$. (Note that we embed a tree that helps us create a cycle of an odd length up to $\frac{1}{2}n$, even though we only need it to be of length up to $\left(\frac{1}{2} - \beta\right) n$. We do this to ensure that eventually we get a cycle of length $\left(\frac{1}{2} - \beta\right) n$ even when looking at G', which have $(1 - \varepsilon')n$ vertices, instead of G, as mentioned at the beginning of Case 2.) In either case we embed a $T_\ell^{(r,h)}$-copy with both leaf-sets, denoted by L_1, L_2, in U_1. Recall that by the definition of the tree $T_\ell^{(r,h)}$ we further know that $|L_1|, |L_2| \geq 5\sqrt{\varepsilon}\left(\frac{1}{2} - \sqrt{8\varepsilon}\right) n$. We are left with the third and last exposure. Let $G_3 = G[A, A^c](p_1)$. Let $P = (x_0, x_1, x_2, x_3)$ be some P_3-copy in M_1. Recall that both endpoints of P, i.e., x_0, x_3, miss at most $4\sqrt{\varepsilon} n$ vertices in $G[A]$, so $d_G(x_i, L_j) \geq 5\sqrt{\varepsilon}\left(\frac{1}{2} - \sqrt{8\varepsilon}\right) n - 4\sqrt{\varepsilon} n \geq \sqrt{\varepsilon} n$, for $i = 0, 3$ and $j = 1, 2$. Similarly to (5.1) and the argument preceeding it, we get that the probability that in G_3 both x_0 has a neighbour in L_1 and x_3 has a neighbour in L_2 is at least $(1 - e^{-a_2})^2$, for some constant $a_2 > 0$. Note further that, as the endpoints of the P_3-copies in M_1 are all distinct, these experiments are all independent. Hence, in total, we get that w.h.p. there exists some P_3-copy in M_1 for which in G_3, both its endpoints have neighbours in L_1 and L_2, one in each. Note that $G(p) \supseteq G[A](p) \cup G_1 \cup G_2 \cup G_3$, so in particular, we get that w.h.p. $G(p)$ contains a cycle of length t.

Subcase 2.2: Assume now that R is η-far from being bipartite, so in particular non-bipartite. We will show that in this case R contains an odd cycle of length at least $(\frac{1}{2} - 130\varepsilon)k$, and an odd cycle of length at most 61. Then, we will deduce the likely existence of the desired cycle in G.

The first step will be to get a subgraph of R which has a large minimum degree. We iteratively remove vertices from R with degree less than $\frac{k}{10}$. Similarly to the argument at the beginning of Case 2, if the process has not stopped after at most $16\varepsilon k$ steps, then we get a graph R'' on $k' = (1-16\varepsilon)k$ vertices and at least $e(R) - \frac{8\varepsilon}{5}k^2 \geq \frac{1}{4}(k')^2 + 1$ edges, so by Theorem 2.2 it contains all cycles of lengths in $[3, \frac{1}{2}(k'+3)]$, and in particular all cycles of odd lengths in $[3, (\frac{1}{2} - 8\varepsilon)k]$, so we proceed as in Case 1. So we may assume that this process ends after at most $16\varepsilon k$ steps, with a graph denoted R', on $k' \geq (1-16\varepsilon)k$ vertices, with $e(R') \geq (\frac{1}{4} - 8\varepsilon)(k')^2$ and $\delta(R') \geq \frac{k'}{10}$. As $\eta > \frac{8\varepsilon}{5}$, R' is still non-bipartite. Hence we may assume that R is a non-bipartite graph on k vertices with minimum degree at least $\frac{k}{10}$, or otherwise we consider R' instead.

We next use the following lemma, which we prove later.

Lemma 5.4. *Let $0 < \delta' < \frac{1}{5}$ and let R be a non-bipartite graph on k vertices for $k \geq \frac{2}{(\delta')^2}$, satisfying $e(R) \geq \left(\frac{1}{4} - \delta'\right)k^2$ and $\delta(R) \geq \frac{k}{10}$. Then R contains an odd cycle of length at least $\left(\frac{1}{2} - 15\delta'\right)k$ and an odd cycle of length at most 61.*

We will show now how to use Lemma 5.4 to complete the proof of Theorem 1.6. By Lemma 5.4 (for $\delta' = 8\varepsilon$), we have in R', and thus in R, an odd cycle of length $b_0 \in [3, 61]$, and an odd cycle of length $b_1 \geq \left(\frac{1}{2} - 120\varepsilon\right)k' \geq \left(\frac{1}{2} - 130\varepsilon\right)k$. Similarly to Case 1, let $C := C(\varepsilon)$ be as given in Lemma 5.2, look at the graph $G(p)$ for $p \geq \frac{C}{n}$, and consider the ε-graph $S := S(G(p), \Pi, \varepsilon)$. Recall that, by Corollary 5.2, w.h.p. $R \subseteq S$. Lemma 3.1 Item 2 and the cycle of length b_0 in R give, w.h.p., cycles of all odd lengths in $[\frac{C_1}{\log(1/\beta)}\log n, 3(1-48\varepsilon)\frac{n}{k}]$ in $G(p)$, where C_1 is the absolute constant from Lemma 3.1. By Lemma 3.1 Item 2, the cycle of length b_1 in R gives, w.h.p., cycles of all odd lengths in $[3(1-48\varepsilon)\frac{n}{k}, \left(\frac{1}{2} - 200\varepsilon\right)n]$ in $G(p)$. Recall that the graph we are looking at might be, instead of G, the graph G' which has $(1-\varepsilon')n$ vertices. As we have $\varepsilon \leq \frac{\beta}{1000}$ and $\varepsilon' = \frac{5}{9}\beta$, we get $\left(\frac{1}{2} - 200\varepsilon\right)(1-\varepsilon')n \geq \left(\frac{1}{2} - \beta\right)n$. Hence, altogether, we get that w.h.p. $G(p)$ contains a cycle of length t, for any odd $t \in [\frac{C_1}{\log(1/\beta)}\log n, \left(\frac{1}{2} - \beta\right)n]$. □

It is left to prove Lemma 5.4. For this proof we use two results. The first is by Erdős and Gallai [**13**], estimating the maximal number of edges in a graph containing no cycle of at least a certain length.

Theorem 5.5 ([**13**], Theorem 2.7). *Let G be an n-vertex graph with more than $\left\lfloor \frac{1}{2}(n-1)(t-1) \right\rfloor$ edges. Then G contains a cycle of length at least t.*

The second result we need is by Moon [**47**] (see also [**14**]), regarding the diameter of a connected graph. For a graph G, let $\mathrm{diam}(G) = \max_{\{u,v\}} d(u,v)$, where $d(u,v)$ is the length of a shortest path between u and v in the graph (and is equal to ∞ if the graph is not connected and u, v are in different connected components).

Theorem 5.6 ([**14**], Theorem 1; [**47**]). *Let R be a connected graph on k vertices with minimum degree $\delta(R) \geq 2$. Then*

$$\operatorname{diam}(R) \leq \left\lceil \frac{3k}{\delta(R)+1} \right\rceil - 1$$

We further note the following.

Claim 5.7. *Let $\kappa(R)$ be the connectivity of the graph R (the minimum number of vertices one must remove from R in order to make it disconnected), and assume that $\kappa(R) = \kappa \leq \delta' k$ and $e(R) \geq \left(\frac{1}{4} - \delta'\right) k^2$. Then R contains cycles of all lengths between 3 and $\left(\frac{1}{2} - 15\delta'\right) k$.*

Proof. Let $\delta' > 0$. Let $Q \subseteq V(R)$ be such that $|Q| = \kappa$, $A \cup Q \cup B = V(R)$, and $e_R(A, B) = 0$. Denote $v(A) = a$ for some a and assume w.l.o.g. that $a \geq \frac{1}{2}(k - \kappa)$. Note that $|B| + \kappa \geq \frac{k}{10}$ (due to the minimum degree condition), and thus $a \leq \frac{9}{10}k$. Then, as we have $e(R) \geq \left(\frac{1}{4} - \delta'\right) k^2$, $e(B) \leq \binom{k-a-\kappa}{2}$, and $e(Q) + e(Q, R \setminus Q) \leq \binom{\kappa}{2} + \kappa(k - \kappa) \leq k\kappa$, we get, for $t = \left(\frac{1}{2} - 15\delta'\right) k$,

$$e(A) \geq e(R) - e(B) - e(Q) - e(Q, R \setminus Q) \geq \binom{t-1}{2} + \binom{a-t+2}{2} + 1.$$

Hence, by Theorem 2.2, $G[A]$, and thus G, contain all odd cycles of lengths between 3 and $\left(\frac{1}{2} - 15\delta'\right) k$. □

Proof of Lemma 5.4. Let $\delta' > 0$. By Claim 5.7 we may assume that $\kappa(R) \geq \delta' k$ (as otherwise we are done). We find an odd cycle of length at most 61, and an odd cycle of length at least $\left(\frac{1}{2} - 3\delta'\right) k$ in R in three steps.

Step I: Find a short odd cycle in R. Let $C_0 = (v_1, \ldots, v_{2t+1}, v_1)$ be a shortest odd cycle in R, for some integer t. By Theorem 5.6 there exists some path $\tilde{P}$ of length at most $\frac{3k}{\delta(R)} \leq 30$ from v_1 to v_{t+1}. Then the union of $\tilde{P}$ with the part of C_0 from v_1 to v_{t+1} of the right parity creates a cycle whose length is at most $t + 31$, so by C_0 being of minumum length we get $2t + 2 \leq 61$.

Step II: Find an odd cycle in R of length in $[\frac{1}{2}\delta' k, \frac{1}{2}\delta' k + 120]$. Let (u, v) be an arbitrary edge of C_0, and denote $R' := R \setminus (V(C_0) \setminus \{u, v\})$. We start by finding a path of length at least $\frac{1}{2}\delta' k$ in R' with endpoints u, v. Note that we have $\kappa(R') \geq \kappa(R) - |V(C_0)| \geq \kappa(R) - 61 > \frac{1}{2}\delta' k$ and $\delta(R') \geq \delta(R) - |V(C_0)| \geq \frac{k}{10} - 61$. Let P be a path of length $\frac{1}{2}\delta' k$ in R' starting at v and avoiding u, and let $w \neq v$ be the other endpoint of this path. This path exists as $\delta(R' - \{u\}) \geq \frac{k}{10} - 62 > \frac{1}{2}\delta' k$. Now look at the graph $R'' := R' - V(P) + \{w\}$, and note that we have $\kappa(R'') \geq \kappa(R') - |V(P)| > 0$ and $\delta(R'') \geq \delta(R') - |V(P)| \geq \frac{k}{10} - 61 - \frac{1}{2}\delta' k > \frac{k}{20}$. By Theorem 5.6 we get that $diam(R'') \leq 60$ and in particular there exists a path P' of length at most 60 between w and u. Then $P \cup P'$ is a path from u to v in R' of length at least $\frac{1}{2}\delta' k$ and at most $\frac{1}{2}\delta' k + 60$. Recall that C_0 is of an odd length, so by concatenating the path $P \cup P'$ with a path between u and v on C_0 of the right parity, we get an odd cycle of length $\ell_1 \in [\frac{1}{2}\delta' k, \frac{1}{2}\delta' k + 120]$ in R. Denote this cycle by C_1.

Step III: Find an odd cycle of length at least $(\frac{1}{2} - 3\delta')k$. Let $R^* := R - V(C_1)$, and note that $e(R^*) \geq e(R) - \binom{\ell_1}{2} - \ell_1(k - \ell_1) \geq \left(\frac{1}{4} - \frac{3}{2}\delta'\right) k^2$. By Theorem 5.5 we get that R^* contains a cycle of length at least $\left(\frac{1}{2} - 2\delta'\right) k$. Denote this cycle by C_2. If C_2 is of an odd length then we are done. Assume that C_2 is of an even length, and recall that $\kappa(R) \geq \delta' k$. Using Menger's Theorem, we find ℓ_1 pairwise vertex-disjoint paths from C_1 to C_2. There exist two such paths for which their endpoints in C_2 are of distance at most $\frac{|V(C_2)|}{|V(C_1)|} \leq \frac{2}{\delta'}$ along C_2. Denote these two paths by P_1 and P_2, their endpoints in C_2 by u_1, u_2, respectively, and their endpoints in C_1 by v_1, v_2, respectively. Further denote by P_3 the long path between u_1 and u_2 on C_2, which has length at least $(\frac{1}{2} - 2\delta')k - \frac{2}{\delta'} \geq (\frac{1}{2} - 3\delta')k$. Recall that C_1 is an odd cycle of length ℓ_1, so by concatenating $P_1 \cup P_3 \cup P_2$ with a path between v_1 and v_2 on C_1 of the right parity, we get an odd cycle of length at least $\left(\frac{1}{2} - 3\delta'\right)k$. □

6. *Further results*

6.1. *Pseudo-random graphs*

As mentioned in the introduction, Theorem 1.2 is also applicable to pseudo-random graphs. For proving Corollary 1.4, we use the Expander Mixing Lemma due to Alon and Chung [**2**] cited below.

THEOREM 6.1 (Expander Mixing Lemma [**2**]). *Let G be a d-regular graph on n vertices where $\lambda \leq d$ is the second largest eigenvalue of its adjacency matrix, in absolute value. Then for any two disjoint subsets of vertices $A, B \subseteq V(G)$ we have*

$$\left|e(A,B) - \tfrac{d}{n}|A||B|\right| \leq \lambda\sqrt{|A||B|}.$$

We now verify that for suitable values of d and λ, an (n, d, λ)-graph is also upper-uniform, proving Corollary 1.4.

Proof of Corollary 1.4. Take η, n_0, γ as given by Theorem 1.2. Let G be an (n, d, λ)-graph with $n \geq n_0$ and $\lambda \leq \delta d$. By the Expander Mixing Lemma (Theorem 6.1), any two subsets $A, B \subseteq V(G)$ satisfy

$$e(A,B) \leq \tfrac{d}{n}|A||B| + \lambda\sqrt{|A||B|}.$$

Setting $\eta = \sqrt{\delta}$ and recalling that $\lambda \leq \delta d$ we get

$$e(A,B) \leq \tfrac{d}{n}|A||B|(1+\eta)$$

for any two subsets $A, B \subseteq V(G)$ satisfying $|A|, |B| \geq \eta n$. It follows that G is $(\frac{d}{n}, \eta)$-upper-uniform. Note, in addition, that an (n, d, λ)-graph G also satisfies $e(G) = \frac{1}{2}dn \geq (1 - \beta/2)\frac{d}{n}\binom{n}{2}$. Hence, the statement follows by Theorem 1.2. □

6.2. *Ramsey-type properties of sparse random graphs*

The purpose of this section is to present some immediate applications of the Key Lemma for other extremal problems in random graphs. Given a graph G with some partition of its vertices Π, Lemma 3.1 allows us to convert a cycle in the corresponding ε-graph (or the reduced graph) to a cycle in G of a prescribed length. Here, we will show how to find a monochromatic cycle in a multicoloured ε-graph, and using the Key Lemma (Lemma 3.1) to convert it to a monochromatic cycle of a prescribed length in G. The method is very similar to the one we used to prove Theorem 1.2. For any colouring of G, we use the colourful version of the Sparse Regularity Lemma to obtain a partition Π which is ε-regular in *every* colour. We then look at corresponding coloured ε-graph, where each edge is coloured by taking the majourity colour in it, and show that it has almost all edges present. Then, we need to use a deterministic Ramsey-type argument to guarantee a long monochromatic cycle in an almost complete coloured graph (the ε-graph). In some cases, the Ramsey-type argument already exists and we just use it as a "black box" for our purposes, and in other cases we prove a relevant Ramsey-type argument. This type of arguments is not always trivial, in fact, in some cases it is not even known for r-coloured complete graphs (for example, r-Ramsey numbers for even cycles). Having this monochromatic cycle in the ε-graph allows us to complete the argument using the Key Lemma.

For graphs G and H, we write $G \to_r H$ if for every r-colouring of the edges of G, there is a monochromatic copy of H (G is r-Ramsey for H). The *r-Ramsey number of H*, denoted by $R_r(H)$, is the minimum n such that $K_n \to_r H$. Instead of asking for a monochromatic copy of a single graph H, we can consider the Ramsey property with respect to some family of graphs $\mathcal{H}$ (that is, Ramsey-universality). For a family of graphs $\mathcal{H}$, we write $G \to_r \mathcal{H}$ if for every r-colouring of the edges of G, there exists a colour i, such that there is a monochromatic copy of every $H \in \mathcal{H}$ in this colour (G is *r-Ramsey-universal* for $\mathcal{H}$).

Below we list several typical properties of random graphs, where G is the random graph $G(n,p)$ for $p = \Omega\left(\frac{1}{n}\right)$, and $\mathcal{H}$ is a family of long cycles. In some cases we give short proofs (using the Key Lemma) of previously known results (see [**27**, **31**]). As we will show, the maximum length of a monochromatic cycle in $G(n,p)$ is w.h.p. asymptotically equal to the one in a coloured (almost) complete graph. This gives another example of the *transference principle* discussed in the introduction. We would like to add that, as in the case of Turán numbers, all of these results can also be proved for upper-uniform graphs and for (n,d,λ)-graphs, with appropriate parameters. It is also worth mentioning that the theorems below give an immediate linear upper bound for the *size-Ramsey number* of cycles. However, in this context, better upper bounds are already known (see [**27**, **31**]).

Long monochromatic even cycles in r-colourings of random graphs For an integer $r \geq 2$ we let

$$\lambda^* := \lambda^*(r) = \sup_{\varepsilon>0} \left\{ \lambda \;\middle|\; \begin{array}{c} \exists k_0 \text{ s.t. } \forall k \geq k_0 \text{ every graph } G \text{ on } k \text{ vertices} \\ \text{with at least } (1-\varepsilon)\binom{k}{2} \text{ edges has } G \to_r P_{\lambda k} \end{array} \right\}.$$

We then obtain the following theorem.

THEOREM 6.2. *Let $r \geq 2$ be an integer. For every $\beta > 0$ there exists $C > 0$ such that for $p \geq \frac{C}{n}$, w.h.p. $G(n,p) \to_r \mathcal{C}$, for $\mathcal{C} = \{C_t \mid A_r \log n \leq t \leq (\lambda^* - \beta)n,\ t \text{ is even}\}$, where $A_r > 0$ is a constant that depends only on r. In particular, w.h.p. $G(n,p) \to_r C_{\lfloor(\lambda^*-\beta)n\rfloor_{even}}$.*

Note that by Theorem 2.1 we have $\lambda^* \geq \frac{1}{r}$. We believe that the value of λ^* is equal to the value of $R_r^{-1}(k)/k$ for $r \geq 2$, where by $R_r^{-1}(k)$ we denote the inverse of the Ramsey function with respect to paths. This would make Theorem 6.2 asymptotically optimal. The values of $R_2^{-1}(k)/k$ and $R_3^{-1}(k)/k$ were shown to be $2/3$ and $1/2$ in [**24**] and in [**26**], respectively. In [**41**] it was proved that $\lambda^*(2) = 2/3$, giving an equality between these two parameters. Later in this section we include a proof sketch showing that for $r = 3$ Theorem 6.2 holds with $\mathcal{C} = \{C_t \mid A_r \log n \leq t \leq (1/2 - \beta)n,\ t \text{ is even}\}$. This gives an asymptotically optimal result for $r = 2$ and $r = 3$ in Theorem 6.2. The value of $R_r^{-1}(k)/k$ is still unknown and is thought to be $\frac{1}{r-1}$ for $r \geq 3$ (see, e.g. [**50**], see also [**33**] for a recent improvement). Nevertheless, any lower bound, λ', on λ^*, gives us w.h.p. $G(n,p) \to_r C_{\lfloor(\lambda'-\beta)n\rfloor_{even}}$.

Long monochromatic odd cycles in r-colourings of random graphs For longer odd cycles, given an integer $r \geq 2$, we let

$$\lambda_o^* := \lambda_o^*(r) = \sup_{\varepsilon>0} \left\{ \lambda \;\middle|\; \begin{array}{c} \exists k_0 \text{ s.t. } \forall k \geq k_0 \text{ every graph } G \text{ on } k \text{ vertices} \\ \text{and at least } (1-\varepsilon)\binom{k}{2} \text{ edges has } G \to_r C_{\lfloor\lambda k\rfloor_{odd}} \end{array} \right\}.$$

Then by using Lemma 3.1 Item 2, we get the following.

THEOREM 6.3. *Let $r \geq 2$ be an integer. For every $\beta > 0$ there exists $C > 0$ such that for $p \geq \frac{C}{n}$, w.h.p. $G(n,p) \to_r \mathcal{C}$, for $\mathcal{C} = \{C_t \mid A_r \log n \leq t \leq (\lambda_o^* - \beta)n,\ t \text{ is odd}\}$, where $A_r > 0$ is a constant that depend only on r. In particular, w.h.p. $G(n,p) \to_r C_{\lfloor (\lambda_o^* - \beta)n \rfloor_{odd}}$.*

The value of $R_r(C_n)$ for an odd n was determined exactly by Jenssen and Skokan in [**32**], and is equal to $2^{r-1}(n-1)+1$. Thus it is plausible to believe that λ_o^* should be asymptotically equal to $\frac{1}{2^{r-1}}$. This is known to hold for the cases $r = 2$ and $r = 3$ due to Łuczak [**44**] (Lemma 8 and Lemma 9 in [**44**]), giving an optimal result in Theorem 6.3 for these two cases. Here we present a Ramsey-type argument (see Lemma 6.7) to show that $\lambda_o^*(r) \geq \frac{1}{r2^{r+4}}$, which differs from the upper bound only by a factor of $2^5 r$. This gives the following.

COROLLARY 6.4. *Let $r \geq 3$ be an integer. For every $\beta > 0$ there exists $C > 0$ such that for $p \geq \frac{C}{n}$, w.h.p. $G(n,p) \to_r \mathcal{C}$, for $\mathcal{C} = \{C_t \mid A_r \log n \leq t \leq (\frac{1}{r2^{r+4}} - \beta)n,\ t \text{ is odd}\}$, where $A_r > 0$ is a constant that depend only on r. In particular, w.h.p. $G(n,p) \to_r C_{\lfloor (\frac{1}{r2^{r+4}} - \beta)n \rfloor_{odd}}$.*

Proof ideas For proving the above theorems, we will use a slightly different variant of Theorem 2.4, the Colourful version of the Sparse Regularity Lemma (see, e.g., [**27**]).

THEOREM 6.5 (Colourful Sparse Regularity Lemma). *For any given $\varepsilon > 0$, and integers $r \geq 1$ and $k_0 \geq 1$, there are constants $\eta = \eta(\varepsilon, k_0) > 0$ and $K_0 = K_0(\varepsilon, k_0) \geq k_0$ such that if $G_1, \ldots, G_r$ are (p,η)-upper-uniform graphs on the vertex set V of size n for large enough n, with $0 < p \leq 1$, there is an (ε, p)-regular partition of V into k parts, $\Pi = (V_1, \ldots, V_k)$, where $k_0 \leq k \leq K_0$, such that at least $(1-\varepsilon)\binom{k}{2}$ pairs (V_i, V_j) with $1 \leq i < j \leq k$ are (ε, G_ℓ, p)-regular, for each $\ell \in [r]$.*

The proofs of Theorem 6.2 and Theorem 6.3 follow the same pattern, which is an easy combination of Theorem 6.5 and Lemma 3.1. We sketch the details of their proofs below.

Proof sketch of Theorem 6.2 and Theorem 6.3. We present here a proof sketch for Theorem 6.2, where similar arguments apply for Theorem 6.3, considering λ_o^* instead of λ^* and an odd cycle instead of a path, mutatis mutandis. Recall that w.h.p. $G = G(n,p)$ is a (p,η)-upper-uniform graph for any $\eta > 0$, and that w.h.p. $e(G) \geq (1-o(1))p\binom{n}{2}$. Let $c : E(G) \to [r]$ be an arbitrary colouring of its edges. Let $\delta := \delta(\beta)$ be such that every r-colouring of every graph on k vertices with at least $(1-\delta)\binom{k}{2}$ edges contains a monochromatic path of length at least $(\lambda^* - \beta/2)k$; such δ exists due to the definition of λ^*.

Define G_i to be the graph of the ith colour, that is, $V(G_i) = V(G)$, $E(G_i) = \{e \in E(G) \mid c(e) = i\}$. Then $G_1, \ldots, G_r$ are (p,η)-upper-uniform graphs. For an appropriate choice of parameters $\varepsilon := \varepsilon(\delta), k_0, \eta, \rho$, where k_0 is chosen also considering the definition of λ^*, by Theorem 6.5 we get an (ε, p)-regular partition of $V(G)$ into k parts, $\Pi = (V_1, \ldots, V_k)$, where $k_0 \leq k \leq K_0$, such that at least $(1-\varepsilon)\binom{k}{2}$ pairs (V_i, V_j) with $1 \leq i < j \leq k$ are (ε, G_ν, p)-regular, for each $\nu \in [r]$.

We now look at the following graph obtained from G, which we denote by $\tilde{R}$. The vertices are $[k]$, and for any two distinct $i, j \in [k]$, we have $ij \in E(\tilde{R})$ if and only if the pair (V_i, V_j) is ε-regular in G_ν for every $\nu \in [r]$, and is of p-density at least $\rho := 2r\varepsilon$ in G. By the choice of ε and by a result similar to Lemma 2.5, we have that $e(\tilde{R}) \geq (1-\delta)\binom{k}{2}$. We colour the edges of $\tilde{R}$ with r colours as follows: an edge ij is coloured with colour $\nu \in [r]$ if $d_{G_\nu,p}(V_i, V_j) \geq \rho/r$ (if there is more than one such colour, we choose one arbitrarily).

By the definition of λ^* and the choice of δ, we find a monochromatic path P of length $(\lambda^* - \beta/2)k$. Assume that this monochromatic path is coloured with the colour ν and look at the graph G_ν. Recall that Π is an (ε, p)-regular partition with respect to G_ν, then for $R_\nu := R(G_\nu, \varepsilon, \rho/r, p)$, we get that P is also contained in R_ν, and thus in $S_\nu = S(G_\nu, \varepsilon, \Pi)$. By the Key Lemma (Lemma 3.1), we get monochromatic even cycles of all lengths from $A_r \log n$ to $(\lambda^* - \beta)n$, all in the same colour. $\square$

For proving the optimality of Theorem 6.2 in the case where $r = 3$ we need to adjust Lemma 3.1, Item 1. Assume that G and S are as in Lemma 3.1, and assume that S contains a tree T with a matching M of size $b/2$ where b is even and T is minimal with respect to containment. Denote the edges of M by $\{v_{2i-1}, v_{2i}\}$ for $i \in [b/2]$. Firstly, note that for $t \in \left[\frac{C_1}{\log(1/\varepsilon)} \log n, 2\varepsilon m\right]$, the proof Lemma 3.1 can be applied as is by considering just one edge of the matching and finding there a cycle of length t.

For $t \in \left[2\varepsilon m, (1 - 48\varepsilon)\frac{b}{k} n\right]$, we combine the argument on connected matchings (see [**44**], and also [**12**, **17**]) with a slight adaptation of the proof of the Key Lemma. Let W be a minimal closed walk on T in which every edge is visited exactly twice, starting at an arbitrarily chosen root v of the tree (such a walk is obtained, e.g., by a standard DFS-based argument, see, e.g., [**49**]). In particular, every edge of M is visited exactly twice in W. Let $V_1, \dots, V_b$ be the clusters that correspond to the vertices $v_1, \dots, v_b$ of the matching M, respectively. Note that by its minimality, any leaf of T is a vertex of an edge in M. Assume w.l.o.g. that for some even integer $0 < b' \le b$ we have that for any $i \in [b'/2]$ the edge $\{v_{2i-1}, v_{2i}\}$ contains a leaf of T, and that for any $i \in [b/2] \setminus [b'/2]$ the edge $\{v_{2i-1}, v_{2i}\}$ contains no leaf of T.

For each $i \in [b'/2] \setminus [b/2]$ we split each of V_{2i-1} and V_{2i} into two subclusters of equal sizes (up to possibly one vertex), U_{2i-1}, U'_{2i-1} and U_{2i}, U'_{2i}, respectively. For convenience, (U_{2i-1}, U_{2i}) corresponds to the first time that the walk W is going through the edge $\{v_{2i-1}, v_{2i}\}$, and (U'_{2i-1}, U'_{2i}) corresponds to the second time. We embed a copy of $T_\ell^{(r,h)}$ in (U_{2i-1}, U_{2i}) and in (U'_{2i-1}, U'_{2i}), for an appropriate choice of parameters r, h, ℓ which we specify later, such that the leaf sets are embedded each in a different subcluster. Denote the corresponding leaf sets by $L_{2i-1}, L_{2i}, L'_{2i-1}, L'_{2i}$ and recall that they are all of size εm. For edges $\{v_{2i-1}, v_{2i}\}$ in M that contain a leaf (say, v_{2i}, for some $i \in [b'/2]$, is a leaf of T), we embed a copy of $T_{\ell'}^{(r',h')}$ in (V_{2i-1}, V_{2i}), for an appropriate (and different) choice of parameters r', h', ℓ' which we specify later, such that both leaf sets are embedded in V_{2i-1} (similarly to the proof of the Key Lemma). Denote these leaf sets by L_{2i-1}, L'_{2i-1}. Note that after embedding copies of $T_\ell^{(r,h)}$ and of $T_{\ell'}^{(r',h')}$ in $V_1, \dots, V_b$, each cluster still contains at least $2\varepsilon m$ vertices not touched by the tree embeddings. We are now left with connecting the leaf sets of the embedded trees, to create a cycle of prescribed length.

As W is a closed walk, we may assume it starts with a vertex of M which is a leaf of T. Hence, we can write W as follows, $(e_1, P_1, e_2, P_2, \dots, e_{b-1}, P_{b-1}, e_b)$, where e_i is an edge of M, oriented according to W, for every $i \in [b]$, and P_i is an oriented path which is a piece of the walk W separating e_i and e_{i+1} and containing no edge of M, for every $i \in [b-1]$. After having embedded trees in the clusters corresponding to edges of M we now replace each path P_i in the walk by a path of the same length and going through the same sequence of clusters. We embed these paths one by one, as follows. Assume for that for some $i \in [b-1]$ we have $P_i = (w_0, \dots, w_t)$ for some $t \ge 1$ (where w_0 is the end vertex of e_i and w_t is the start vertex of e_{i+1}), and let $W_0, \dots, W_t$ be the clusters corresponding to the vertices of P_i. We start with a leaf set in W_0 of the tree corresponding to the current traversal of e_i, and we carefully choose the next vertex of the path in each cluster of the path P_i, using the ε-property we have for any two adjacent clusterts in T, so that we avoid all vertices that have already been used for embedding tress or previously embedded paths. Note that we can always embed such paths as in each cluster W_j, $j \in \{0, \dots, t\}$, we had at least $2\varepsilon m$ avaliable vertices to begin with, and in previous steps of the path ebmeddings we used at most $|T| \le k$ of them. Doing this for every piece of the walk W, we obtain a cycle of length $2|T| - b + \frac{b'}{2}(2h' + \ell') + \frac{b-b'}{2}(2h+\ell) \in \left[2\varepsilon m, (1-48\varepsilon)\frac{b}{k}n\right]$. By choosing r, h, ℓ and r', h', ℓ' appropriately, we get a cycle of any desired length in this interval.

Thus, the proof of Theorem 6.2 with $r = 3$ and $\mathcal{C} = \{C_t \mid A_r \log n \le t \le (1/2 - \beta)n,\ t \text{ is even}\}$ follows from the statement below.

Lemma 6.6 ([**17**], Lemma 8). *Let $0 < \varepsilon < 0.00025$ and let k be large enough. Let G be a graph on k vertices with at least $(1-\varepsilon^7)\binom{k}{2}$ edges. Then, for every 3-colouring of the edges of G, there is a monochromatic component containing a matching saturating at least $(1/2 - 4.5\varepsilon)k$ vertices.*

The proof of Corollary 6.4 follows immediately from Theorem 6.3 and the next argument about the existence of a long monochromatic odd cycle in an almost compete r-coloured graph.

Lemma 6.7. *Let $r \ge 3$ be an integer and let $0 < \beta < 1/2^{r+1}$. Let k be large enough as a function of r, and let G be a graph on k vertices with at least $(1-\beta)\binom{k}{2}$ edges. Let $c : E(G) \to [r]$ be an r-colouring of its edges. Then there exists a monochromatic odd cycle of length at least $\frac{k}{r2^{r+4}}$.*

Proof. Let $\varepsilon = \frac{1}{r2^{r+2}}$. For $i \in [r]$, let $G_i \subseteq G$ be the graph obtained by the edges coloured by colour i according to the colouring c. We first show that there is a colour $i \in [r]$ for which G_i is ε-far from being bipartite. Assume that G_i is ε-close to being bipartite, for every $i \in [r]$. Then, as $\chi(H) \le \Pi_{i\in[r]}\chi(H_i)$ for any graph H and any partition of its edges, we obtain that G contains a subgraph G', on the same set of vertices, with $e(G') \ge (1-\beta)\binom{k}{2} - \varepsilon r k^2 \ge (1 - \beta - 2\varepsilon r)\binom{k}{2}$ such that G' is 2^r-colourable. Note that since $e(G') \ge (1-\beta-2\varepsilon r)\binom{k}{2}$, then by Turán's theorem there is a clique in G' of size $\frac{1}{\beta + 2\varepsilon r}$ and thus $\chi(G') > \frac{1}{\beta+2\varepsilon r}$. Since $\beta < \frac{1}{2^r} - 2\varepsilon r$ we get a contradiction.

Let i_0 be such that G_{i_0} is ε-far from being bipartite.

We next find a subgraph in G_{i_0} of a linear size which is 2-connected and is $\frac{\varepsilon}{3}$-far from being bipartite. For this, we define an auxiliary graph H. For each vertex $v \in V(G_{i_0})$ we look at the maximal 2-connected subgraph (a *block*, sometimes also called a *biconnected component*) that contains v (might also be a single vertex or a bridge). Let $B_1, \dots, B_h$ be those blocks (note that $|B_i \cap B_j| \le 1$) and write $|B_i| = b_i$. Let $U = \{u_1, \dots, u_t\}$ be the intersection vertices, that is, for every $s \in [t]$, there are $i, j \in [h]$ such that $B_i \cap B_j = u_s$. Let H be the following bipartite graph (also called the *block-cut tree* or the *block decomposition graph*, see, e.g., [**58**] p. 155). $V(H) = [h]\dot\cup[t]$, and $E(G) = \{ij \mid u_j \in B_i\}$. By definition, H is a forest and $\sum b_i \ge k$. Moreover, we have $\sum b_i \le 2k$. Indeed, we look at a leaf v_1 in H. By the definition it corresponds to a block B_{j_1}. Remove this block from the graph G_{i_0}. Then we obtain a graph with $h-1$ blocks and $k - b_{j_1}$ vertices. By induction, $\sum_{i\in[h]\setminus\{j_1\}} b_i \le 2(k - b_{j_1})$. Since v_1 was a leaf, we obtain that $\sum_{i\in[h]} b_i \le 2(k-b_{j_1}) + 2b_{j_1} = 2k$. Assume that each block B_j is such that at most $\frac{\varepsilon}{3}kb_j$ edges should be removed from it in order to make it bipartite (that is, is at most $\frac{\varepsilon k}{3b_j}$-far from being bipartite). Then, as H is bipartite, at most $\frac{\varepsilon}{3}k\sum b_j \le \frac{2}{3}\varepsilon k^2$ edges can be removed from G_{i_0} to make it bipartite. This is a contradiction. Then there is $B_j \subseteq G_{i_0}$ and is $\frac{\varepsilon k}{3b_j}$-far from being bipartite. In particular this means that $\binom{b_j}{2} > \frac{2}{3}\varepsilon k b_j$, so we obtain B_j which is 2-connected, with $b_j \ge \frac{4}{3}\varepsilon k$.

We now look at $B := B_j$, a subgraph of G_{i_0} which has $k' := b_j \ge \frac{4}{3}\varepsilon k$ vertices, is 2-connected, and is $\frac{\varepsilon k}{3k'}$-far from being bipartite. Then in particular $e(B) \ge \frac{2}{3}\varepsilon k k'$. Let C_1 be an odd cycle in B. If $v(C_1) \geq \frac{1}{3}\varepsilon k$ then we are done. Otherwise, let $B'' := B \setminus V(C_1)$ be the graph obtained by removing the vertices of C_1 from B. Then $v(B'') = v(B) - v(C_1) \geq k' - \frac{1}{3}\varepsilon k$ and $e(B'') \geq \frac{1}{3}\varepsilon k k'$. By Theorem 5.5 we have a cycle C_2 of length at least $\frac{2}{3}\varepsilon k$. If C_2 is odd then we are done. Otherwise, by 2-connectivity, there are two vertex-disjoint paths P_1, P_2 from C_1 to C_2. Let P_3 be the longer path on C_2 that connects P_1 and P_2. Then $v(P_1 \cup P_2 \cup P_3) \geq \frac{1}{3}\varepsilon k$. Since C_1 is odd, there are two paths that connect P_1 and P_2 along C_1, one is even and one is odd. We choose the one that creates an odd cycle together with $P_1 \cup P_2 \cup P_3$. This gives an odd cycle of length at least $\frac{1}{3}\varepsilon k$. □

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