# Avoider-Enforcer: The Rules of the Game

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**Abstract**

An Avoider-Enforcer game is played by two players, called Avoider and Enforcer, on a hypergraph $\mathcal{F} \subseteq 2^X$. The players claim previously unoccupied elements of the board $X$ in turns. Enforcer *wins* if Avoider claims all vertices of some element of $\mathcal{F}$, otherwise Avoider wins. In a more general version of the game a *bias* $b$ is introduced to level up the players' chances of winning; Avoider claims one element of the board in each of his moves, while Enforcer responds by claiming $b$ elements. This traditional set of rules for Avoider-Enforcer games is known to have a shortcoming: it is not bias monotone.

We relax the traditional rules in a rather natural way to obtain bias monotonicity. We analyze this new set of rules and compare it with the traditional ones to conclude some surprising results. In particular, we show that under the new rules the threshold bias for both the connectivity and Hamiltonicity games, played on the edge set of the complete graph $K_n$, is asymptotically equal to $n/\log n$. This coincides with the asymptotic threshold bias of the same game played by two "random" players.

## 1 Introduction

In this paper we consider Avoider-Enforcer games. To motivate our investigation we start with a short discussion of their widely studied ancestors, Maker-Breaker games.

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**Biased Maker-Breaker games.** Let $p$ and $q$ be positive integers and let $\mathcal{F} \subseteq 2^X$ be a hypergraph over the vertex set $X$. In a $(p : q)$ *Maker-Breaker game* $\mathcal{F}$, two players, called Maker and Breaker, take turns selecting previously unclaimed vertices of $X$ (with Maker going first). Maker selects $p$ vertices per turn and Breaker selects $q$ vertices per turn. If the number of unclaimed vertices is strictly less than $p$ (or $q$) before a move of Maker (or Breaker, respectively) then he must claim all the remaining free vertices. The integers $p, q$ are called the *biases* of the respective players, the members of the family $\mathcal{F}$ are called the *winning sets* and the base set $X$ is called the *board*. The game ends when every element of the board has been claimed by one of the players. Maker wins the game if he claims all the vertices of some winning set; otherwise Breaker wins. Note that there is no possibility of a draw. The parameters $\mathcal{F}, p, q$ unambiguously determine the outcome of the $(p : q)$ Maker-Breaker game $\mathcal{F}$, that is, determine whether Maker has a strategy to win against an arbitrary Breaker, or Breaker has a strategy to beat an arbitrary Maker. In the former case we say that the game is a *Maker's win*, whereas in the latter case we say that the game is a *Breaker's win*.

Chvátal and Erdős [7] studied Maker-Breaker games played on the edge set of the complete graph $K_n$ as the board. They have come to realize that natural graph games are often "easily" won by Maker when played in a fair fashion (that is, with $p = q = 1$). They explored a more general question: What is the largest bias $b$ of Breaker, against which Maker can still win a particular game, if his bias is 1? For such a question to make sense, one would like to have the following property: if the $(1 : b)$ game $\mathcal{F}$ is a Breaker's win for some integer $b$, then the $(1 : b')$ game $\mathcal{F}$ is also a Breaker's win for any $b' \geq b$. It is easy to see that this holds for any family $\mathcal{F}$. More generally, Maker-Breaker games are *bias monotone*, that is, claiming more elements of the board per turn cannot "harm" a player. Formally, if Maker wins the $(p : q)$ Maker-Breaker game $\mathcal{F}$ for some hypergraph $\mathcal{F}$ and positive integers $p, q$, then he also wins the $(p + 1 : q)$ and the $(p : q - 1)$ games (the analogous statement for Breaker's win holds as well).

For a family $\mathcal{F}$ of sets, let the *threshold bias* $b_{\mathcal{F}}$ be the non-negative integer for which Maker has a winning strategy in the $(1 : b)$ game $\mathcal{F}$ if and only if $b < b_{\mathcal{F}}$. Note that, by the aforementioned monotonicity, $b_{\mathcal{F}}$ is well-defined for any (monotone increasing) family $\mathcal{F}$ (unless $\mathcal{F} = \emptyset$ or $\mathcal{F}$ contains a hyperedge of size at most one).

Chvátal and Erdős [7] have initiated the study of the biased graph games "connectivity", "Hamiltonicity", and "triangle", where the families of winning sets are the family $\mathcal{T} = \mathcal{T}(n) \subseteq 2^{E(K_n)}$ of all $n$-vertex connected graphs, the family $\mathcal{H} = \mathcal{H}(n) \subseteq 2^{E(K_n)}$ of all $n$-vertex Hamiltonian graphs, and the family $\mathcal{K}_{K_3} = \mathcal{K}_{K_3}(n) \subseteq 2^{E(K_n)}$ of all $n$-vertex graphs containing a triangle, respectively (the parameter $n$ is routinely suppressed in our notation). They showed that $b_{\mathcal{T}} = \Theta\left(\frac{n}{\log n}\right)$ and noted the remarkable phenomenon that this threshold bias is of the same order of magnitude as the threshold bias of the "connectivity game" in which both players play randomly, rather than cleverly. This fact is a consequence of the classic work of Erdős and Rényi on random graphs, and therefore the phenomenon is often referred to as the "random graph intuition".

In [1], Beck has studied the unbiased clique game. He has proved that the size of the largest clique the first player can build in an unbiased game is almost exactly the same as the size of the largest clique in the random graph $G(n, 1/2)$. Subsequently, Beck [2] has shown that the random graph intuition is valid for the Hamiltonicity game as well; in particular $b_{\mathcal{H}} = \Theta\left(\frac{n}{\log n}\right)$. The current best estimate,

$$(\log 2 - o(1)) \frac{n}{\log n} \leq b_{\mathcal{H}},$$

is due to Krivelevich and Szabó [13]. Here, and throughout the paper, log stands for the natural logarithm.

For the connectivity game Gebauer and Szabó [9] have recently shown that the random graph intuition is correct even asymptotically, that is,

$$b_{\mathcal{T}} = (1 + o(1)) \frac{n}{\log n}.$$

**Biased Avoider-Enforcer games.** Avoider-Enforcer games are the *misère* version of Maker-Breaker games. Generally speaking, a *misère* game is played according to its conventional rules, except that it is played to "lose". This concept has been extensively studied in combinatorial game theory, see, e.g., [6].

The following problem of Beck (see [5, Open problem 20.2]) motivated most of our research on Avoider-Enforcer games.

*Consider the Reverse Hamiltonian Game, played on the edges of $K_n$, where Avoider takes 1 and Enforcer takes $f$ edges per move; Enforcer wins if at the end Avoider's graph contains a Hamiltonian cycle. Is it true that, if $f = c_0 n / \log n$ for some absolute positive constant $c_0$ and $n$ large enough, then Enforcer can win the game?*

While this question was answered positively in [13] (following progress in [10]), several new problems have surfaced in the process.

Let us now give the formal definition of Avoider-Enforcer games as defined in the literature (see, e.g., [14, 3, 5, 4]). Let $p$ and $q$ be positive integers and let $\mathcal{F} \subseteq 2^X$ be a hypergraph. In a $(p : q)$ Avoider-Enforcer game $\mathcal{F}$ two players, called Avoider and Enforcer, take turns selecting previously unclaimed elements of $X$ (with Avoider going first). Avoider selects $p$ vertices per move and Enforcer selects $q$ vertices per move. If the number of unclaimed vertices is strictly less than $p$ (or $q$) before a move of Avoider (or Enforcer, respectively), then he must claim all of the remaining free vertices. The game ends when every element of the board has been claimed by one of the players. Avoider wins the game if he does *not* claim all the vertices of any hyperedge of $\mathcal{F}$; otherwise Enforcer wins. Fittingly, we call the members of the family $\mathcal{F}$ *losing sets.* Since there is no possibility of a draw, the parameters $\mathcal{F}, p, q$ unambiguously determine whether the $(p : q)$ Avoider-Enforcer game $\mathcal{F}$ is an *Avoider's win* or an *Enforcer's win.*

Similarly to Maker-Breaker games, one would like to define for every family $\mathcal{F}$ the Avoider-Enforcer threshold bias $f_{\mathcal{F}}$ as the non-negative integer for which Enforcer wins the $(1:b)$ game $\mathcal{F}$ if and only if $b < f_{\mathcal{F}}$. Somewhat surprisingly, unlike for Maker-Breaker games, such a threshold does not exist in general for Avoider-Enforcer games (see [10]). Even more discouragingly, we cannot establish the existence of a threshold bias even for such a natural graph game as Hamiltonicity. In fact, the smallest bias $b$ for which we can show that the $(1:b)$ game $\mathcal{H}$ is an Avoider's win is the trivial $b = n/2$ (with this bias, Avoider will have less than $n$ edges at the end of the game, and will thus win irregardless of his strategy). This is in striking contrast to the Maker-Breaker counterpart of the Hamiltonicity game, where the order of magnitude of the threshold bias is known.

Regarding the random graph intuition, the main motivation behind the aforementioned problem of Beck, we face yet another surprise for Avoider-Enforcer games. In [10] it was shown that, in general, the random graph intuition is not true in a strong sense, as for the connectivity game the Avoider-Enforcer threshold bias $f_{\mathcal{T}}$ does exist for every $n$ and is equal to $\lfloor \frac{n-1}{2} \rfloor$. That is, the threshold bias of the Maker-Breaker connectivity game is $(1+o(1))\frac{n}{\log n}$ while the threshold bias of its misère version is of linear order!

Following [10], let us introduce some relevant terminology. For a hypergraph $\mathcal{F}$ we define the *lower threshold bias* $f_{\mathcal{F}}^-$ to be the largest integer such that Enforcer can win the $(1:b)$ game $\mathcal{F}$ for every $b \leq f_{\mathcal{F}}^-$, and the *upper threshold bias* $f_{\mathcal{F}}^+$ to be the smallest non-negative integer such that Avoider can win the $(1:b)$ game $\mathcal{F}$ for every $b > f_{\mathcal{F}}^+$. Except for certain degenerate cases, $f_{\mathcal{F}}^-$ and $f_{\mathcal{F}}^+$ always exist and satisfy $f_{\mathcal{F}}^- \leq f_{\mathcal{F}}^+$. Observe that whenever $f_{\mathcal{F}}^- = f_{\mathcal{F}}^+$, the threshold bias $f_{\mathcal{F}}$ of the Avoider-Enforcer game $\mathcal{F}$ does exist and satisfies $f_{\mathcal{F}} = f_{\mathcal{F}}^+$.

In order to overcome the non-monotonicity of Avoider-Enforcer games and, as a consequence, the lack of a well-defined threshold bias, we offer a modification of the rules of Avoider-Enforcer games. We refer to the new rules as *monotone rules*, while the original set of rules will be referred to as *strict rules*. In this new setting of Avoider-Enforcer games everything remains the same as before except that we allow both players to claim *more* elements per turn than their respective bias. Formally, in a monotone $(p:q)$ Avoider-Enforcer game $\mathcal{F} \subseteq 2^X$, Avoider claims *at least* $p$ elements of $X$ per turn and Enforcer claims *at least* $q$ elements of $X$ per turn. It is easy to see that Avoider-Enforcer games with these rules *are* bias monotone. Hence, one can define the threshold bias $f_{\mathcal{F}}^{mon}$ of the monotone game $\mathcal{F}$ as the non-negative integer for which Enforcer has a winning strategy in the $(1:b)$ game if and only if $b \leq f_{\mathcal{F}}^{mon}$.

Our relaxation of the rules of Avoider-Enforcer games is inspired by the seemingly plausible assumption that "taking more edges cannot possibly help a player in an Avoider-Enforcer game". The presumed analogy to Maker-Breaker games further supports the idea of monotone rules, since the analogous relaxation of the rules of Maker-Breaker games does not change the outcome of the game – it is known that allowing a player to claim less edges than his respective bias in a Maker-Breaker game cannot help him. Formally, in a monotone $(p:q)$

Maker-Breaker game $\mathcal{F}$, Maker claims *at most* $p$ elements per turn and Breaker claims *at most* $q$ elements per turn. Then, if we denote the threshold bias of the monotone game by $b_{\mathcal{F}}^{mon}$, it is easy to see that $b_{\mathcal{F}}^{mon} = b_{\mathcal{F}}$ for every family $\mathcal{F}$.

One may wonder about the relationship between a biased Avoider-Enforcer game played according to the strict rules and the same game played according to the monotone rules. Is it true that our relaxation of the rules has no significant effect, other than making the game bias-monotone? Is it plausible to believe that even if there is some alternation in the identity of the winner of a strict game $\mathcal{F}$, the inequalities

$$f_{\mathcal{F}}^{-} \leq f_{\mathcal{F}}^{mon} \leq f_{\mathcal{F}}^{+}, \tag{1}$$

should hold for every family $\mathcal{F}$?

Unexpectedly, neither of the inequalities (1) is true in general. In fact, (1) does not even hold for such a natural graph game as connectivity, which is even bias monotone under the strict rules (see Theorem 1.1 below, and Theorem 1.5 in [10]).

Let $k$ be a positive integer and let $\mathcal{D}_k \subseteq 2^{E(K_n)}$ denote the hypergraph containing the edge sets of all graphs on $n$ vertices with minimum degree at least $k$. The main result of our paper is the following theorem.

**Theorem 1.1** *If $b \geq \frac{n-1}{\log(n-2)-1}$ and $n$ is sufficiently large, then Avoider has a winning strategy in the monotone $(1 : b)$ game $\mathcal{D}_1$. Therefore,*

$$f_{\mathcal{D}_1}^{mon} \leq (1 + o(1))\frac{n}{\log n}.$$

As was mentioned earlier, Theorem 1.1 coupled with Theorem 1.5 of [10] exemplifies that in the connectivity game Avoider *does benefit* from having the possibility of taking *more* than one edge in each move. As proved in [10], when playing according to the strict rules, Avoider can only win if the bias of Enforcer is at least as large as $\lfloor \frac{n-1}{2} \rfloor + 1$, so Avoider will have strictly less than $n - 1$ edges at the end. On the other hand, when playing according to the monotone rules, Avoider can avoid building a connected graph even if Enforcer's bias is as small as $\Theta\left(\frac{n}{\log n}\right)$.

**Corollary 1.2** *Inequality* (1) *does not hold in general, not even in the special case where the threshold bias for the strict game exists.*

Combined with the results of [13], Theorem 1.1 also has the following important corollary. Let $\mathcal{C}_k \subseteq 2^{E(K_n)}$ denote the hypergraph containing the edge sets of all $k$-connected spanning subgraphs of $K_n$.

**Corollary 1.3**

$$f_{\mathcal{D}_k}^{mon}, f_{\mathcal{T}}^{mon}, f_{\mathcal{C}_k}^{mon}, f_{\mathcal{H}}^{mon} = (1 + o(1))\,\frac{n}{\log n}. \tag{2}$$

Corollary 1.3 states that the random graph intuition holds *asymptotically* for all of the above games. Note that for some of these games, such as "Hamiltonicity" or "$k$-connectivity", where $k \geq 2$, currently we do *not* have such tight results for the Maker-Breaker version.

The upper bound in equation (2) follows from Theorem 1.1, whereas the lower bound was essentially proved in [13]. For the latter, we must observe that the proof in [13] depends on the use of a general sufficient condition for Avoider's win from [10] for strict Avoider-Enforcer games. Minor changes to its proof show that the same sufficient condition for Avoider's win holds for the monotone game as well (we omit the straightforward details).

**Theorem 1.4** *[10, Theorem 1.1] Avoider wins the $(p : q)$ game $\mathcal{F}$ (both with strict and monotone rules), provided that*

$$\sum_{D \in \mathcal{F}} \left(1 + \frac{1}{p}\right)^{-|D|} < \left(1 + \frac{1}{p}\right)^{-p} .$$

**Avoiding small graphs.** All the games discussed in Corollary 1.3 have one common property – the size of the losing sets grows with $n$. The extreme opposite of these are games with losing sets of constant size, in particular games in which Enforcer wants to make Avoider claim a copy of some fixed graph $H$. Let $\mathcal{K}_H \subseteq 2^{E(K_n)}$ consist of the edge sets of the subgraphs of $K_n$ which contain $H$ as a subgraph. In the $\mathcal{K}_H$ game, one property of the monotone rules seems to play an important role. Namely, Avoider will surely lose the game if he claims a copy of $H^-$ (a copy of $H$ with one edge missing), for which the missing edge is still unclaimed and there are "many", that is, at least $b$, additional unclaimed edges. Since this is not the case when playing with the strict rules, one may expect this to influence the outcome of the game in Enforcer's favor, especially when the losing sets are small. In particular, it may be reasonable to compare the outcomes and strategies of both players in the strict $H^-$ game and the monotone $H$ game. We will analyze the smallest non-trivial cases of the $H$-game on $K_n$; namely, the monotone triangle game (that is, when $H = K_3$), and the strict 2-path game (that is, when $H = P_3 = K_3^-$ is the path of length 2).

The 2-path game $\mathcal{K}_{P_3}$ is an example of an Avoider-Enforcer game for which the strict threshold bias does not exist, but inequality (1) holds.

**Theorem 1.5** *$f^+_{\mathcal{K}_{P_3}} = \binom{n}{2} - 2$, $f^-_{\mathcal{K}_{P_3}} = \Theta\left(n^{3/2}\right)$, and $f^{mon}_{\mathcal{K}_{P_3}} = \binom{n}{2} - \lfloor \frac{n}{2} \rfloor - 1$.*

We note that while the third statement of this theorem is easy to verify, the first two will be proved in Section 3.2.

For the Maker-Breaker triangle game Chvátal and Erdős [7] proved that the threshold bias $b_{\mathcal{K}_{K_3}}$ is of order $\sqrt{n}$; the dissimilarity with the monotone Avoider-Enforcer threshold bias is striking.

**Theorem 1.6**

$$f^{mon}_{\mathcal{K}_{K_3}} = \Theta\left(n^{3/2}\right) \ .$$

It was proved in [12] that the threshold bias for the Maker-Breaker non-$k$-colorability game is of order $n$ for every fixed $k \geq 2$. Theorem 1.6 may suggest that the threshold bias for the monotone Avoider-Enforcer $k$-coloring game is of superlinear order in $n$, as it provides such a result for $k = 2$.

For the sake of simplicity and clarity of presentation, we do not make a particular effort to optimize the constants obtained in theorems we prove. We also omit floor and ceiling signs whenever these are not crucial. Many of our results are asymptotic in nature and whenever necessary we assume that $n$ is sufficiently large. Our graph-theoretic notation is standard and follows that of [15]. In particular, if $G = (V, E)$ is a graph and $u \in V$, then the *$G$-degree of* $u$ is the number of neighbors $u$ has in $G$.

The rest of this paper is organized as follows: In Section 2 we prove Theorem 1.1, and in Section 3 we prove Theorem 1.6 and Theorem 1.5. In Section 4 we present some conclusions and open problems.

## 2 Isolating a vertex

**Proof of Theorem 1.1.** We present a winning strategy for Avoider. At any point of the game, let $A$ be the set of vertices that have positive degree in Avoider's graph. Avoider will make sure that $A$ grows by at most two vertices in each round and after each of his moves there is no free edge within $A$.

*Avoider's strategy:* As long as $|V \setminus A| \geq 5$, Avoider does the following. Let $M \subseteq V \setminus A$ denote the subset of vertices, which have the smallest E-degree (that is, degree in Enforcer's graph) among the vertices of $V \setminus A$. If there is an unclaimed edge $uv$ such that $v \in M$ and $u \in A$, then Avoider claims *all* unclaimed edges $wv$ for which $w \in A$. Otherwise, if all edges with one endpoint in $A$ and the other in $M$ were already claimed by Enforcer, then Avoider claims an arbitrary free edge $uw$ as well as all free edges with one endpoint in $\{u, w\}$ and the other in $A$.

If $|V \setminus A| \leq 4$, then Avoider changes his strategy. He chooses an arbitrary vertex $z \in V \setminus A$ of maximum E-degree and claims all of the remaining free edges, except for the ones which are incident with $z$.

First we show that this strategy is well-defined, that is, unless Avoider has already won, he can always follow it.

For as long as $|V \setminus A| \geq 5$ Avoider can follow his strategy unless Enforcer has already claimed all edges incident with $V \setminus A$, in which case Avoider had already won.

Since $A$ is increased by 1 or 2 in each round, the first time Avoider encounters $|V \setminus A| \leq 4$, the value of $|V \setminus A|$ is either 4 or 3. Let $z \in V \setminus A$ be an arbitrary vertex of maximal E-degree among the vertices of $V \setminus A$. We will show that at that point there must be at least one free edge which is not incident with $z$. Assume for the sake of contradiction that every free edge is incident with $z$, then for all other vertices $u \in (V \setminus A) \setminus \{z\}$, the only free incident edge can be $zu$. In fact, $zu$ must be free for all $u \in (V \setminus A) \setminus \{z\}$ as, otherwise, Enforcer would have claimed all edges incident with $u$ and thus Avoider would have won already. Then, since $|(V \setminus A) \setminus \{z\}| \geq 2$, the E-degree of $z$ is at most $n-3$, while $d_E(u) \geq n-2$ for any $u \in (V \setminus A) \setminus \{z\}$, contradicting the maximality of the E-degree of $z$.

We have thus proved that Avoider can follow his strategy until $|V \setminus A|$ first drops below 5, and for one additional move. In the following we will prove that he will not have to play another move; that is, there will be *at most* $b$ free edges left on the board, each of them incident with $z$. Enforcer must claim all of them in his next move and so the game ends with Avoider's win.

Assume now that Avoider plays the game against some fixed strategy of Enforcer. For clarity, we introduce an indexing of the set $A$. For $i \geq 0$, let $A_i$ be the set of those vertices that have a positive degree in Avoider's graph just before his $(i+1)$st move. Let $d_i^*$ be the average degree of vertices of $V \setminus A_i$ in Enforcer's graph, that is,

$$d_i^* = \frac{\sum_{v \in V \setminus A_i} d_E(v)}{|V \setminus A_i|}.$$

Let $g$ be the smallest integer, such that $|V \setminus A_{g-1}| \leq 4$.

**Claim 2.1** *For every* $1 \leq j \leq g-1$*, we have*

$$d_j^* \geq \min\left\{ \sum_{i=2}^{|A_j|} \frac{b}{n-i},\ n-1-b \right\}.$$

Before proving this claim, let us show that it readily implies Theorem 1.1. By definition, before Avoider's move in round $g$, we have $|V \setminus A_{g-1}| = 3$ or $|V \setminus A_{g-1}| = 4$. Hence by Claim 2.1 we have either

$$d_E(z) \geq d_{g-1}^* \geq n-1-b,$$

or

$$d_E(z) \geq d_{g-1}^* \geq \sum_{i=2}^{|A_{g-1}|} \frac{b}{n-i} \geq \sum_{i=2}^{n-4} \frac{b}{n-i} = \sum_{j=4}^{n-2} \frac{b}{j} > b(\log(n-2)-2) \geq n-1-b.$$

In his $g$th move, Avoider claims all edges other than the ones which are incident with $z$. Then, at most $b$ free edges will remain, all of them incident with $z$. Enforcer must claim all of them, thus isolating $z$ in Avoider's graph.

*Proof of Claim 2.1.* We proceed by induction on $i$. For $i = 1$ the statement is certainly true as after Avoider's first move we have $|A_1| = 2$, and on his first move, Enforcer has claimed at least $b$ edges. Each of these edges has at least one endpoint in $V \setminus A_1$, entailing $d_1^* \geq \frac{b}{n-2}$.

Next, assume that after the $l$th move of Enforcer, where $1 \leq l \leq g-2$, the statement is true. We show that it remains true after the next round. We distinguish between two cases.

*Case 1.* There exists an unclaimed edge $uv$ such that $u \in A_l$, and $v \in V \setminus A_l$ satisfies $d_E(v) = \min\{d_E(w) : w \in V \setminus A_l\}$.
In this case we have $|A_{l+1}| = |A_l| + 1$. Assume first that $\sum_{i=2}^{|A_l|} \frac{b}{n-i} \leq n-1-b$, then by induction we have $d_l^* \geq \sum_{i=2}^{|A_l|} \frac{b}{n-i}$. The vertex $v$ was of minimum degree in Enforcer's graph on $V \setminus A_l$, and so the value of $d_l^*$ was not decreased by Avoider's move. In his counter move, Enforcer has claimed at least $b$ edges. Each of these edges has at least one endpoint in $V \setminus A_{l+1}$, since Avoider made sure that all edges spanned by $A_{l+1}$ are already claimed. Hence, the value of $d_l^*$ was increased by at least $\frac{b}{n-|A_{l+1}|} = \frac{b}{n-(|A_l|+1)}$. Therefore, after both players have made their $(l+1)$st move, we have

$$d_{l+1}^* \geq d_l^* + \frac{b}{n-(|A_l|+1)} \geq \sum_{i=2}^{|A_{l+1}|} \frac{b}{n-i}.$$

Next, assume that $\sum_{i=2}^{|A_l|} \frac{b}{n-i} > n-1-b$. By the induction hypothesis and Avoider's strategy, $d_{l+1}^* \geq d_l^* \geq n-1-b$ holds in this case.

*Case 2.* All edges $uv$ such that $u \in A_l$, and $v \in V \setminus A_l$ satisfying $d_E(v) = \min\{d_E(w) : w \in V \setminus A_l\}$, were already claimed by Enforcer.
Then, the degree of every vertex of $V \setminus A_l \supset V \setminus A_{l+1}$ in Enforcer's graph is at least $|A_l|$, implying $d_{l+1}^* \geq |A_l|$. It follows that if $|A_l| \geq n-1-b$, then we are done. Assume now that $|A_l| < n-1-b$. The size of $A_l$ is increased by either one or two in Avoider's $(l+1)$st move. Hence, after this move, we have $d_{l+1}^* \geq |A_l| \geq |A_{l+1}| - 2 > \sum_{i=2}^{|A_{l+1}|} \frac{b}{n-i}$, where the last inequality clearly holds for $A_{l+1}$ of size $3 \leq |A_{l+1}| \leq n-b$ as each summand is at most 1, and the sum of the first two is $\frac{b}{n-2} + \frac{b}{n-3} < 1$. $\square$

**Remark.** Our proof of Theorem 1.1 is some sort of a dynamic version of the *Box Game* defined in [7]. It is interesting to note that Avoider's strategy here is similar to Maker's strategy in the Box Game which means that, when the hyperedges are pairwise disjoint, a player who wants to claim a complete hyperedge and a player who wants to avoid one, will essentially choose the same strategy.

**Remark.** The first phase of Avoider's strategy resembles in a way some strategies used in Nim-like games, as in every move Avoider attaches vertices to $A$ by claiming all free edges between one or two vertices of $V \setminus A$ and $A$. Hence, his opponent is forced to touch a vertex outside of $A$ in every move.

# 3 Avoiding and enforcing a small subgraph

## 3.1 Triangle game

**Proof of Theorem 1.6.** First, assume that $b > n^{3/2}$. In his strategy, Avoider will always claim exactly one edge per turn. For as long as possible he claims independent edges, that is, he greedily builds a matching of maximum possible size. We will prove that when he can no longer extend his matching, the game is almost over, that is, Avoider will claim at most one more edge. This suffices to prove our claim as a union of a matching and a single edge is bipartite. Let $e$ denote the number of edges in Avoider's matching; clearly $e \leq \left\lceil \frac{\binom{n}{2}}{b+1} \right\rceil$. At the point when Avoider cannot further extend his matching, Enforcer must have claimed at least $\binom{n-2e}{2}$ edges (every edge which is not incident with Avoider's matching). It follows that the number of unclaimed edges is at most $\binom{2e}{2} + 2e(n-2e) < b$ and so Avoider will win.

Next, assume that $b \leq \frac{1}{5}n^{3/2}$. Due to the bias-monotonicity of Avoider-Enforcer games with monotone rules, it is enough to present a winning strategy for Enforcer for $b = \frac{1}{5}n^{3/2}$. Set $t = n^{1/2}$.

The game is divided into two phases. The first phase lasts as long as Avoider's graph $A$ is a matching with at most $t$ edges; the first move of Avoider which violates this condition starts the second phase (it is therefore possible that the first phase will not take place at all).
A vertex is called *good* if it is isolated in $A$. A good vertex $v$ is called *fulfilled* if Enforcer claimed all edges between $v$ and other good vertices; a good vertex which is not fulfilled is called *unfulfilled*. Note that once a vertex becomes fulfilled it stays that way until the end of the first phase.
In each round of the first phase Enforcer acts as follows. He spots an unfulfilled vertex $v$ and claims the unclaimed edges between $v$ and all other good vertices; hence $v$ becomes fulfilled. Enforcer continues fulfilling unfulfilled vertices this way, for as long as he has not yet claimed at least $b$ edges in his move. Note that in each of his moves during the first phase Enforcer claims at least $b$ but no more than $b + n$ edges.
In the second phase Enforcer will make at most two moves. His course of play depends on Avoider's graph.

*Case 1.* $A$ contains a vertex $x$ of degree at least 2.
In this case Enforcer needs only one more move. He spots the edge $xy$ that is claimed by Avoider the *earliest* among all $A$-edges incident to $x$. Let $xz$ be another, arbitrary edge claimed by Avoider. In his move Enforcer claims all unclaimed edges except for $yz$.

*Case 2.* $A$ is a matching $M'$ consisting of more than $t$ edges.
In this case Enforcer will need at most two more moves. Let $e_1, e_2, \ldots, e_t$ denote the first $t$ edges of $M'$ that were claimed by Avoider (breaking ties arbitrarily), let $M = \{e_1, \ldots, e_t\}$, and let $V_M$ be the set of vertices covered by the edges of $M$. Enforcer claims every unclaimed edge with both endpoints in $V \setminus V_M$;

denote the number of such edges by $r$. If $r < b$, then Enforcer also claims $2\lceil\frac{b-r}{2}\rceil$ additional edges between $V_M$ and $V \setminus V_M$ as follows. He spots $\lceil\frac{b-r}{2}\rceil$ pairs $(u, ww')$, with $u \in V \setminus V_M$ and $ww' \in M$, such that both $uw$ and $uw'$ are unclaimed and claims these edges. Note that in this turn Enforcer claims at most $b+1$ edges between $V_M$ and $V \setminus V_M$. After Avoider's next move Enforcer spots an arbitrary vertex $x$ of $A$-degree at least 2, say $xy$ and $xz$ were claimed by Avoider, and claims all unclaimed edges except for $yz$.

In the following we will show that, unless he has already won, Enforcer is always able to follow his strategy while playing against an arbitrary strategy of Avoider. We will also show that Enforcer does *not* claim the edge $yz$, which implies that the above strategy of Enforcer is a winning one, since in both cases Avoider must occupy the triangle on the vertices $x, y$ and $z$.

First we show that Enforcer can always follow his strategy during the first phase. During the first phase at most $t$ edges were claimed by Avoider and thus there are at least $n - 2t$ good vertices. Since Enforcer claims at most $t(b+n)$ edges during the entire first phase, there are at least $\binom{n-2t}{2} - t(b+n) > b+n$ unclaimed edges which are spanned by good vertices. It follows that Enforcer has a legitimate move at any point during the first phase.

Let the second phase start in round $g$; note that $1 \leq g \leq t+1$. The following claims will be useful in showing that Enforcer can follow his strategy during the second phase and that the edge $yz$ is not claimed by Enforcer.

**Claim 3.1** $(i)$ *If $vw$ is an isolated edge in Avoider's graph, then neither $v$ nor $w$ was ever a fulfilled vertex.*
$(ii)$ *At least one endpoint of every edge claimed by Enforcer in the first phase is fulfilled.*

*Proof of Claim 3.1.* $(i)$ Before Avoider claims the edge $vw$, both of the endpoints were good. If one of them was fulfilled, then the edge $vw$ would have already been claimed by Enforcer.
$(ii)$ Clear from the strategy of Enforcer. □

**Claim 3.2** *At the end of the first phase the following is true for every good vertex $u$. For any pair $v, w$ such that either $vw \in E(A)$ or both $v$ and $w$ are good, we have that either both edges $vu$ and $wu$ are unclaimed or they were both claimed by Enforcer.*

*Proof of Claim 3.2.* We prove the assertion of the claim by induction on the number of rounds. The claim is clearly true at the beginning of the game. Now suppose the claim is true before the $i$th move of Avoider (in the first phase). In his $i$th move, Avoider claims independent edges between good and unfulfilled vertices. For these pairs the statement was true by the induction hypothesis and Avoider's move does not change that. Enforcer fulfills several good vertices in his $i$th move; this also preserves the correctness of the statement. □

Let us look at the board just after Avoider's first move of the second phase (that is, his $g$th move).

*Case 1.* There is a vertex $x$ of degree at least 2 in $A$.
Recall the selection of $y$: $xy$ was claimed by Avoider the earliest among the $A$-edges incident to $x$. We can assume that $yz$ was not claimed by Avoider, as otherwise he had already lost. For a contradiction, assume that $yz$ was claimed by Enforcer in the first phase. By Claim 3.1($ii$) at least one of the endpoints of $yz$ was fulfilled by the end of the first phase, let $w \in \{y, z\}$ be the first one to be fulfilled and let $\bar{w}$ be the other vertex of $\{y, z\}$. Then by Claim 3.1($i$), $xw$ must have been claimed by Avoider only in the second phase. Hence $xw$ was free at the time Enforcer was fulfilling $w$, he still did not claim it while he did claim $w\bar{w} = yz$. That is, at that time $\bar{w}$ must have been good, while $x$ must have been not good. So $x\bar{w}$ was not yet claimed by Avoider, but there must have been another edge $xu$ already claimed by Avoider contradicting Enforcer's choice of the $A$-neighbor $y$ of $x$. It follows that the edge $yz$ is indeed unclaimed at this point.

By the Mantel-Turán theorem we can also assume that $|A| \leq n^2/4$, as otherwise Avoider has already lost. In the first phase, Enforcer has claimed at most $t(b+n)$ edges, so right after Avoider's $g$th move at least $\binom{n}{2} - n^2/4 - t(b+n) \geq b+1$ edges are still unclaimed. Enforcer can thus follow his strategy, claim all free edges except $yz$ and make Avoider lose on his next move.

*Case 2.* Avoider's graph is a matching $M'$ containing more than $t$ edges.
Recall that $e_1, e_2, \ldots, e_t$ are the first $t$ edges of $M'$, $M = \{e_1, \ldots, e_t\}$, and $V_M$ is the set of vertices covered by the edges of $M$.

Note first that by Claim 3.2, for any pair $(u, xy)$, $u \in V \setminus V_M$ and $xy \in M$, we have that either both $ux$ and $uy$ are unclaimed or both of them were already claimed by Enforcer. Hence, Enforcer can claim the unclaimed edges between $V_M$ and $V \setminus V_M$ *in pairs*, as required by his strategy.

We will now show that just before Enforcer's first move of the second phase, there are at least $3(b+1)$ unclaimed edges between $V_M$ and $V \setminus V_M$. This is enough to ensure that Enforcer will be able to follow his strategy for the remainder of the game. In his $g$th move Enforcer takes at most $b+1$ edges between $V_M$ and $V \setminus V_M$. In Avoider's subsequent move, Avoider cannot claim more than half of the unclaimed edges between $V_M$ and $V \setminus V_M$ without losing immediately. Indeed, by Claim 3.2 and Enforcer's strategy for his $g$th move, if $vw \in M$ and $u \in V \setminus V_M$, then $uv$ is unclaimed if and only if $uw$ is unclaimed, and by claiming both of them, Avoider would build a triangle on $u, v, w$. Hence after Avoider's $(g+1)$st move there are at least $b+1$ free edges between $V_M$ and $V \setminus V_M$, enough for Enforcer to make his last move.

To follow his strategy in his last move, Enforcer also needs a vertex $x \in V_M$ of $A$-degree at least 2. Such a vertex will be created by Avoider in his $(g+1)$st move, as after Enforcer's $g$th move none of the edges in $V \setminus V_M$ are free. Then Enforcer will win after Avoider's next move, as he could not have claimed $yz$. Indeed, if $\{y, z\} \subseteq V_M$, this follows from Claim 3.1, parts ($i$) and ($ii$), and otherwise from Claim 3.2 and Enforcer's strategy for his $g$th move.

It remains to show that at least $3(b+1)$ edges between $V_M$ and $V \setminus V_M$ are free after Avoider's $g$th move. In fact we will prove that at least one third of the $|V_M||V \setminus V_M|$ edges between $V_M$ and $V \setminus V_M$ are unclaimed. This will conclude the proof of the theorem as $2t(n-2t)/3 \geq 3(b+1)$ for large enough $n$.

Let $m = \lceil t/2 \rceil$, and let $e_1, e_2, \dots, e_m$ be the first $m$ edges to be claimed by Avoider. Assume that Avoider claimed $e_m$ in move $g' \leq g$. After round $g'-1$, we denote the number of edges claimed by Enforcer by $\ell$, and the number of fulfilled vertices by $k$. We have that $\ell \leq m(b+n) = \left(\frac{1}{10} + o(1)\right) n^2$. On the other hand, by Enforcer's strategy we get $\ell \geq \binom{k}{2} + k(n-2m-k)$. Combining the two inequalities gives $k^2 - 2(1-o(1))kn + \left(\frac{1}{5} + o(1)\right) n^2 > 0$ from which we infer that $k < n/9$.

Thus there are at least $n - 2t - n/9$ unfulfilled vertices $U$ before round $g'$. At this moment the edges between these vertices and the endpoints of the $e_i$, $1 \leq i \leq m$ are free by Claim 3.1 *(i)* and *(ii)*. These edges stay unclaimed throughout the first phase, since even if some vertices of $U$ become fulfilled, they will not be connected to an endpoint of $e_i$, as these vertices are not good anymore. Hence all these edges are unclaimed at the end of the first phase and thus the number of unclaimed edges between $V_M$ and $V \setminus V_M$ is at least $2m(n-2t-n/9) > |V_M||V \setminus V_M|/3$. □

## 3.2 $P_3$-game

The bounds on the strict thresholds in Theorem 1.5 follow readily from the following lemma.

**Lemma 3.3** *Let $n$ be sufficiently large and $r$ be the remainder of the integer division of $\binom{n}{2}$ by $b+1$.*

*(i) If $b < \frac{1}{5}n^{3/2}$, then Enforcer wins the $(1:b)$ 2-path game with strict rules, independently of the value of $r$.*

*(ii) Let $b > 2n^{3/2}$. If $0 < r < 2(n-2)$, then Enforcer wins the $(1:b)$ 2-path game with strict rules, but if $r > n^{3/2}$, then Avoider wins this game.*

**Proof of Lemma 3.3**

*(i)* Let $b < \frac{1}{5}n^{3/2}$ be an integer. We give a strategy for Enforcer to win the strict $(1:b)$ $P_3$-game.

Let $m = \left\lfloor \frac{1}{5} \frac{\binom{n}{2}}{b+1} \right\rfloor$. At any point of the game, let $H$ denote the graph consisting of all edges that were previously claimed (by either Avoider or Enforcer). During the first $m$ rounds Enforcer claims $mb$ edges according to the following simple strategy. He claims an arbitrary free edge $uv$, such that both endpoints $u$ and $v$ have $H$-degree strictly less than $n/2$.

¿From round $m+1$ on, Enforcer changes his strategy. We say that an unclaimed edge is a *threat*, if it is adjacent to an edge of Avoider. Enforcer identifies an arbitrary set $T$ of $b+1$ threats and in his following moves he claims arbitrary free edges with the only restriction that he claims an edge from $T$ only if all other edges are claimed.

It is clear that if Enforcer is able to always act according to the above strategy, then he wins the game. Indeed, in the move right after Enforcer claimed his first edge from $T$, Avoider must also occupy an edge of $T$ and thus create a copy of $P_3$.

To finish the proof of part $(i)$, we show that Enforcer can follow his strategy while playing against an arbitrary fixed strategy of Avoider. At any point during this game let $l$ denote the number of vertices $v$ satisfying $d_H(v) \geq \frac{n}{2}$. Then up to round $m$ we have

$$\frac{1}{5}\binom{n}{2} \geq m(b+1) \geq \frac{1}{2} \cdot \frac{n}{2} \cdot l,$$

entailing $l \leq \frac{2}{5}(n-1) < \frac{n}{2} - 1$. That is, up to round $m$, there are more than $\frac{n}{2}+1$ vertices of $H$-degree less than $\frac{n}{2}$. We conclude that there are two vertices $u$ and $v$ of $H$-degree less than $\frac{n}{2}$, such that the edge $uv$ is unclaimed and is thus available for Enforcer to claim.

Unless Avoider had already lost, Avoider's graph after his $(m+1)$st move is a matching consisting of exactly $m+1$ edges. Each edge of Avoider creates $2(n-2)$ potential threats and each actual threat is created by at most two edges of Avoider. Hence, the number of threats after Avoider's $(m+1)$st move is at least

$$\frac{(m+1)(2(n-2))}{2} \geq \frac{\binom{n}{2}}{5(b+1)}(n-2) \geq \frac{(n-2)^3}{10(b+1)} > b,$$

as $b < \frac{1}{5}n^{3/2}$ and $n$ is sufficiently large. We conclude that after Avoider's $(m+1)$st move Enforcer can find a set $T$ of $b+1$ threats and this enables him to follow his strategy from round $m+1$ onward.

*(ii)* Let $b > 2n^{3/2}$. By the definition of the strict game, Enforcer in his last move claims the last unclaimed $r-1$ edges. If $n^{3/2} < r$, then Avoider's strategy throughout the game is to claim arbitrary edges which are not threats. This is always possible as long as the number of threats is less than $r$. The total number of edges played by Avoider is at most

$$\frac{\binom{n}{2}}{b+1} + 1 \leq 1 + \frac{\sqrt{n}}{4}.$$

Hence the number of threats at any point of the game is at most $\left(1+\frac{\sqrt{n}}{4}\right)(2n-2) < r$, and Avoider wins the game.

On the other hand, if $0 < r < 2(n-2)$, then Enforcer wins the game. Indeed, already on his first move, Avoider creates $2(n-2)$ threats, which Enforcer can avoid taking until the very end. □

# 4 Concluding remarks and open problems

**Strict vs. monotone rules.** In this paper, we have shown that the outcome of strict Avoider-Enforcer games can differ substantially from the outcome of monotone Avoider-Enforcer games (even when the strict game is bias-monotone).

A natural question one may ask is: *Which set of rules is "better" than the other?*

The advantage of monotone rules is of course the existence of a threshold bias for every game. Moreover, some of the obtained results concerning the threshold bias of the monotone Avoider-Enforcer game tend to show great similarity to their Maker-Breaker analogues.

The benefit of the strict rules lies in their applicability to Maker-Breaker games (see, e.g., [12]) or to discrepancy type games (see, e.g., [5, 8, 11]). In these applications, in order to provide a strategy for Maker or for Breaker, one defines an auxiliary Avoider-Enforcer game which models the original Maker-Breaker game, and uses the winning strategy of Avoider or Enforcer in the auxiliary game. Clearly, in this situation the monotone rules are useless.

It would be very interesting to study further the differences between the two sets of rules. Finding additional examples that support the random graph intuition for monotone rules would be particularly desirable. The strict versions of the Avoider-Enforcer planarity, $k$-colorability and minor games have already been analyzed in [12]; it would be worthwhile to investigate them in the monotone setting.

**Avoiding small graphs.** Another possible line of research is the further study of the monotone $H$-game for some fixed graph $H$ on at least four vertices.

It seems that in the strict $(1 : b)$ $H$-game the remainder $r$ of the integer division of $\binom{n}{2}$ by $b+1$ plays a significant role. Namely, if $r = 1$, then Avoider is to claim the last edge of the game, and he will therefore lose if at any point of the game there is an unclaimed edge which completes an Avoider's copy of $H^-$ into an $H$.

On the other hand, if $r$ is large, and it can be as large as $b$, then Avoider will lose the game only if at some point of the game there are $r$ unclaimed edges such that each of them completes an Avoider's copy of $H^-$ into an $H$.

Since $r$ can change drastically with small changes of $b$, we believe that in any strict $H$-game the gap between $f^-_{\mathcal{K}_H}$ and $f^+_{\mathcal{K}_H}$ will be substantial.

**Conjecture 4.1** *For every graph $H$, the thresholds $f^-_{\mathcal{K}_H}$ and $f^+_{\mathcal{K}_H}$ are not of the same order.*

This conjecture has been verified for the $P_3$-game in Theorem 1.5.

As mentioned earlier, there are reasons to believe that the monotone $H$-game would behave similarly to the strict $H^-$-game. We are curious whether the fact that the thresholds $f^-_{\mathcal{K}_{P_3}}$ and $f^{mon}_{\mathcal{K}_{K_3}}$ (as obtained in Theorem 1.5 and Theorem 1.6 respectively) are of the same order is merely a coincidence, in particular since $P_3 = K_3^-$ and the respective players' optimal strategies exhibit similarities.

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