Natural Language Processing
Language models

Based on slides from Michael Collins, Chris Manning, Richard Soccer, Dan Jurafsky, Abigail See
Plan

- Problem definition
- n-gram models
- Evaluation
- Estimation
  - Interpolation
  - Discounting
Motivations

• Define a probability distribution over sentences

• Why?

\[ p(01010 \mid \sim) \propto p(\sim \mid 01010) \cdot p(01010) \]

• Machine translation

  • P(“high winds”) > P(“large winds”)

• Spelling correction

  • P(“The office is fifteen minutes from here”) > P(“The office is fifteen minuets from here”)

• Speech recognition (that's where it started!)

  • P(“recognize speech”) > P(“wreck a nice beach”)

• And more!
Motivations

I'll meet you at the cafe.
Motivations

what is the weather
what is the meaning of life
what is the dark web
what is the xfl
what is the doomsday clock
what is the weather today
what is the keto diet
what is the american dream
what is the speed of light
what is the bill of rights
Motivations

• **Philosophical**: a model that is good at predicting the next word, must know something about language and the world

• A good representations for any NLP task
  
  • paper 1
  
  • paper 2
Motivations

• Techniques will be useful later
  • How to encode a sentence?
Problem definition

• Given a finite vocabulary
  \[ \mathcal{V} = \{ \text{the, a, man, telescope, Beckham, two, \ldots} \} \]

• We have an infinite language \( L \), which is \( \mathcal{V}^* \) concatenated with the special symbol STOP

  the STOP
  a STOP
  the fan STOP
  the fan saw Beckham STOP
  the fan saw saw STOP
  the fan saw Beckham play for Real Madrid STOP
Problem definition

- **Input**: a training set of example sentences
  - Currently: roughly hundreds of billions of words.

- **Output**: a probability distribution $p$ over $L$
  \[
  \sum_{x \in L} p(x) = 1, \ p(x) \geq 0 \text{ for all } x \in L
  \]

- $p(\text{“the STOP”}) = 10^{-12}$
- $p(\text{“the fan saw Beckham STOP”}) = 2 \times 10^{-8}$
- $p(\text{“the fan saw saw STOP”}) = 10^{-15}$
A naive method

• Assume we have N training sentences

• Let $x_1, x_2, \ldots, x_n$ be a sentence, and $c(x_1, x_2, \ldots, x_n)$ be the number of times it appeared in the training data.

• Define a language model:

$$p(x_1, \ldots, x_n) = \frac{c(x_1, \ldots, x_n)}{N}$$
A naive method

• Assume we have N training sentences

• Let $x_1, x_2, \ldots, x_n$ be a sentence, and $c(x_1, x_2, \ldots, x_n)$ be the number of times it appeared in the training data.

• Define a language model:

$$p(x_1, \ldots, x_n) = \frac{c(x_1, \ldots, x_n)}{N}$$

• No generalization!
Markov processes

• Markov processes:

  • Given a sequence of $n$ random variables:

  • We want a sequence probability model

  \[ X_1, X_2, \ldots, X_n, \ n = 100, \ X_i \in \mathcal{V} \]

  \[ p(X_1 = x_1, X_2 = x_2, \ldots X_n = x_n) \]

  • There are $|\mathcal{V}|^n$ possible sequences
First-order Markov process

Chain rule

\[ p(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = \]

\[ p(X_1 = x_1) \prod_{i=2}^{n} p(X_i = x_i \mid X_1 = x_1, \ldots, X_{i-1} = x_{i-1}) \]
First-order Markov process

Chain rule

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Markov assumption

\[ p(X_i = x_i \mid X_1 = x_1, \ldots, X_{i-1} = x_{i-1}) = p(X_i = x_i \mid X_{i-1} = x_{i-1}) \]
Second-order Markov process

Relax independence assumption:

\[ p(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = \]
\[ p(X_1 = x_1) \times p(X_2 = x_2 \mid X_1 = x_1) \]
\[ \times \prod_{i=3}^{n} p(X_i = x_i \mid X_{i-2} = x_{i-2}, X_{i-1} = x_{i-1}) \]

Simplify notation

\[ x_0 = *, x_{-1} = * \]
Second-order Markov process

Relax independence assumption:

\[
p(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = \\
p(X_1 = x_1) \times p(X_2 = x_2 \mid X_1 = x_1) \\
\times \prod_{i=3}^{n} p(X_i = x_i \mid X_{i-2} = x_{i-2}, X_{i-1} = x_{i-1})
\]

Simplify notation

\[x_0 = *, \ x_{-1} = *
\]

Is this reasonable?
Detail: variable length

- Probability distribution over sequences of any length
- Define always \( X_n = \text{STOP} \), and obtain a probability distribution over all sequences
- **Intuition:** at every step you have probability \( \alpha_h \) to stop (conditioned on history) and \( (1-\alpha_h) \) to keep going

\[
p(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = \\
\prod_{i=1}^{n} p(X_i = x_i \mid X_{i-2} = x_{i-2}, X_{i-1} = x_{i-1})
\]
Trigram language model

• A trigram language model contains
  
  • A vocabulary $V$
  
  • Non-negative parameters $q(w|u,v)$ for every trigram, such that

  $$w \in V \cup \{\text{STOP}\}, \quad u, v \in V \cup \{\ast\}$$

• The probability of a sentence $x_1, \ldots, x_n$, where $x_n=\text{STOP}$ is

  $$p(x_1, \ldots, x_n) = \prod_{i=1}^{n} q(x_i \mid x_{i-1}, x_{i-2})$$
Example

\[ p(\text{the dog barks STOP}) = q(\text{the} \mid *, *) \times \]
\[ q(\text{dog} \mid *, \text{the}) \times \]
\[ q(\text{barks} \mid \text{the, dog}) \times \]
\[ q(\text{STOP} \mid \text{dog, barks}) \times \]
Limitation

• Markovian assumption is false

He is from France, so it makes sense that his first language is…

• We would want to model longer dependencies
Sparseness

- Maximum likelihood for estimating $q$

- Let $c(w_1, \ldots, w_n)$ be the number of times that $n$-gram appears in a corpus

$$q(w_i \mid w_{i-2}, w_{i-1}) = \frac{c(w_{i-2}, w_{i-1}, w_i)}{c(w_{i-2}, w_{i-1})}$$
Sparseness

- Maximum likelihood for estimating q
  - Let $c(w_1, \ldots, w_n)$ be the number of times that n-gram appears in a corpus
  
  $$ q(w_i \mid w_{i-2}, w_{i-1}) = \frac{c(w_{i-2}, w_{i-1}, w_i)}{c(w_{i-2}, w_{i-1})} $$

- If vocabulary has 20,000 words —> Number of parameters is $8 \times 10^{12}$!

- Most sentences will have zero or undefined probabilities
Berkeley restaurant project sentences

- can you tell me about any good cantonese restaurants close by
- mid priced that food is what i’m looking for
- tell me about chez pansies
- can you give me a listing of the kinds of food that are available
- i’m looking for a good place to eat breakfast
- when is caffe venezia open during the day
<table>
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<th>eat</th>
<th>chinese</th>
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<th>lunch</th>
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## Bigram probabilities

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<td>0</td>
<td>0</td>
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</table>
What did we learn

• \( p(\text{English} \mid \text{want}) < p(\text{Chinese} \mid \text{want}) \) - people like Chinese stuff more when it comes to this corpus

• \( p(\text{to} \mid \text{want}) = 0.66 \) - English behaves in a certain way

• \( p(\text{eat} \mid \text{to}) = 0.28 \) - English behaves in a certain way
Evaluation: perplexity

- Test data: \( S = (s_1, s_2, \ldots, s_M) \)
- Parameters are not estimated from \( S \)
- A good language model has high \( p(S) \) and low perplexity
- Perplexity is the normalized inverse log-probability of \( S \)

\[
p(S) = \prod_{i=1}^{M} p(s_i \mid s_1, \ldots, s_{i-1})
\]

\[
\log_2 p(S) = \sum_{i=1}^{M} \log_2 p(s_i \mid s_1, \ldots, s_{i-1})
\]

\[
\text{perplexity} = 2^{-l}, \quad l = \frac{1}{M} \sum_{i=1}^{M} \log_2 p(s_i \mid s_1, \ldots, s_{i-1})
\]

- \( M \) is the number of words in the corpus
Evaluation: perplexity

• Say we have a vocabulary \( V \) and \( N = |V| + 1 \) and a trigram model with uniform distribution

• \( q(w \mid u, v) = \frac{1}{N}. \)

\[
l = \frac{1}{M} \sum_{i=1}^{m} \log_2 p(s_i \mid s_1, \ldots, s_{i-1})
\]

\[
= \frac{1}{M} \log \left( \frac{1}{N} \right)^M = \log \frac{1}{N}
\]

perplexity = \( 2^{-l} = 2^{\log N} = N \)

• Perplexity is the “effective” vocabulary size.
Typical values of perplexity

• When $|V| = 50,000$
  
  • trigram model perplexity: 74 ($<< 50000$)
  
  • bigram model: 137
  
  • unigram model: 955
Evaluation

• Extrinsic evaluations: MT, speech, spelling correction, …
History

- Shannon (1950) estimated the perplexity score that humans get for printed English (we are good!)
  - Test your perplexity
Estimating parameters

• Recall that the number of parameters for a trigram model with $|V| = 20,000$ is $8 \times 10^{12}$, leading to zeros and undefined probabilities

$$q(w_i \mid w_{i-2}, w_{i-1}) = \frac{c(w_{i-2}, w_{i-1}, w_i)}{c(w_{i-2}, w_{i-1})}$$
Over/under-fitting

- Given a corpus of length $M$:
  - Trigram model:
    \[
    q(w_i \mid w_{i-2}, w_{i-1}) = \frac{c(w_{i-2}, w_{i-1}, w_i)}{c(w_{i-1}, w_i)}
    \]
  - Bigram model:
    \[
    q(w_i \mid w_{i-1}) = \frac{c(w_{i-1}, w_i)}{c(w_{i-1})}
    \]
  - Unigram model:
    \[
    q(w_i) = \frac{c(w_i)}{M}
    \]
Unigram

To him swallowed confess hear both. Which. Of save on trail for are ay device and rote life have
Every enter now severally so, let
Hill he late speaks; or! a more to leg less first you enter
Are where exeunt and sighs have rise excellency took of.. Sleep knave we. near; vile like

Bigram

What means, sir. I confess she? then all sorts, he is trim, captain.
Why dost stand forth thy canopy, forsooth; he is this palpable hit the King Henry. Live king. Follow.
What we, hath got so she that I rest and sent to scold and nature bankrupt, nor the first gentleman?

Trigram

Sweet prince, Falstaff shall die. Harry of Monmouth’s grave.
This shall forbid it should be branded, if renown made it empty.
Indeed the duke; and had a very good friend.
Fly, and will rid me these news of price. Therefore the sadness of parting, as they say, ’tis done.

Quadrigram

King Henry. What! I will go seek the traitor Gloucester. Exeunt some of the watch. A great banquet serv’d in;
Will you not tell me who I am?
It cannot be but so.
Indeed the short and the long. Marry, ’tis a noble Lepidus.
In a shocking finding, scientist discovered a herd of **unicorns** living in a remote, previously unexplored valley, in the Andes Mountains. Even more surprising to the researchers was the fact that the unicorns spoke perfect English.

The scientist named the population, after their distinctive horn, Ovid’s Unicorn. These four-horned, silver-white unicorns were previously unknown to science.

Now, after almost two centuries, the mystery of what sparked this odd phenomenon is finally solved.

Dr. Jorge Pérez, an evolutionary biologist from the University of La Paz, and several companions, were exploring the Andes Mountains when they found a small valley, with no other animals or humans. Pérez noticed that the valley had what appeared to be a natural fountain, surrounded by two peaks of rock and silver snow.

Pérez and the others then ventured further into the valley. “By the time we reached the top of one peak, the water looked blue, with some crystals on top,” said Pérez.

Pérez and his friends were astonished to see the unicorn herd. These creatures could be seen from the air without having to move too much to see them – they were so close they could touch
Linear interpolation

\[
q_{LI}(w_i \mid w_{i-2}, w_{i-1}) = \lambda_1 \times q(w_i \mid w_{i-2}, w_{i-1}) \\
+ \lambda_2 \times q(w_i \mid w_{i-1}) \\
+ \lambda_3 \times q(w_i)
\]

\[
\lambda_i \geq 0, \quad \lambda_1 + \lambda_2 + \lambda_3 = 1
\]

- Combine the three models to get all benefits
Linear interpolation

- Need to verify the parameters define a probability distribution

\[
\sum_{w \in \mathcal{V}} q_{LI}(w \mid u, v) = \sum_{w \in \mathcal{V}} \lambda_1 \times q(w \mid u, v) + \lambda_2 \times q(w \mid v) + \lambda_3 \times q(w) \\
= \lambda_1 \sum_{w \in \mathcal{V}} q(w \mid u, v) + \lambda_2 \sum_{w \in \mathcal{V}} q(w \mid v) + \lambda_3 \sum_{w \in \mathcal{V}} \times q(w) \\
= \lambda_1 + \lambda_2 + \lambda_3 = 1
\]
Estimating coefficients

• Use validation/development set (intro to ML!)

• Partition training data to training (90%?) and dev (10%?) data and optimize the coefficient to minimize the perplexity (the measure we care about!) on the development data
Linear interpolation

\[ q_{LI}(w_i \mid w_{i-2}, w_{i-1}) = \lambda_1 \times q(w_i \mid w_{i-2}, w_{i-1}) \]
\[ + \lambda_2 \times q(w_i \mid w_{i-1}) \]
\[ + \lambda_3 \times q(w_i) \]
\[ \lambda_i \geq 0, \ \lambda_1 + \lambda_2 + \lambda_3 = 1 \]
Linear interpolation

\[ \Pi(w_{i-2}, w_{i-1}) = \begin{cases} 
1 & c(w_{i-2}, w_{i-1}) = 0 \\
2 & 1 \leq c(w_{i-2}, w_{i-1}) \leq 2 \\
3 & 3 \leq c(w_{i-2}, w_{i-1}) \leq 10 \\
4 & \text{otherwise} 
\end{cases} \]

\[ q_{LI}(w_i \mid w_{i-2}, w_{i-1}) = \lambda_{\Pi(w_{i-2}, w_{i-1})} \times q(w_i \mid w_{i-2}, w_{i-1}) + \lambda_{\Pi(w_{i-2}, w_{i-1})} \times q(w_i \mid w_{i-1}) + \lambda_{\Pi(w_{i-2}, w_{i-1})} \times q(w_i) \]

\[ \lambda_{\Pi(w_{i-2}, w_{i-1})} \geq 0 \]

\[ \lambda_1^{\Pi(w_{i-2}, w_{i-1})} + \lambda_2^{\Pi(w_{i-2}, w_{i-1})} + \lambda_3^{\Pi(w_{i-2}, w_{i-1})} = 1 \]
Many more n-gram methods

• Backoff methods

• Other advanced smoothing techniques
Unknown words

• What if we see a completely new word at test time?
  • \( p(s) = p(S) = 0 \rightarrow \) infinite perplexity

• Solution: create a special token \(<unk>\)
  • Fix vocabulary \( V \) and replace any word in the training set not in \( V \) with \(<unk>\)
  • Train
  • At test time, use \( p(<unk>) \) for words not in the training set
Summary and re-cap

- **Text probability**: decompose with the chain rule
- Use a Markov independence assumption
- Smooth estimates to do better on rare events
- Many attempts to improve LMs using syntax but not easy
- More complex smoothing methods exist for handling rare events (Kneser-Ney, Good-Turing…)
Problem

- Our estimator $q(w | u, v)$ is based on a one-hot representation. There is no relation between words.
  - $p(\text{played} | \text{the, actress})$
  - $p(\text{played} | \text{the actor})$
- Can we use distributed representations?

Neural networks
Neural networks

- Feed-forward neural networks for language modeling
- Recurrent neural networks for language modeling
- Vanishing/exploding gradients
Neural networks: history

• Proposed in the mid-20th century
  • Criticism from Minsky (“Perceptrons”)

• Back-propagation appeared in the 1980s
  • In the 1990s SVMs appeared and were more successful

• Since the beginning of this decade show great success in speech, vision, language, robotics…
Motivation 1

Can we learn representations from raw data?
Motivation 2

Can we learn non-linear decision boundaries

Actually very similar to motivation 1
A single neuron

- A neuron is a computational unit of the form

$$f_{w,b}(x) = f(w^T x + b)$$

- $x$: input vector
- $w$: weights vector
- $b$: bias
- $f$: activation function

A single neuron
A computational unit with $n$ (3) inputs and 1 output and parameters $W, b$

Bias unit corresponds to intercept term
A single neuron

- If $f$ is sigmoid, a neuron is logistic regression

- Let $x, y$ be a binary classification training example

$$p(y = 1 \mid x) = \frac{1}{1 + e^{-w^\top x - b}} = \sigma(w^\top x + b)$$

Provides a linear decision boundary
Single layer network

- Perform logistic regression in parallel multiple times (with different parameters)

\[ y = \sigma(\hat{w}^\top \hat{x} + b) = \sigma(w^\top x) \]

\[ x, w \in \mathbb{R}^{d+1}, x_{d+1} = 1, w_{d+1} = b \]
Single layer network

- L1: input layer
- L2: hidden layer
- L3: output layer
- Output layer provides the prediction
- Hidden layer is the **learned** representation
Multi-layer network

Repeat:
Matrix notation

\[ a_1 = f(W_{11}x_1 + W_{12}x_2 + W_{13}x_3 + b_1) \]
\[ a_2 = f(W_{21}x_1 + W_{22}x_2 + W_{23}x_3 + b_2) \]
\[ a_3 = f(W_{31}x_1 + W_{32}x_2 + W_{33}x_3 + b_3) \]
\[ z = Wx + b, \quad a = f(z) \]
\[ x \in \mathbb{R}^4, \quad z \in \mathbb{R}^3, \quad W \in \mathbb{R}^{3\times4} \]
Language modeling with NNs

• Keep the Markov assumption

• Learn a probability $q(u \mid v, w)$ with distributed representations
Language modeling with NNs

\[ e(w) = W_e \cdot w, \quad W_e \in \mathbb{R}^{d \times |\mathcal{V}|}, \quad w \in \mathbb{R}^{1 \times |\mathcal{V}|} \]

\[ h(w_{i-2}, w_{i-1}) = \sigma(W_h[e(w_{i-2}); e(w_{i-1})]), \quad W_h \in \mathbb{R}^{m \times 2d} \]

\[ f(z) = \text{softmax}(W_o \cdot z), \quad W_o \in \mathbb{R}^{1 \times m}, \quad z \in \mathbb{R}^{m \times 1} \]

\[ p(w_i \mid w_{i-2}, w_{i-1}) = f(h(w_{i-2}, w_{i-1}))_i \]
Loss function

• Minimize negative log-likelihood

• When we learned word vectors, we were basically training a language model

\[ L(\theta) = - \sum_{i=1}^{T} \log p_{\theta}(w_i \mid w_{i-2}, w_{i-1}) \]
Advantages

• If we see in the training set

   *The cat is walking in the bedroom*

• We can hope to learn

   *A dog was running through a room*

• We can use n-grams with n > 3 and pay only a linear price

• Compared to what?
Parameter estimation

- We train with SGD
- How to efficiently compute the gradients?
  - **backpropagation** (Rumelhart, Hinton and Williams, 1986)
  - Proof in intro to ML, will repeat algorithm only and give an example here
Backpropagation

• Notation:

\[ W_t \]: weight matrix at the input of layer \( t \)
\[ z_t \]: output vector at layer \( t \)
\[ x = z_0 \]: input vector
\[ y \]: gold scalar
\[ \hat{y} = z_L \]: predicted scalar
\[ l(y, \hat{y}) \]: loss function
\[ v_t = W_t \cdot z_{t-1} \]: pre-activations
\[ \delta_t = \frac{\partial l(y, \hat{y})}{\partial z_t} \]: gradient vector
Backpropagation

- Run the network forward to obtain all values $v_t, z_t$

- Base:
  \[ \delta_L = l'(y, z_L) \]

- Recursion:
  \[
  \delta_t = W_{t+1}^\top (\sigma'(v_{t+1}) \circ \delta_{t+1})
  \]
  \[
  \sigma'(v_{t+1}), \delta_{t+1} \in \mathbb{R}^{d_{t+1} \times 1}, W_{t+1} \in \mathbb{R}^{d_{t+1} \times d_t}
  \]

- Gradients:
  \[
  \frac{\partial l}{\partial W_t} = (\delta_t \circ \sigma'(v_t))z_{t-1}^\top
  \]
Bigram LM example

- Forward pass:

\[ z_0 \in \mathbb{R}^{|\mathcal{V}| \times 1} : \text{one-hot vector input} \]

\[ z_1 = W_1 \cdot z_0, W_1 \in \mathbb{R}^{d_1 \times |\mathcal{V}|}, z_1 \in \mathbb{R}^{d_1 \times 1} \]

\[ z_2 = \sigma(W_2 \cdot z_1), W_2 \in \mathbb{R}^{d_2 \times d_1}, z_2 \in \mathbb{R}^{d_2 \times 1} \]

\[ z_3 = \text{softmax}(W_3 \cdot z_2), W_3 \in \mathbb{R}^{|\mathcal{V}| \times d_2}, z_3 \in \mathbb{R}^{|\mathcal{V}| \times 1} \]

\[ l(y, z_3) = \sum_i y^{(i)} \log z_3^{(i)} \]
Bigram LM example

• Backward pass:

\[ \sigma'(v_3) \circ \delta_3 = (z_3 - y) \]

\[ \delta_2 = W_3^\top (\sigma'(v_3) \circ \delta_3) = W_3^\top (z_3 - y) \]

\[ \delta_1 = W_2^\top (\sigma'(v_2) \circ \delta_2) = W_2^\top (z_2 \circ (1 - z_2) \circ \delta_2) \]

\[ \frac{\partial l}{\partial W_3} = (\delta_3 \circ \sigma'(v_3)) z_2^\top = (z_3 - y) z_2^\top \]

\[ \frac{\partial l}{\partial W_2} = (\delta_2 \circ \sigma'(v_2)) z_1^\top = (\delta_2 \circ z_2 \circ (1 - z_2)) z_1^\top \]

\[ \frac{\partial l}{\partial W_1} = (\delta_1 \circ \sigma'(v_1)) z_0^\top = \delta_1 z_0^\top \]
Summary

• Neural nets can improve language models:
  • better scalability for larger N
  • use of word similarity
  • complex decision boundaries
• Training through backpropagation
Summary

• Neural nets can improve language models:
  • better scalability for larger N
  • use of word similarity
  • complex decision boundaries
  • Training through backpropagation

But we still have a Markov assumption

He is from France, so it makes sense that his first language is…
Recurrent neural networks

Input: \( w_1, \ldots, w_{t-1}, w_t, w_{t+1}, \ldots, w_T \), \( w_i \in \mathbb{R}^V \)

Model: \( x_t = W^{(e)} \cdot w_t \), \( W^{(e)} \in \mathbb{R}^{d \times V} \)
\[
    h_t = \sigma(W^{(hh)} \cdot h_{t-1} + W^{(hx)} \cdot x_t), \quad W^{(hh)} \in \mathbb{R}^{D_h \times D_h}, \quad W^{(hx)} \in \mathbb{R}^{D_h \times d}
\]
\[
    \hat{y}_t = \text{softmax}(W^{(s)} \cdot h_t), \quad W^{(s)} \in \mathbb{R}^{V \times D_h}
\]
Recurrent neural networks
Recurrent neural networks

- Can exploit long range dependencies
- Each layer has the same weights (weight sharing/tying)
- What is the loss function?
Recurrent neural networks

- Can exploit long range dependencies
- Each layer has the same weights (weight sharing/tying)
- What is the loss function?

\[ J(\theta) = \sum_{t=1}^{T} \text{CE}(y_t, \hat{y}_t) \]
Recurrent neural networks

- Component:
  - model? RNN (saw before)
  - Loss? sum of cross-entropy over time
  - Optimization? SGD
  - Gradient computation? back-propagation through time
## Comparison

<table>
<thead>
<tr>
<th>Model</th>
<th>Similarity</th>
<th>Unlimited horizon</th>
</tr>
</thead>
<tbody>
<tr>
<td>n-gram</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>FFNN LM</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>RNN LM</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>
Home assignment 2

- Implement:
  - n-gram language model
  - FFNN language model (+ backpropagation)
  - RNN language model (using standard libraries)
Class 4
Language models

- Estimate $p(x)$ for a sentence $x$. Commonly factorized as $p(x_n \mid x_1...x_{n-1})$
Recap: n-gram LM

• Given a corpus of length M:

  • Trigram model:
    \[ q(w_i \mid w_{i-2}, w_{i-1}) = \frac{c(w_{i-2}, w_{i-1}, w_i)}{c(w_{i-1}, w_i)} \]

  • Bigram model:
    \[ q(w_i \mid w_{i-1}) = \frac{c(w_{i-1}, w_i)}{c(w_{i-1})} \]

  • Unigram model:
    \[ q(w_i) = \frac{c(w_i)}{M} \]
Recap: n-gram LM

\[ \Pi(w_{i-2}, w_{i-1}) = \begin{cases} 
1 & c(w_{i-2}, w_{i-1}) = 0 \\
2 & 1 \leq c(w_{i-2}, w_{i-1}) \leq 2 \\
3 & 3 \leq c(w_{i-2}, w_{i-1}) \leq 10 \\
4 & \text{otherwise} 
\end{cases} \]

\[ q_{LI}(w_i \mid w_{i-2}, w_{i-1}) = \lambda_1^{\Pi(w_{i-2}, w_{i-1})} \times q(w_i \mid w_{i-2}, w_{i-1}) + \lambda_2^{\Pi(w_{i-2}, w_{i-1})} \times q(w_i \mid w_{i-1}) + \lambda_3^{\Pi(w_{i-2}, w_{i-1})} \times q(w_i) \]

\[ \lambda_i^{\Pi(w_{i-2}, w_{i-1})} \geq 0 \]

\[ \lambda_1^{\Pi(w_{i-2}, w_{i-1})} + \lambda_2^{\Pi(w_{i-2}, w_{i-1})} + \lambda_3^{\Pi(w_{i-2}, w_{i-1})} = 1 \]
Recap: n-gram LM

- Training: count stuff on the training set
- Tuning: optimize linear interpolation coefficients over development set (minimizing perplexity)
- Testing: on the test set (don’t look!!!)
Recap: Feed-forward LMs

\[ e(w) = W_e \cdot w, \quad W_e \in \mathbb{R}^{d \times |\mathcal{V}|}, w \in \mathbb{R}^{|\mathcal{V}| \times 1} \]

\[ h(w_{i-2}, w_{i-1}) = \sigma(W_h[e(w_{i-2}); e(w_{i-1})]), \quad W_h \in \mathbb{R}^{m \times 2d} \]

\[ f(z) = \text{softmax}(W_o \cdot z), \quad W_o \in \mathbb{R}^{|\mathcal{V}| \times m}, z \in \mathbb{R}^{m \times 1} \]

\[ p(w_i | w_{i-2}, w_{i-1}) = f(h(w_{i-2}, w_{i-1}))_i \]

\[ p(\text{laughed} | \text{the, dog}) \]
Recap: Recurrent LMs

Input:  $w_1, \ldots, w_{t-1}, w_t, w_{t+1}, \ldots, w_T$, $w_i \in \mathbb{R}^V$

Model:  $x_t = W^{(e)} \cdot w_t$, $W^{(e)} \in \mathbb{R}^{d \times V}$

$h_t = \sigma(W^{(hh)} \cdot h_{t-1} + W^{(hx)} \cdot x_t)$, $W^{(hh)} \in \mathbb{R}^{D_h \times D_h}$, $W^{(hx)} \in \mathbb{R}^{D_h \times d}$

$\hat{y}_t = \text{softmax}(W^{(s)} \cdot h_t)$, $W^{(s)} \in \mathbb{R}^{V \times D_h}$
Recap: Loss function and training

- Maximize log probability of running text of length T
- Optimize parameters using backpropagation
Recap: Comparison

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- **Problem**: RNNs don’t do better than FFNN
Training RNNs

• Capturing long-range dependencies with RNNs is difficult

  • Vanishing/exploding gradients

  • Small changes in hidden layer value in step $k$ can cause huge/minuscule changes to values in hidden layer $t$ for $t \gg k$
Explanation

Consider a simple linear RNN with no input:

\[ h_t = W \cdot h_{t-1} \]
\[ h_t = W^t \cdot h_0 \]
\[ h_t = (Q\Lambda Q^{-1})^t \cdot h_0 \]
\[ h_t = (Q\Lambda^t Q^{-1}) \cdot h_0 \]

where \( W = Q\Lambda Q^{-1} \) is an eigendecomposition

so if \( h_0 = a_1 q_1 + \cdots + a_n q_n \)

then \( h_t = a_1 \lambda_1^t q_1 + \cdots + a_n \lambda_n^t q_n \)

- Some eigenvalues will explode and some will shrink to zero
  - Stretch input in the direction of \( q_1 \)
Explanation

\[ \frac{\partial J^{(4)}}{\partial h^{(1)}} = \frac{\partial h^{(2)}}{\partial h^{(1)}} \times \frac{\partial h^{(3)}}{\partial h^{(2)}} \times \frac{\partial h^{(4)}}{\partial h^{(3)}} \times \frac{\partial J^{(4)}}{\partial h^{(4)}} \]

What happens if these are small?

Vanishing gradient problem: When these are small, the gradient signal gets smaller and smaller as it backpropagates further.
Explanation

\[ h_t = W^{(hh)} \sigma(h_{t-1}) + W^{(hx)} x_t + b \]

\[ \mathcal{L} = \sum_{t=1}^{T} \mathcal{L}_t = \sum_{t=1}^{T} \mathcal{L}(h_t) \]
Explanation

\[ k=2, \ t=4 \]

\[
\begin{array}{c}
\text{h}_1 \\
\frac{\partial^+ h_2}{\partial \theta} \\
\frac{\partial \ell}{\partial \theta} = \sum_{t=1}^{T} \frac{\partial \ell_t}{\partial \theta} \\
\frac{\partial \ell_t}{\partial \theta} = \sum_{k=1}^{t} \frac{\partial \ell_t}{\partial h_t} \frac{\partial h_k}{\partial h_t} \frac{\partial^+ h_k}{\partial \theta} \\
\frac{\partial h_t}{\partial h_k} = \prod_{i=k+1}^{t} \frac{\partial h_i}{\partial h_{i-1}} = \prod_{i=k+1}^{t} W^{(hh)} \text{diag}(\sigma'(h_{i-1}))
\end{array}
\]
Assume: $\sigma'(\cdot) \leq \gamma, \lambda_1 \leq \frac{1}{\gamma}$.

$\lambda_1$ is the absolute value of the largest eigenvalue of $W^{(hh)}$

\[
\forall k \left\| \frac{\partial h_k}{\partial h_{k-1}} \right\| \leq \left\| W^{(hh)} \right\| \cdot \left\| \text{diag}(\sigma'(h_k)) \right\| < \frac{1}{\gamma} < 1
\]

Let $\eta$ be a constant such that $\forall k \left\| \frac{\partial h_k}{\partial h_{k-1}} \right\| \leq \eta < 1$

\[
\left\| \frac{\partial \mathcal{L}_t}{\partial h_t} \prod_{i=k+1}^t \frac{\partial h_i}{\partial h_{i-1}} \right\| \leq \eta^{t-k} \left\| \frac{\partial \mathcal{L}_t}{\partial h_t} \right\|
\]

so long-term influence vanishes to zero.
Solutions

• Exploding gradient: gradient clipping
  • Re-normalize gradient to be less than C (this is no longer the true gradient in size but it is in direction)
  • Exploding gradients are easy to detect
• The problem is with the model!
  • Change it (LSTMs, GRUs)
Illustration
LSTMs and GRUs
LSTMs and GRUs

• **Bottom line:** use vector addition and not matrix-vector multiplication. Allows for better propagation of gradients to the past
Gated Recurrent Unit (GRU)

- **Main insight**: add learnable gates to the recurrent unit that control the flow of information from the past to the present

- **Vanilla RNN**:
  \[
  h_t = f(W^{(hh)} h_{t-1} + W^{(hx)} x_t)
  \]

- **Update and reset gates**:
  \[
  z_t = \sigma(W^{(z)} h_{t-1} + U^{(z)} x_t)
  \]
  \[
  r_t = \sigma(W^{(r)} h_{t-1} + U^{(r)} x_t)
  \]

Cho et al., 2014
Gated Recurrent Unit (GRU)

\[ z_t = \sigma(W^{(z)} x_t + U^{(z)} h_{t-1}) \]
\[ r_t = \sigma(W^{(r)} x_t + U^{(r)} h_{t-1}) \]
\[ \tilde{h}_t = \tanh(W x_t + r_t \circ U h_{t-1}) \]
\[ h_t = z_t \circ h_{t-1} + (1 - z_t) \circ \tilde{h}_t \]

- Use the gates to control information flow
  - If \( z=1 \), we simply copy the past and ignore the present (note that gradient will be 1)
  - If \( z=0 \), then we have an RNN like update, but we are also free to reset some of the past units, and if \( r=0 \), then we have no memory of the past
  - The + in the last equation is crucial
Illustration

Diagram with nodes labeled $\tilde{h}_t$, $\tilde{h}_t$, $h_{t-1}$, $h_t$, $r_t$, $Z_t$, and $X_t$. Arrows connect these nodes in a specific pattern.
More GRU Intuition

Vanilla-RNN

Registers $h$

Execution
1. Read the whole register $h$
2. Update the whole register

$h \leftarrow \tanh(W[x] + Uh + b)$

GRU

Registers $h$

Execution
1. Select a readable subset $r$
2. Read the subset $r \odot h$
3. Select a writable subset $u$
4. Update the subset

$h \leftarrow u \odot \tilde{h} + (1 - u_t) \odot h$
Long short term memory (LSTMs)

- $z$ has been split into $i$ and $f$
- There is no $r$
- There is a new gate $o$ that distinguishes between the memory and the output.
- $c$ is like $h$ in GRUs
- $h$ is the output

\[
\begin{align*}
    i_t &= \sigma(W^{(i)}x_t + U^{(i)}h_{t-1}) \\
    f_t &= \sigma(W^{(f)}x_t + U^{(f)}h_{t-1}) \\
    o_t &= \sigma(W^{(o)}x_t + U^{(o)}h_{t-1}) \\
    \tilde{c}_t &= \tanh(W^{(c)}x_t + U^{(c)}h_{t-1}) \\
    c_t &= f_t \circ c_{t-1} + i_t \circ \tilde{c}_t \\
    h_t &= o_t \circ \tanh(c_t)
\end{align*}
\]
Illustration

Write some new cell content

Forget some cell content

Compute the forget gate

Compute the input gate

Compute the new cell content

Compute the output gate

Output some cell content to the hidden state
## Results

<table>
<thead>
<tr>
<th>Model</th>
<th>Test Perplexity</th>
<th>Number of Params [Billions]</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>SIGMoid-RNN-2048 (Ji et al., 2015a)</strong></td>
<td>68.3</td>
<td>4.1</td>
</tr>
<tr>
<td><strong>Interpolated KN 5-gram, 1.1B n-grams (Chelba et al., 2013)</strong></td>
<td>67.6</td>
<td>1.76</td>
</tr>
<tr>
<td><strong>Sparse Non-Negative Matrix LM (Shazeer et al., 2015)</strong></td>
<td>52.9</td>
<td>33</td>
</tr>
<tr>
<td><strong>RNN-1024 + MaxEnt 9-gram features (Chelba et al., 2013)</strong></td>
<td>51.3</td>
<td>20</td>
</tr>
<tr>
<td><strong>LSTM-512-512</strong></td>
<td>54.1</td>
<td>0.82</td>
</tr>
<tr>
<td><strong>LSTM-1024-512</strong></td>
<td>48.2</td>
<td>0.82</td>
</tr>
<tr>
<td><strong>LSTM-2048-512</strong></td>
<td>43.7</td>
<td>0.83</td>
</tr>
<tr>
<td><strong>LSTM-8192-2048 (No Dropout)</strong></td>
<td>37.9</td>
<td>3.3</td>
</tr>
<tr>
<td><strong>LSTM-8192-2048 (50% Dropout)</strong></td>
<td>32.2</td>
<td>3.3</td>
</tr>
<tr>
<td><strong>2-Layer LSTM-8192-1024 (Big LSTM)</strong></td>
<td>30.6</td>
<td>1.8</td>
</tr>
<tr>
<td><strong>Big LSTM+CNN Inputs</strong></td>
<td><strong>30.0</strong></td>
<td><strong>1.04</strong></td>
</tr>
<tr>
<td><strong>Big LSTM+CNN Inputs + CNN Softmax</strong></td>
<td>39.8</td>
<td>0.29</td>
</tr>
<tr>
<td><strong>Big LSTM+CNN Inputs + CNN Softmax + 128-dim Correction</strong></td>
<td>35.8</td>
<td>0.39</td>
</tr>
<tr>
<td><strong>Big LSTM+CNN Inputs + Char LSTM Predictions</strong></td>
<td>47.9</td>
<td>0.23</td>
</tr>
</tbody>
</table>
Summary

- Language modeling is a fundamental NLP task used in machine translation, spelling correction, speech recognition, etc.
- Traditional models use n-gram counts and smoothing
- Feed-forward take into account word similarity to generalize better
- Recurrent models can potentially learn to exploit long-range interactions
- Neural models dramatically reduced perplexity
- Recurrent networks are now used in many other NLP tasks (bidirectional RNNs, deep RNNs)