A Sublogarithmic Approximation for Highway and Tollbooth Pricing

Iftah Gamzu

Joint work with Danny Segev

OUTLINE

- introduction
- new algorithm
- conclusions

INTRODUCTION

Haverhill
Lawrence
Lowell
Lynn
Medford
Malden
Waltham
Boston
Brookline
Framingham
Worcester
Quincy
Weymouth
Chicopee
Springfield
Brockton
Taunton
Fall River
New Bedford
495
1
95
93
3
7
202
91
190
20
290
90
84
395
44
195
6
Toll Roads
US Highways
Interstate Highways
0
25 KM
25 Miles
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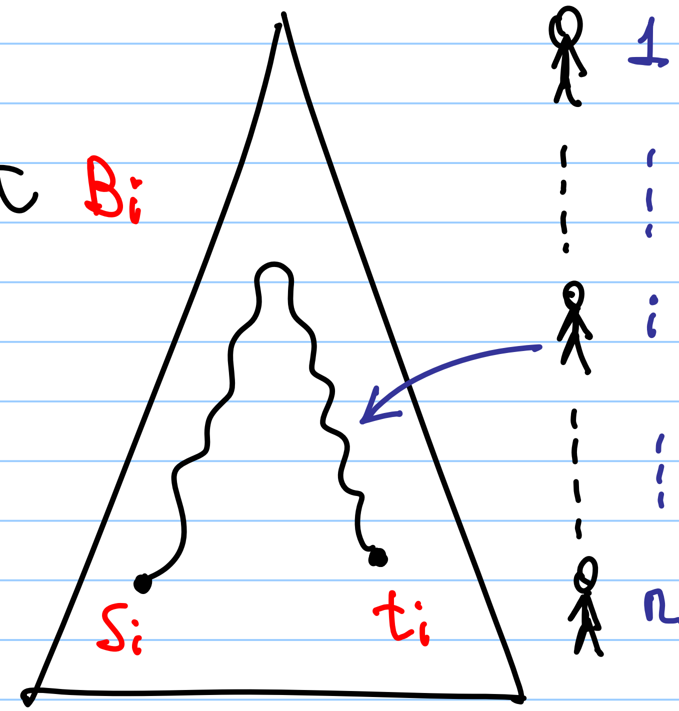
TOLLBOOTH PRICING ON TREES

- Input: tree $T=(V,E)$ on m edges
  n customers
  customer i: path P_i + budget B_i



TOLLBOOTH PRICING ON TREES

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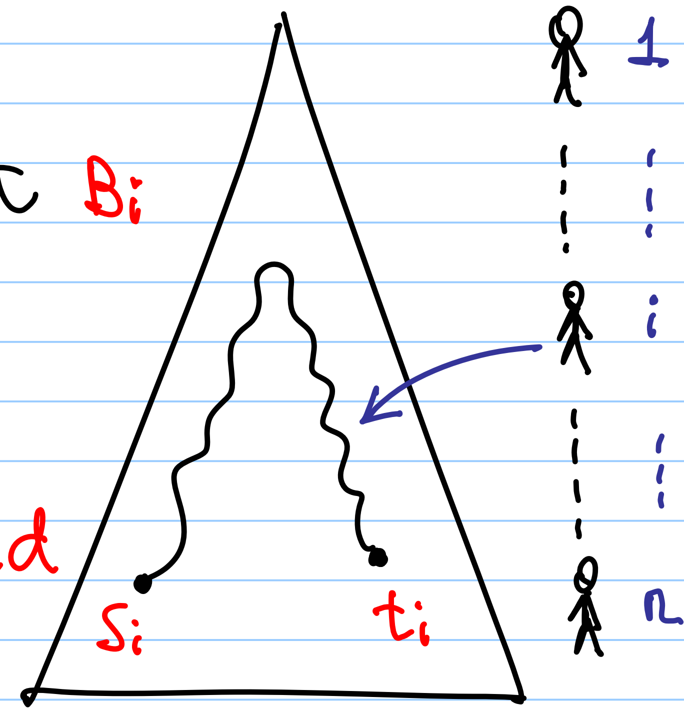
  customer i: path P_i + budget B_i
- goal: assign a per-unit price to each edge such that the overall revenue is maximized



TOLLBOOTH PRICING ON TREES

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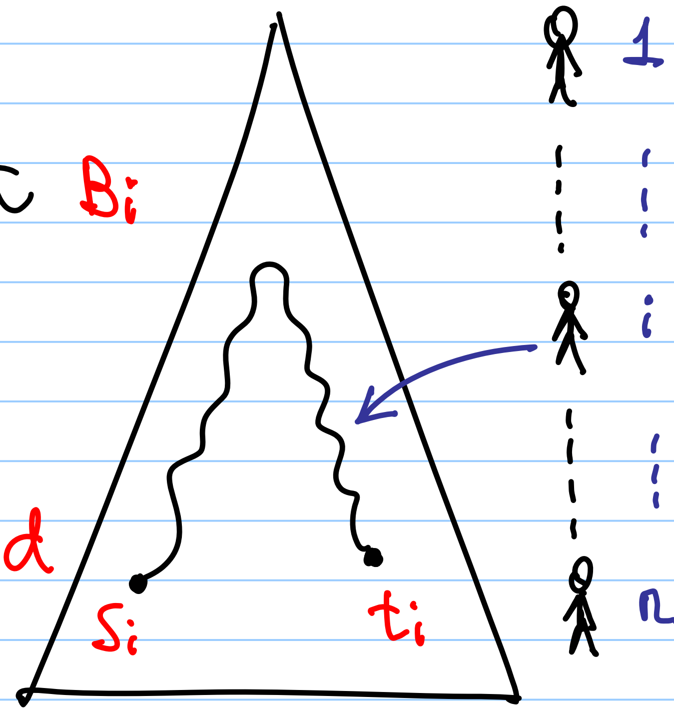
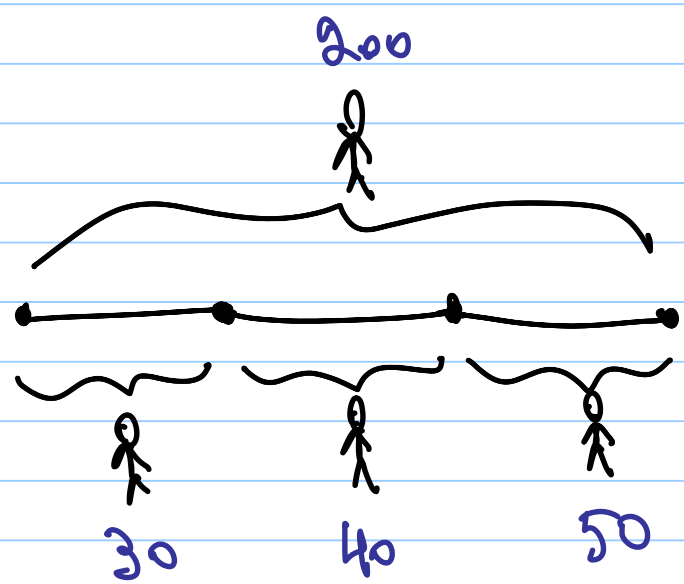
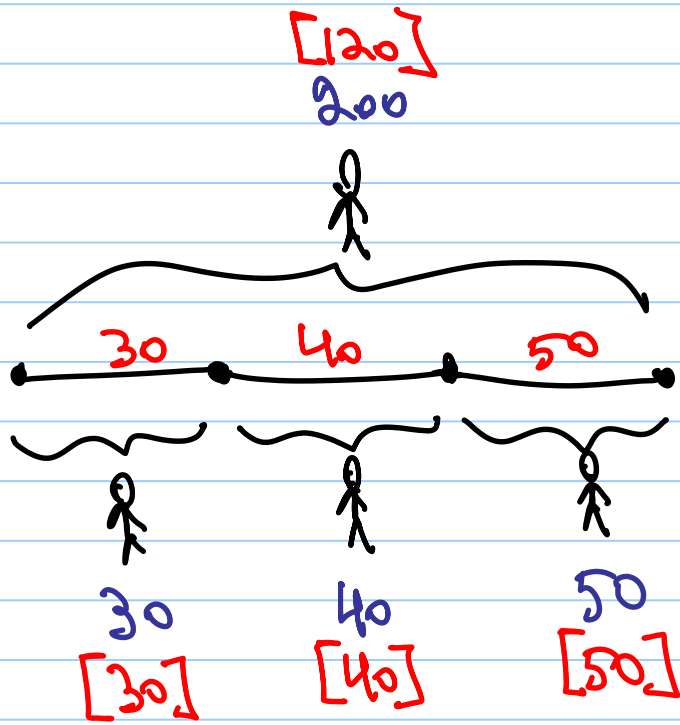


- revenue due to customer i:
  - total price of P_i, when it does not exceed B_i
  - no revenue at all, otherwise

EXAMPLE

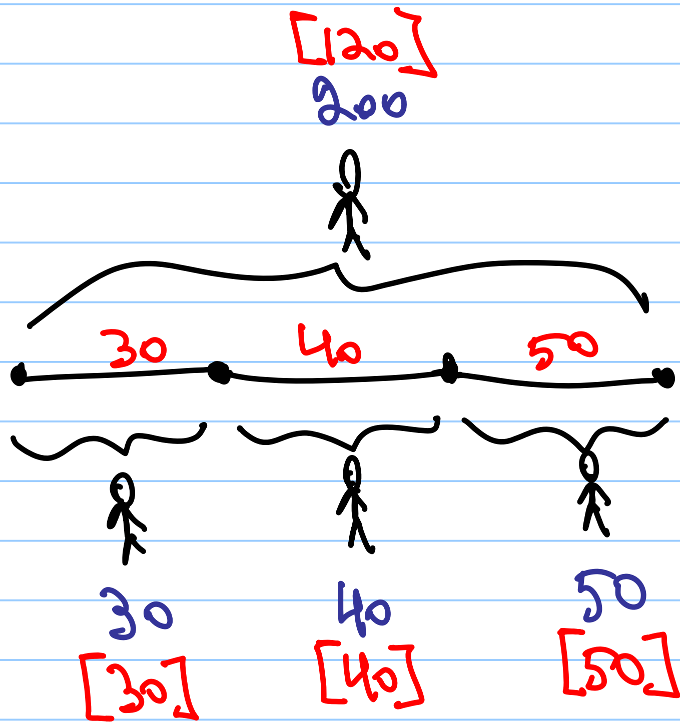


EXAMPLE

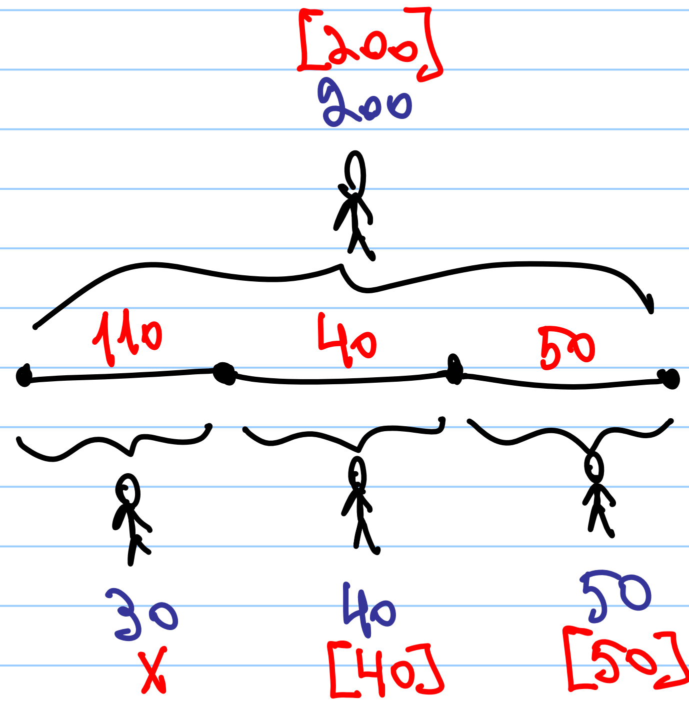


total: 240

EXAMPLE



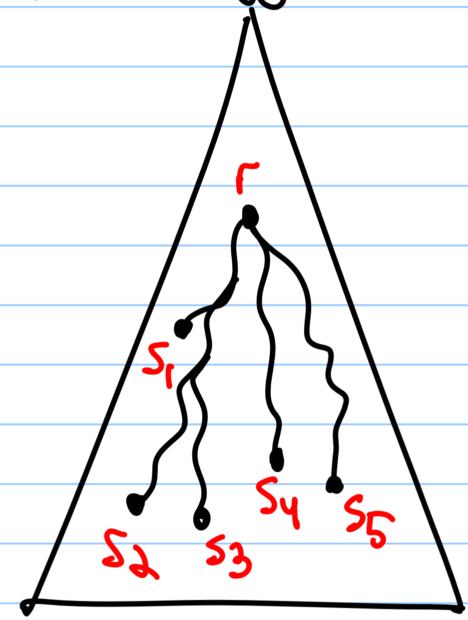
total : 240



total : 290

PREVIOUS WORK

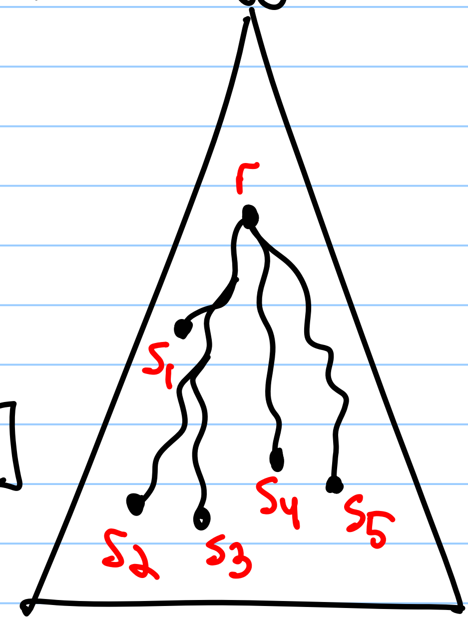
- Tollbooth problem (on trees):
  - APX-hard: Guruswami et al [SODA '05]
  - exact algorithm for single-source case
  - $O(\log m)$-approx: Elbassioni et al. [SAGT '09]



PREVIOUS WORK

- Tollbooth problem (on trees):
  - APX-hard: Guruswami et al [SODA '05]
  - exact algorithm for single-source case
  - $O(\log m)$-approx: Elbassioni et al. [SAGT '09]

- Highway Problem (on simple paths):
  - NP-hard: Briest and Krysta [SODA '06]
  - $O(\log m)$-approx: Balcan and Blum [EC '06]

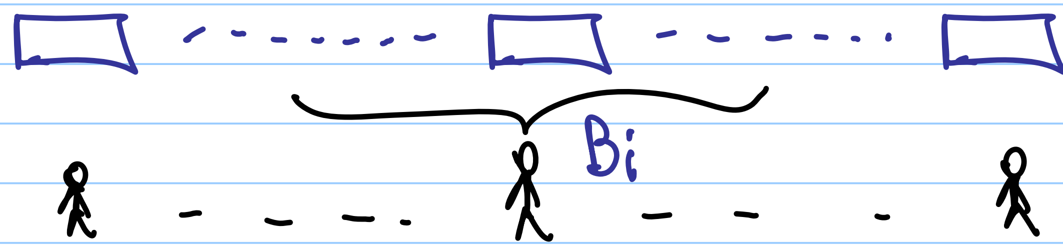


PREVIOUS WORK

- single-minded unlimited supply pricing

m products

n customers



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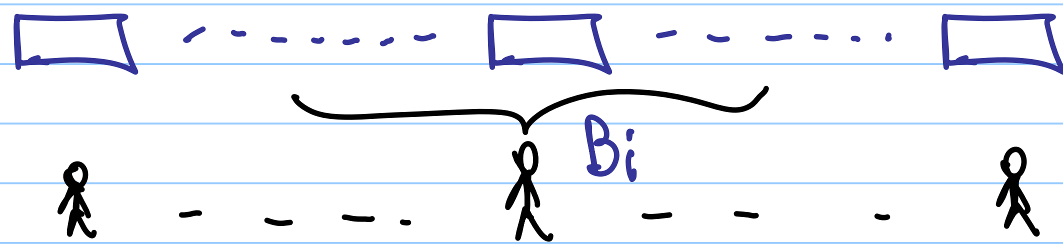


- lots of work; will mention only:
  - $O(\log m + \log n)$-approx: Guruswami et al. [SODA '05]
  - $\Omega(\log m)$ inapproximability: Demaine et al. [SODA '06]

    plausible hardness assumption

MAIN RESULT

deterministic algorithm with ratio

$$O(\log m / \log\log m)$$

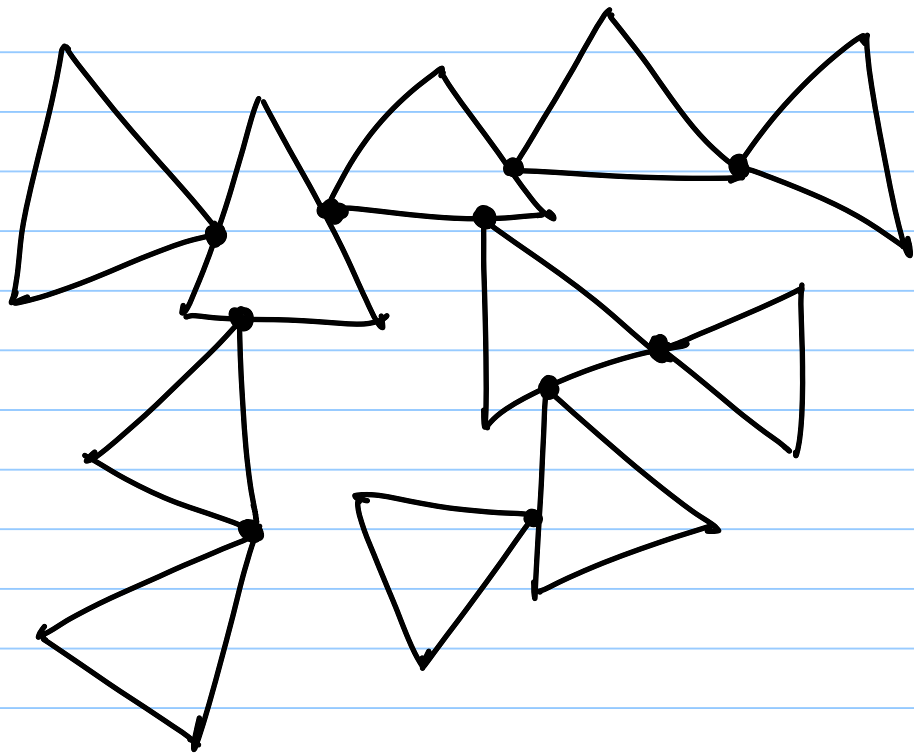
for tollbooth pricing on trees

THE ALGORITHM:

GENERAL IDEA

ALMOST BALANCED K-DECOMPOSITIONS

- any tree $T=(V,E)$ can be decomposed into k edge-disjoint subtrees, each containing $O(|E|/k)$ edges



ALMOST BALANCED K-DECOMPOSITIONS

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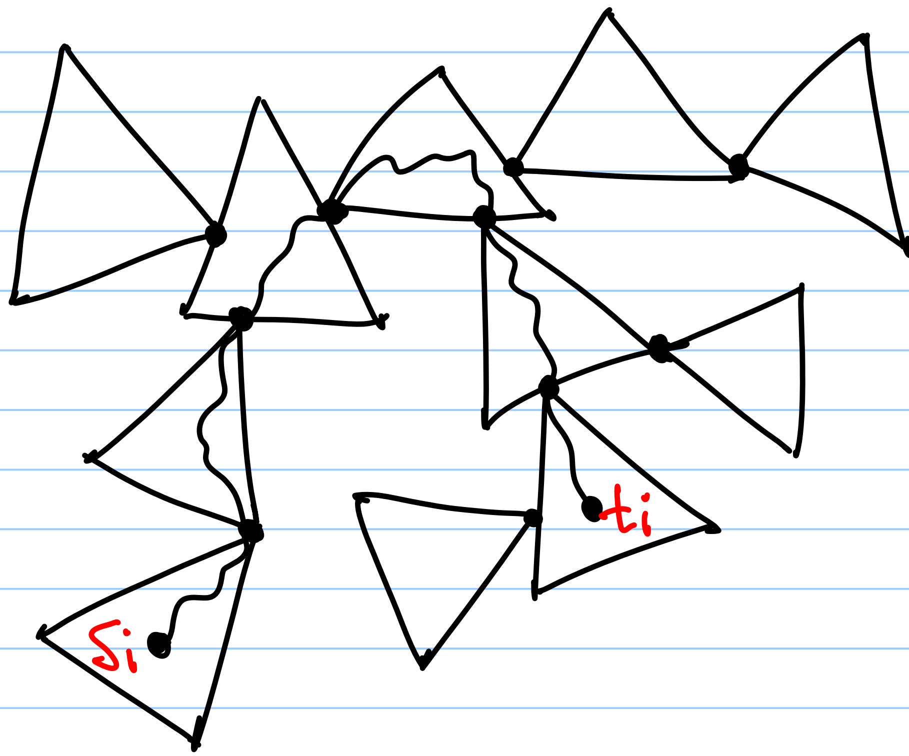
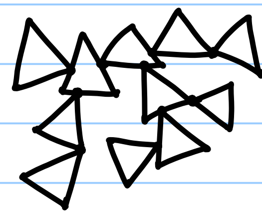


- customer i is separated when s_i and t_i reside in different subtrees

REPEATED K-DECOMPOSITIONS



pick $k = \sqrt{\log m}$

REPEATED K-DECOMPOSITIONS

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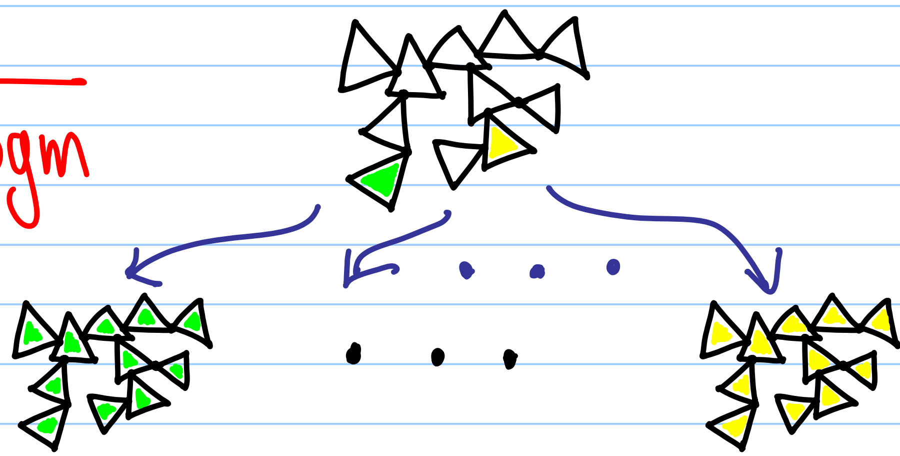

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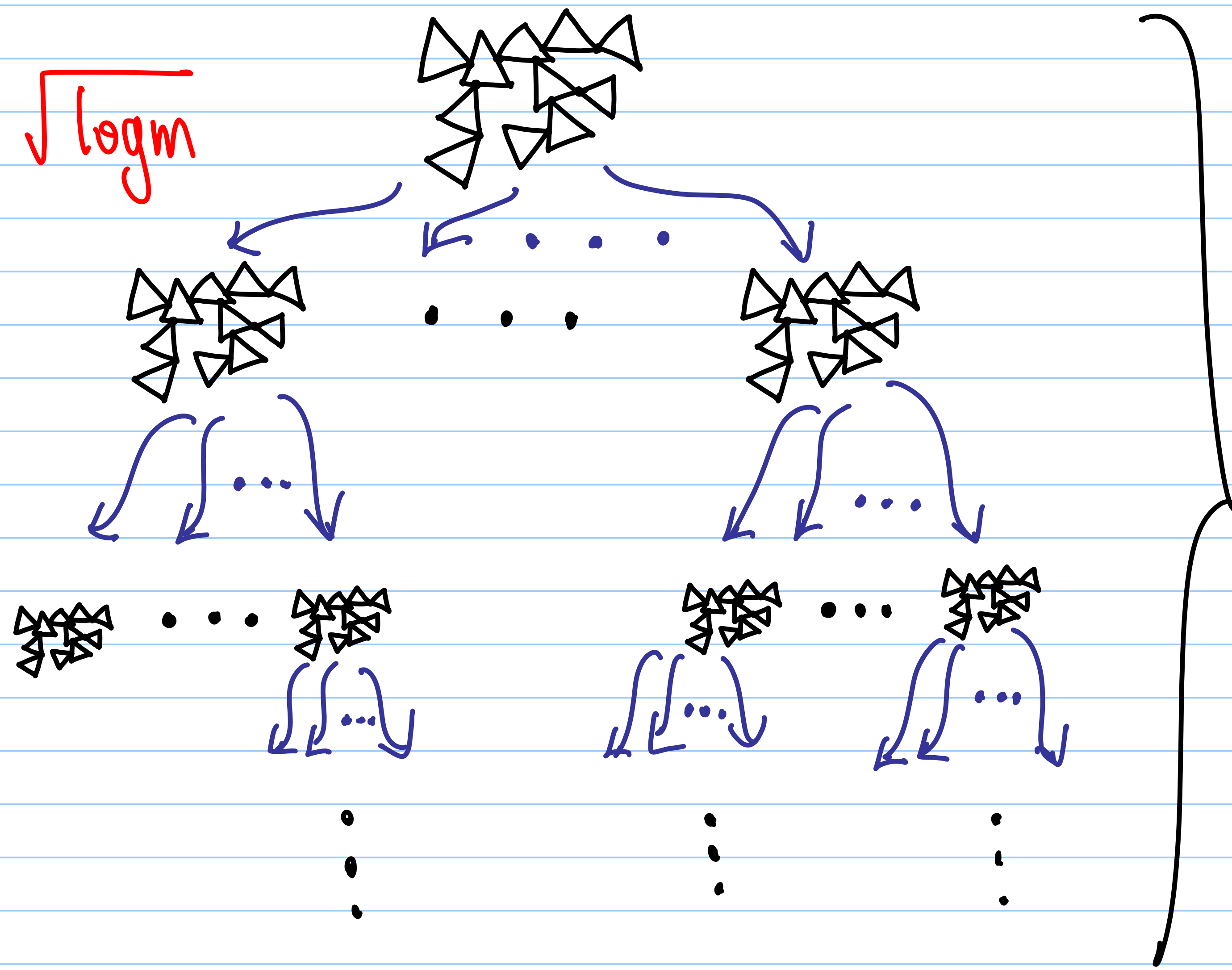
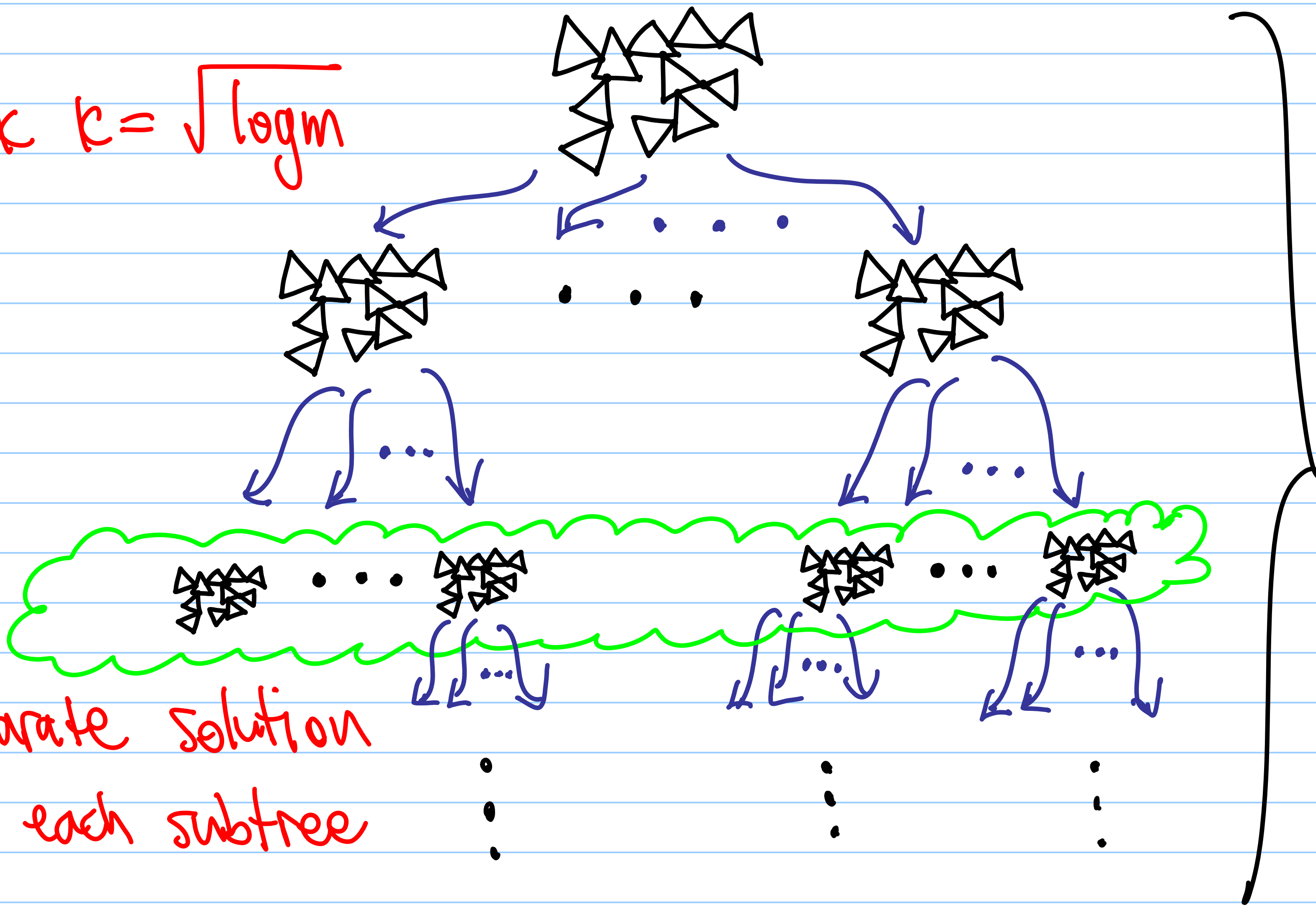


$O\left(\frac{\log m}{\log \log m}\right)$ levels

REPEATED K-DECOMPOSITIONS

pick $k = \sqrt{\log m}$



$O\left(\frac{\log m}{\log \log m}\right)$ levels

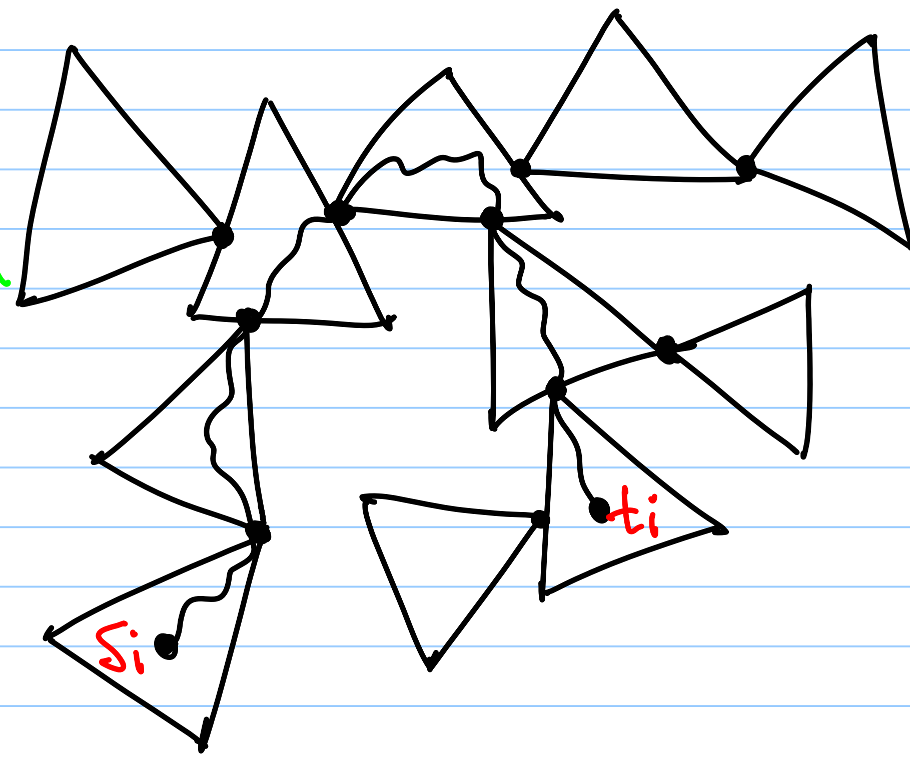
separate solution for each subtree

NEED O(1)-APPROX FOR...

- Input: tree $T=(V,E)$, decomposed into $T_1, \ldots, T_k$
  for each customer i: path P_i + budget B_i

- additional structure:
  customer paths are separated

- goal: find pricing scheme
  $p: E \to \mathbb{R}_+$ that maximizes
  overall revenue



CONSTANT-FACTOR APPROXIMATION

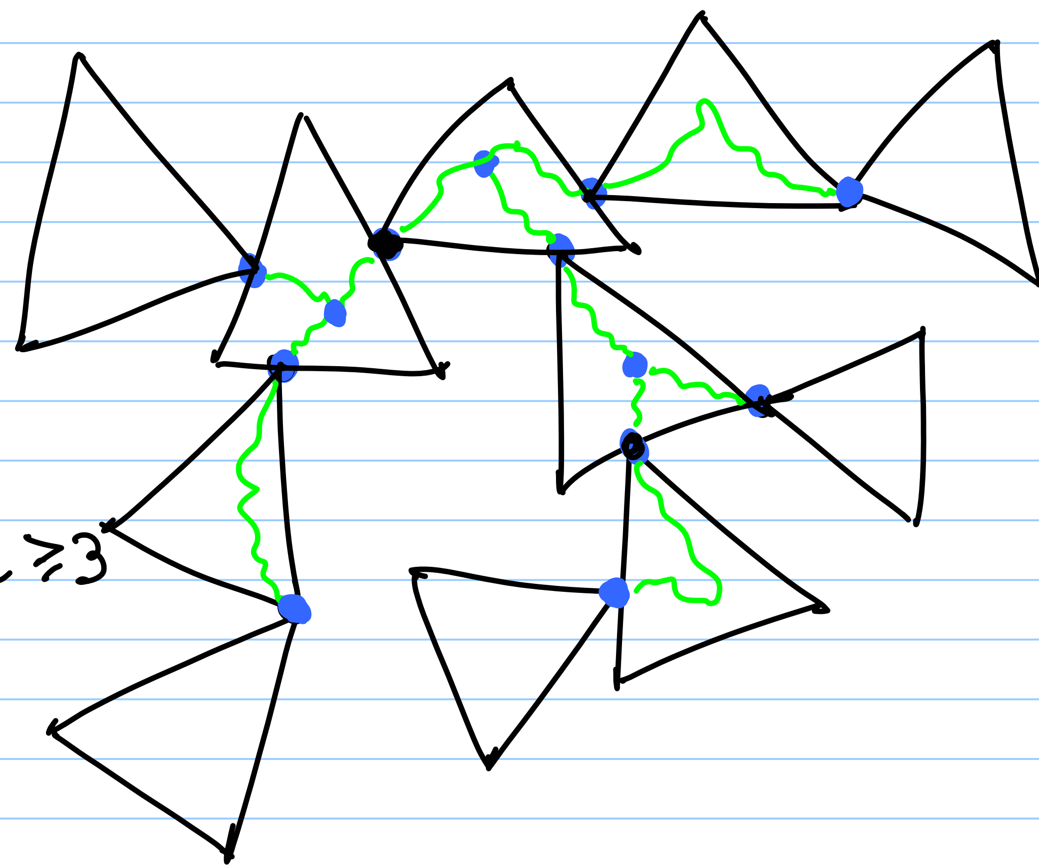
FOR

K-DECOMPOSITIONS

NOTATION AND TERMINOLOGY

- skeleton $\mathcal{S}$ = minimal subtree that spans all intersection vertices

- special vertices =
intersection vertices +
vertices in $\mathcal{S}$ with degree ≥ 3



CASE ANALYSIS

- revenue from customer i :

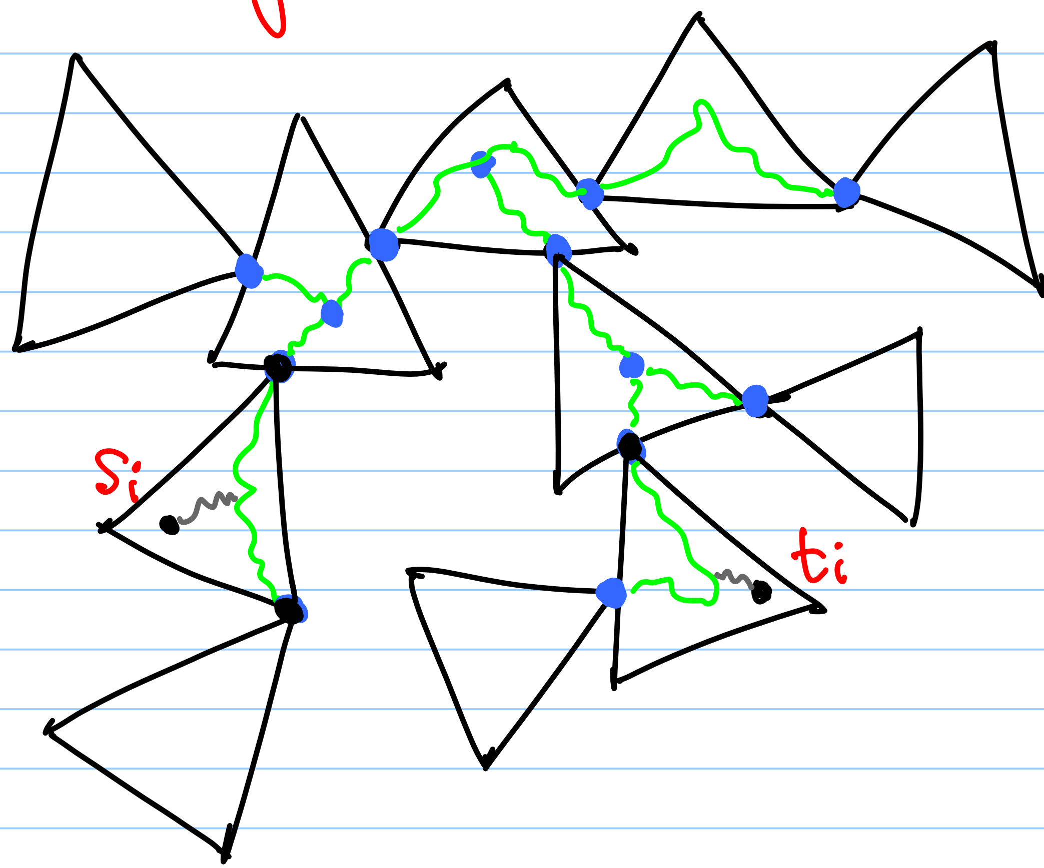
  outside skeleton + along skeleton

- case I :

  total revenue outside skeleton $\geq \frac{OPT}{2}$

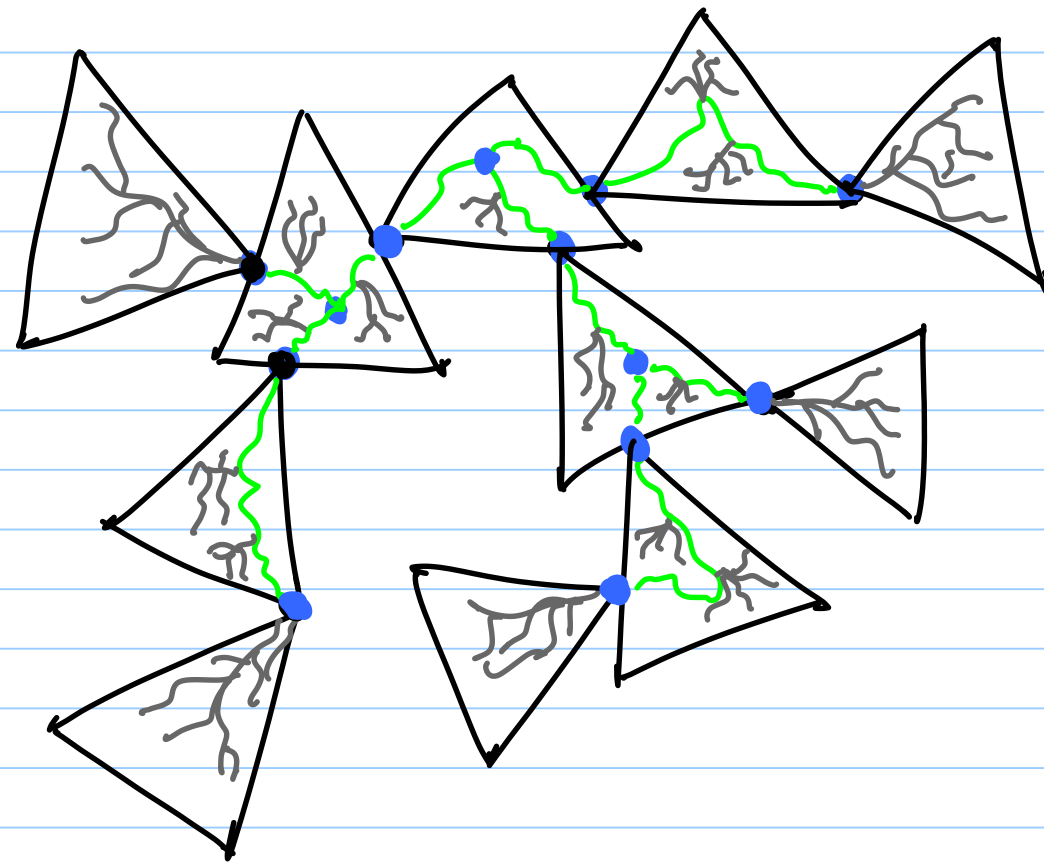
- case II :

  total revenue along skeleton $\geq \frac{OPT}{2}$



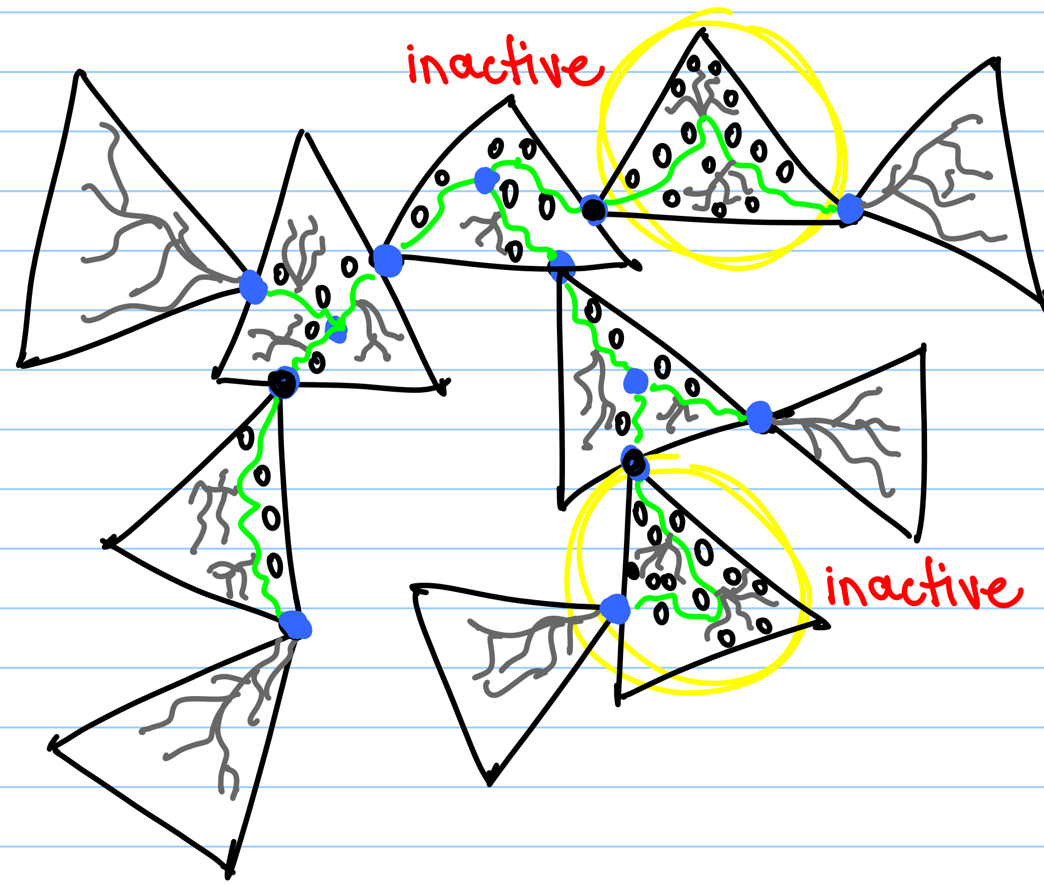
CASE I: HIGH REVENUE OUTSIDE SKELETON

- randomly mark each of $T_1, \ldots, T_k$ as active / inactive



CASE I: HIGH REVENUE OUTSIDE SKELETON

- randomly mark each of $T_1, \dots, T_k$ as active / inactive
- skeleton and inactive tree edges are sold for free



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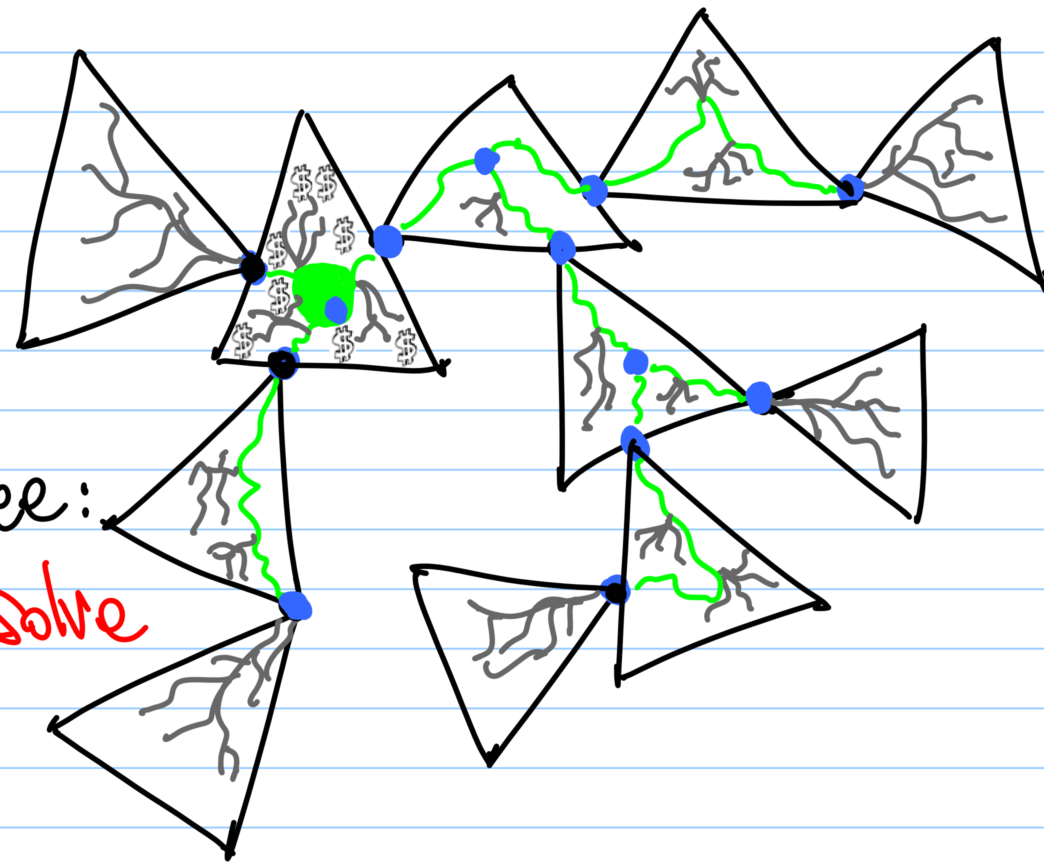


- for each active subtree: contract skeleton and solve single-source instance

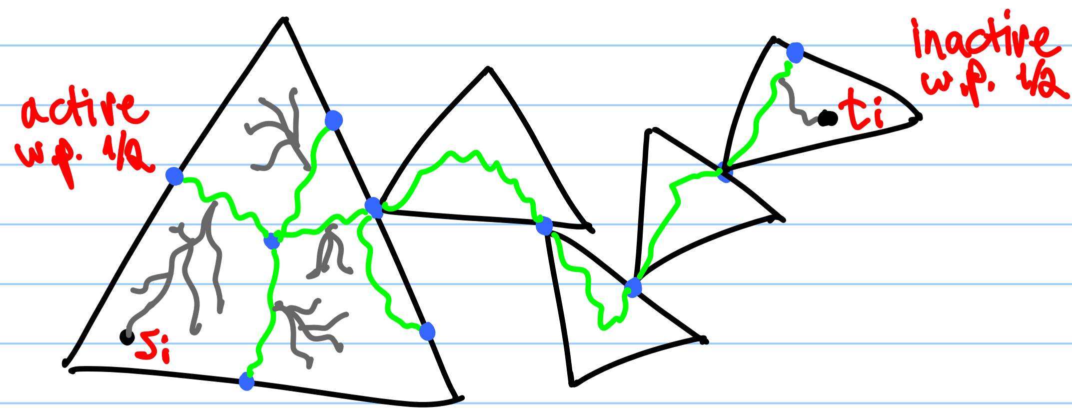
CASE I: ANALYSIS

claim:

$$E\begin{bmatrix}\text{revenue from outside} \\ \text{skeleton of each subtree}\end{bmatrix} \geq \frac{1}{4} \cdot \begin{bmatrix}\text{same revenue} \\ \text{in OPT}\end{bmatrix}$$

proof:

For each subtree that we solve a single-source instance, our outside-skeleton revenue $\geq$ same revenue in OPT



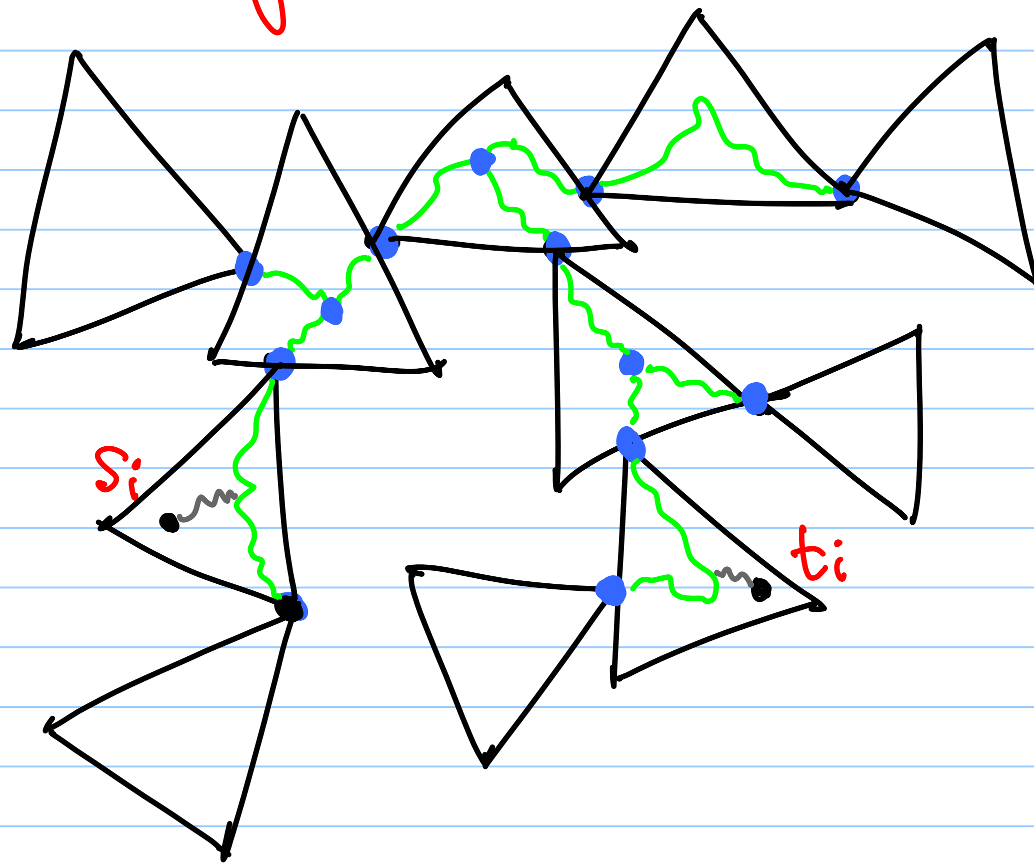
CASE ANALYSIS

- revenue from customer i :

  outside skeleton + along skeleton

- ~~case I:~~

  ~~total revenue outside skeleton~~ $\cancel{\geq \frac{OPT}{2}}$

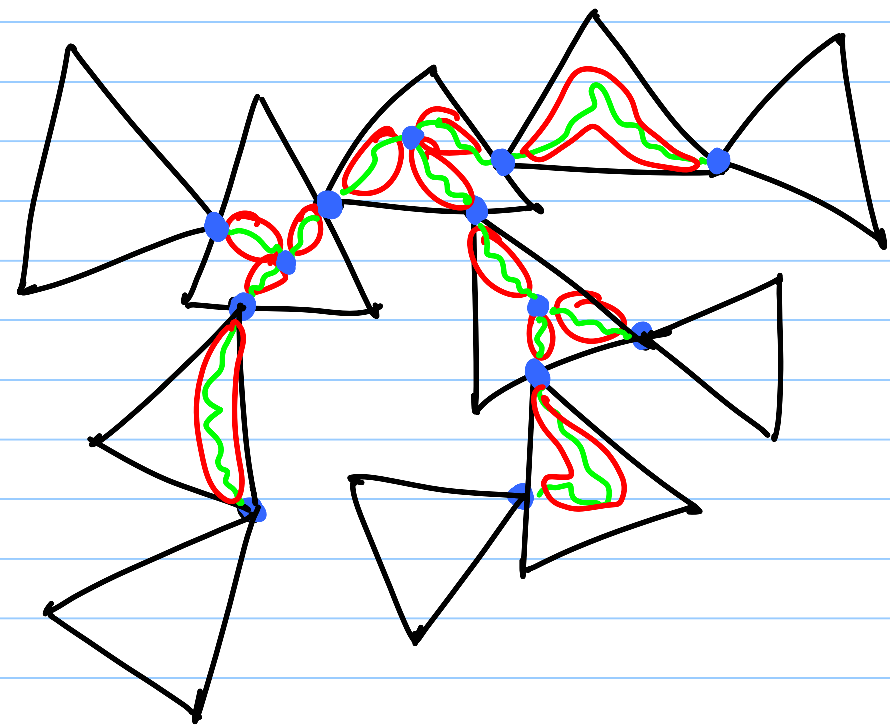


- case II :

  total revenue along skeleton $\geq \frac{OPT}{2}$

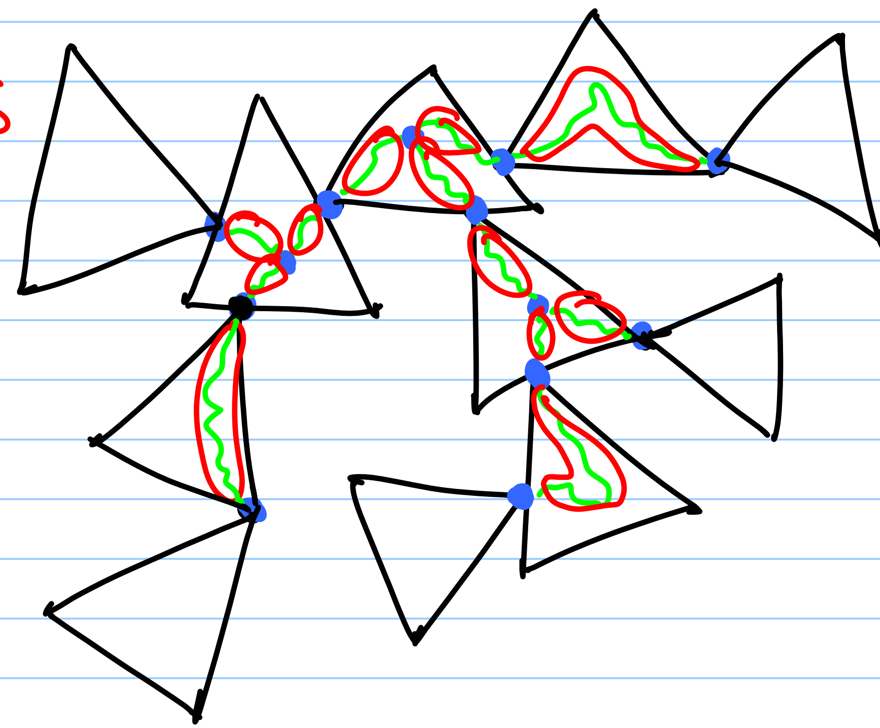
CASE II: HIGH REVENUE ALONG SKELETON

- all non-skeleton edges are sold for free



CASE II: HIGH REVENUE ALONG SKELETON

- all non-skeleton edges are sold for free
- $\Sigma(S')$ = set of segments



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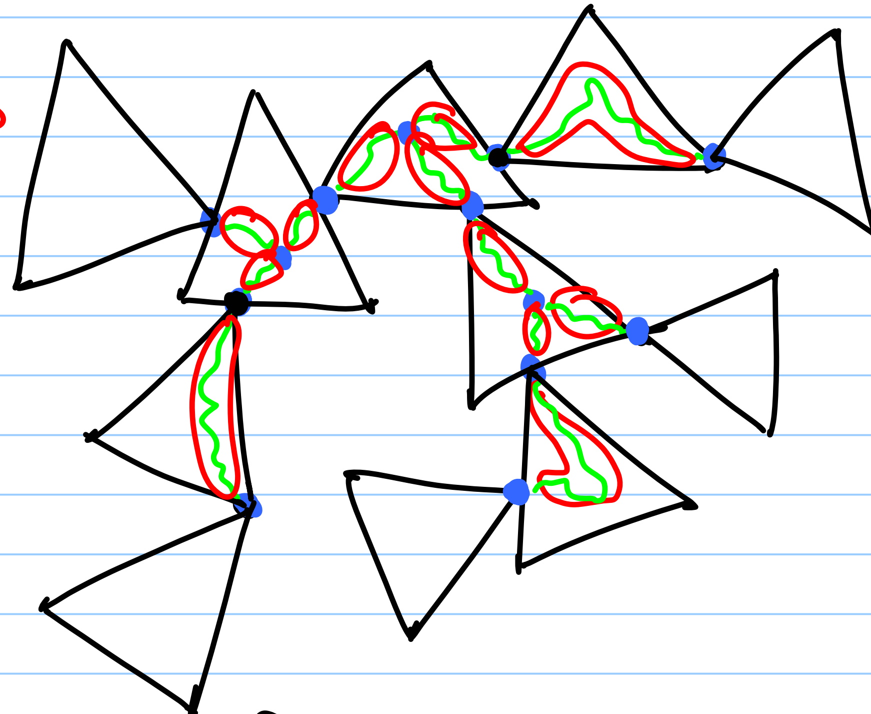


- Claim (very informal): there exists a "near-optimal" pricing scheme $\tilde{p}: E \to \mathbb{R}_+$ and a set of prices Δ, where:
  1. $\tilde{p}(\sigma) \in \Delta$ for each segment σ
  2. $|\Delta| = O(\log(nm))$

SEGMENT GUESSING

- assume that $\tilde{p}(\sigma)$ is known, for all $\sigma \in \Sigma(\xi^i)$
- number of guesses is:

$$|\Delta|^{|\Sigma(\xi^i)|} = [O(\log(nm))]^{O(k)}$$

$$= [O(\log(nm))]^{O(\sqrt{\log m})}$$

$$= o(nm)$$

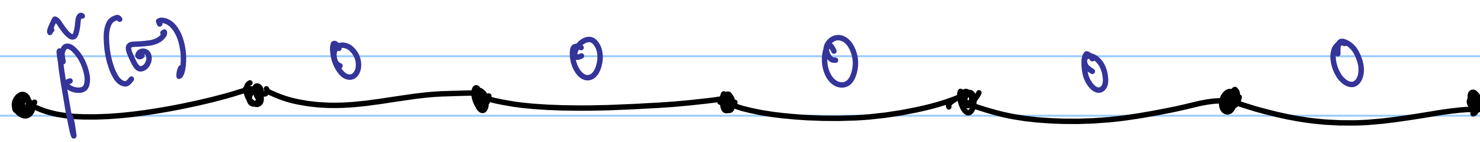
- remaining task: for each segment, spread $\tilde{p}(\sigma)$ over its edges

RANDOMIZED ASSIGNMENT

4 options, with equal probabilities:

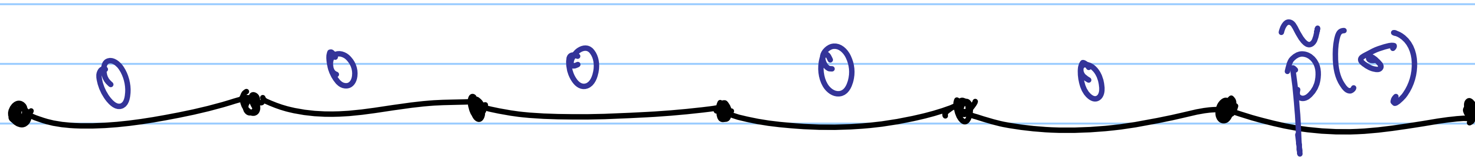
(1) concentrate price on leftmost edge

σ:



(2) concentrate price on rightmost edge

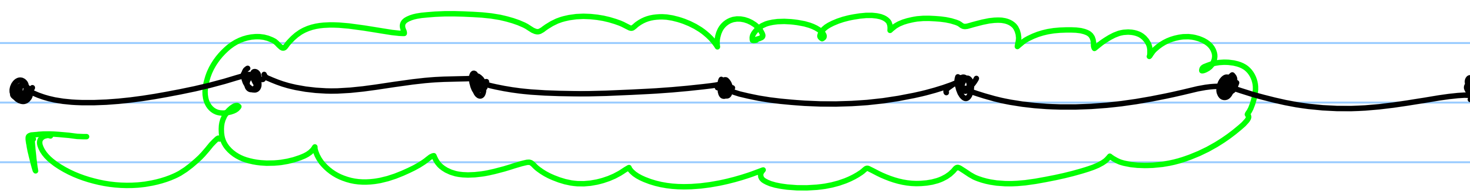
σ:



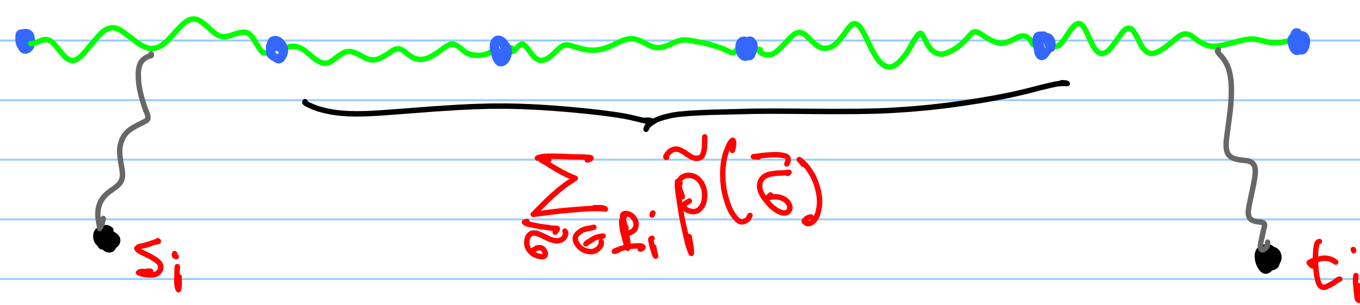
RANDOMIZED ASSIGNMENT - CONTINUED

(3) solution to left-side single-source instance

σ:



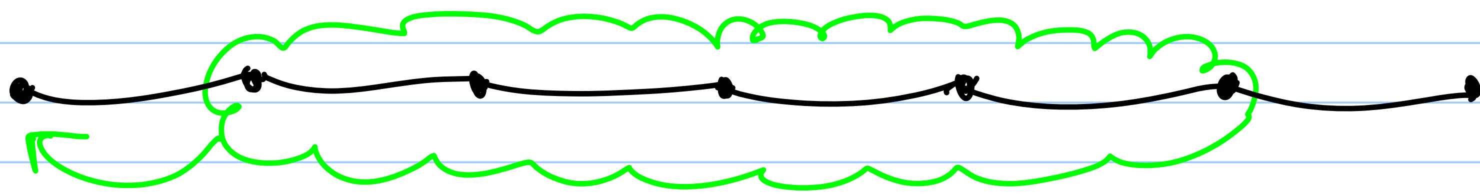
new budget of customer i: $\min\{\tilde{p}(\sigma), B_i - \sum_{\bar{\sigma} \in \mathcal{R}_i} \tilde{p}(\bar{\sigma})\}$



RANDOMIZED ASSIGNMENT - CONTINUED

(3) solution to left-side single-source instance

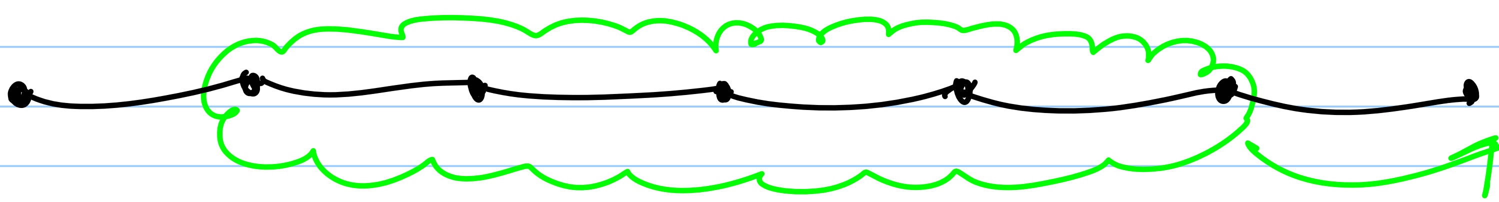
$\tilde{\sigma}$:



new budget of customer i: $\min\{\tilde{p}(\sigma),\ B_i - \sum_{\tilde{\sigma} \in R_i} \tilde{p}(\bar{\sigma})\}$

(4) solution to right-side single-source instance

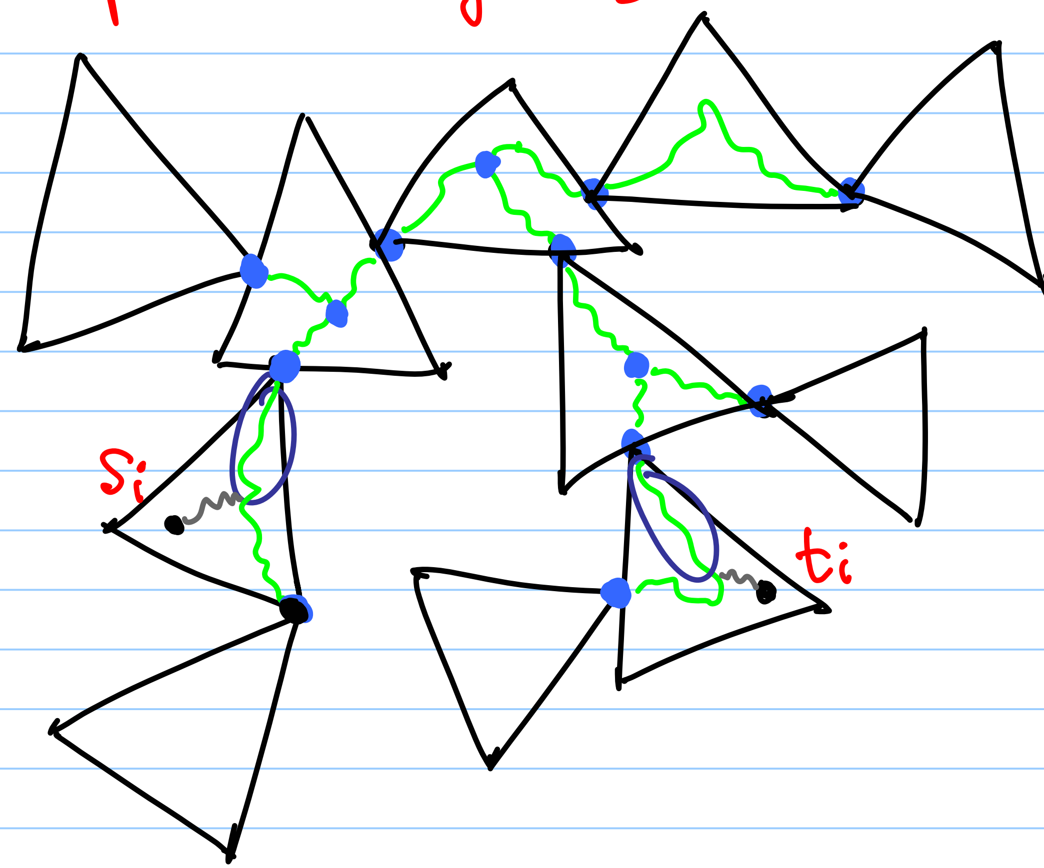
$\tilde{\sigma}$:



CASE II: ANALYSIS

- revenue from customer i along skeleton

  complete segments + partial segments



CASE II: ANALYSIS

- revenue from customer i along skeleton

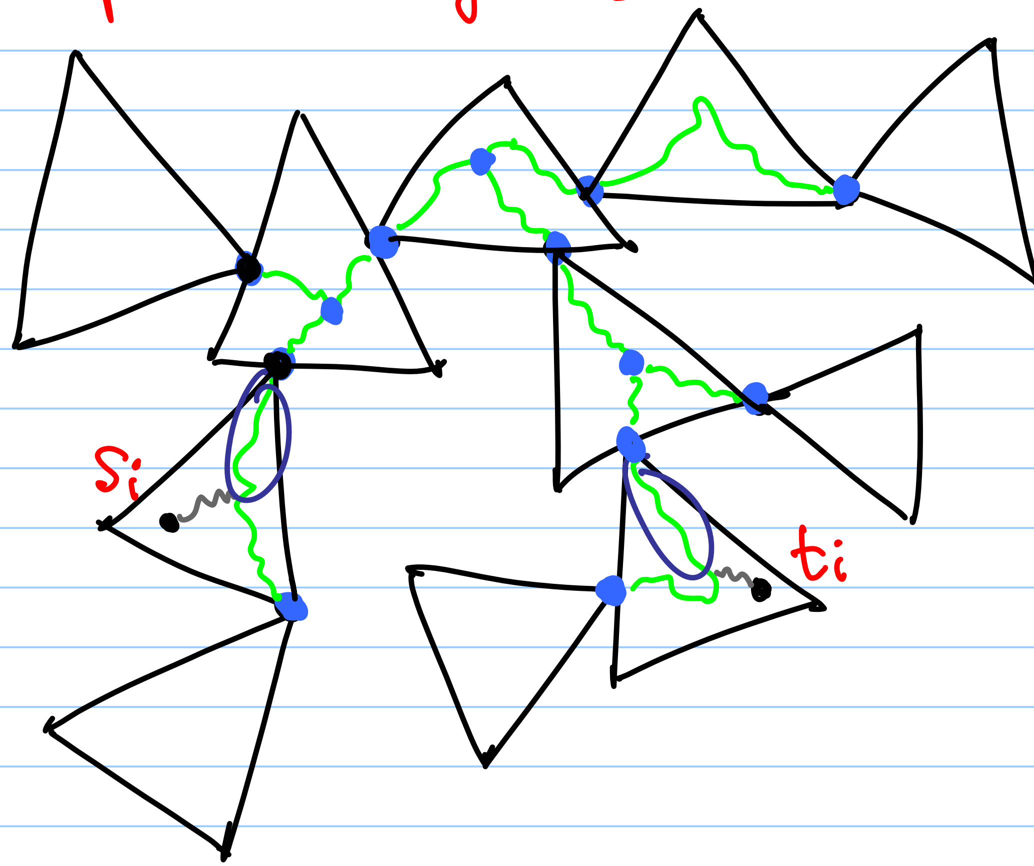
  complete segments + partial segments

- case I:

  total revenue on complete segments $\geq \frac{OPT}{4}$

- case II:

  total revenue on partial segments $\geq \frac{OPT}{4}$

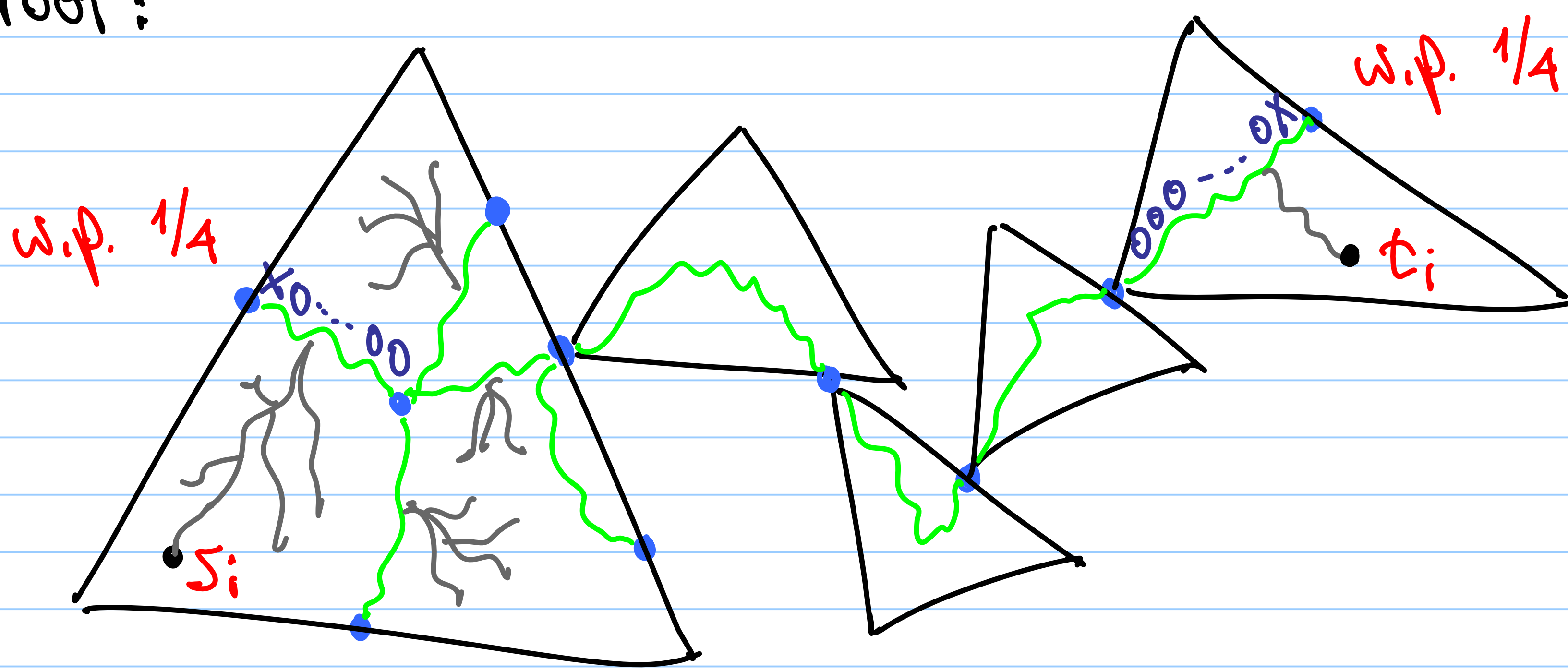


ANALYSIS: COMPLETE SEGMENTS

claim:

$$E\left[\begin{array}{l}\text{revenue from}\\ \text{each customer}\end{array}\right] \geq \frac{1}{16}\cdot\left[\begin{array}{l}\text{same revenue}\\ \text{in OPT}\end{array}\right]$$

proof:



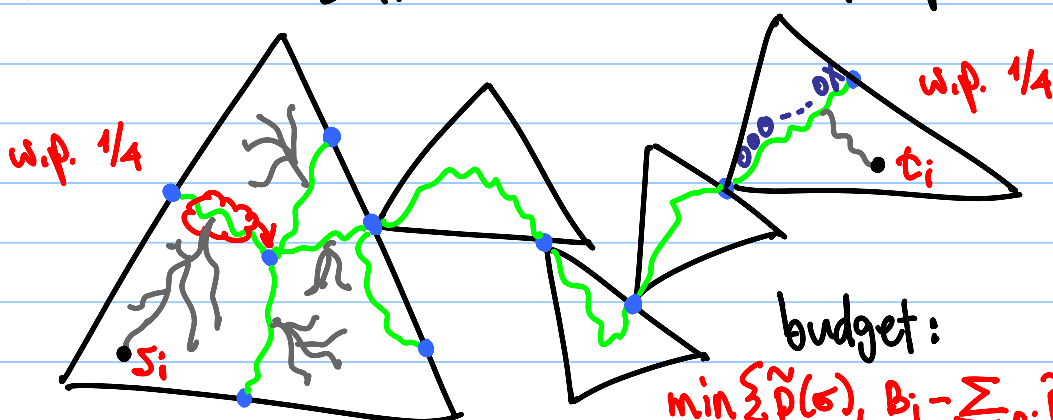
ANALYSIS: PARTIAL SEGMENTS

claim:

$$E\left[\begin{array}{c}\text{revenue from}\\ \text{each right-segment}\end{array}\right] \geq \frac{1}{16}\cdot\left[\begin{array}{c}\text{same revenue}\\ \text{in OPT}\end{array}\right]$$

proof:

For each right-side single-source instance we solve, our revenue $\geq$ same revenue in OPT

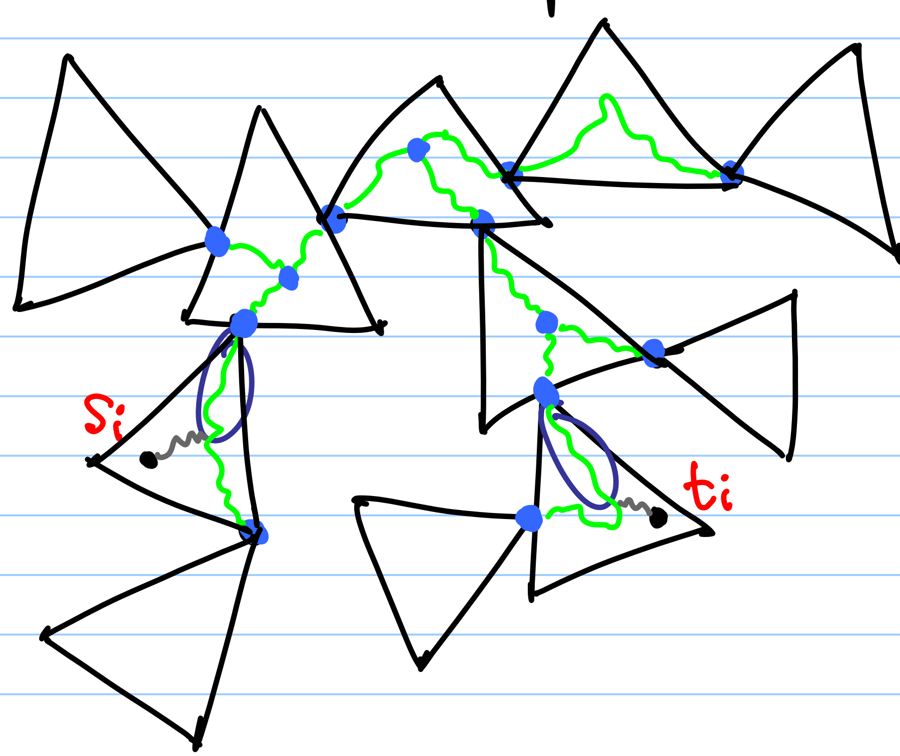


budget:

$\min\{\tilde{p}(\sigma),\ B_i - \sum_{\sigma \in R_i} \tilde{p}(\sigma)\}$

CASE II: BOTTOM LINE

Claim: the randomized price assignment guarantees a constant fraction of the optimal revenue along the skeleton in expectation



CASE ANALYSIS

- revenue from customer i :

  outside skeleton + along skeleton

- ~~case I:~~

  ~~total revenue outside skeleton~~ $\cancel{\geq \frac{OPT}{2}}$

- ~~case II:~~

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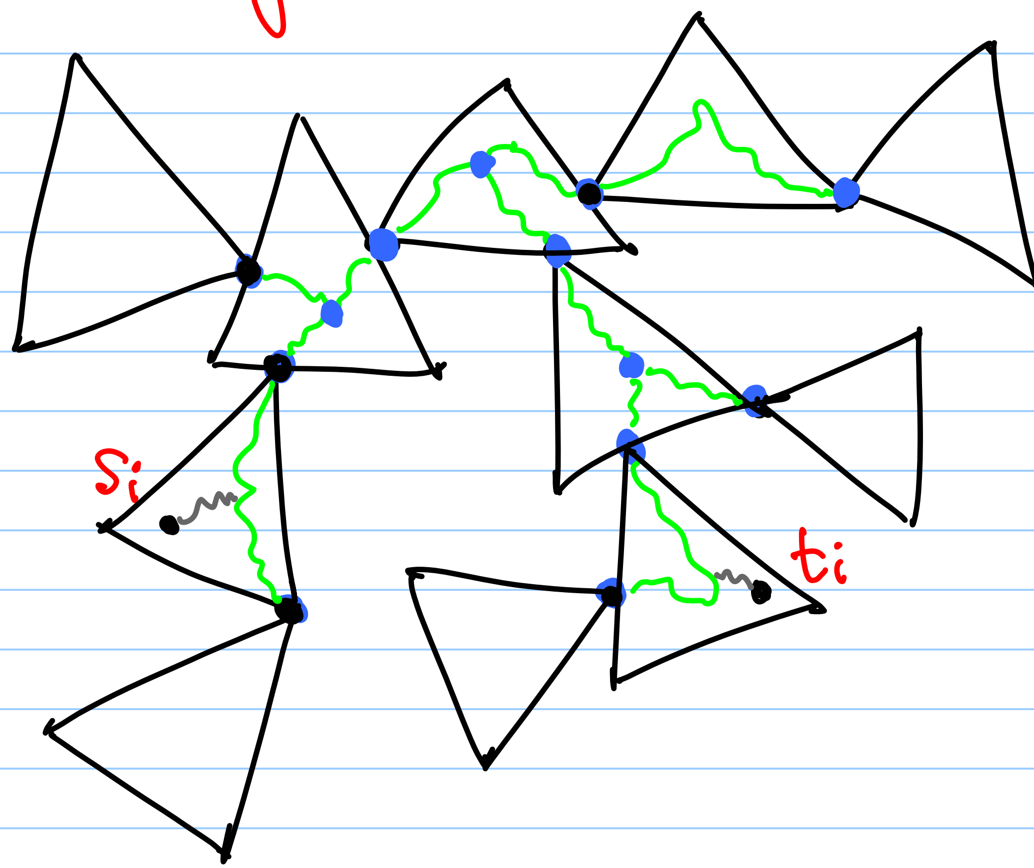
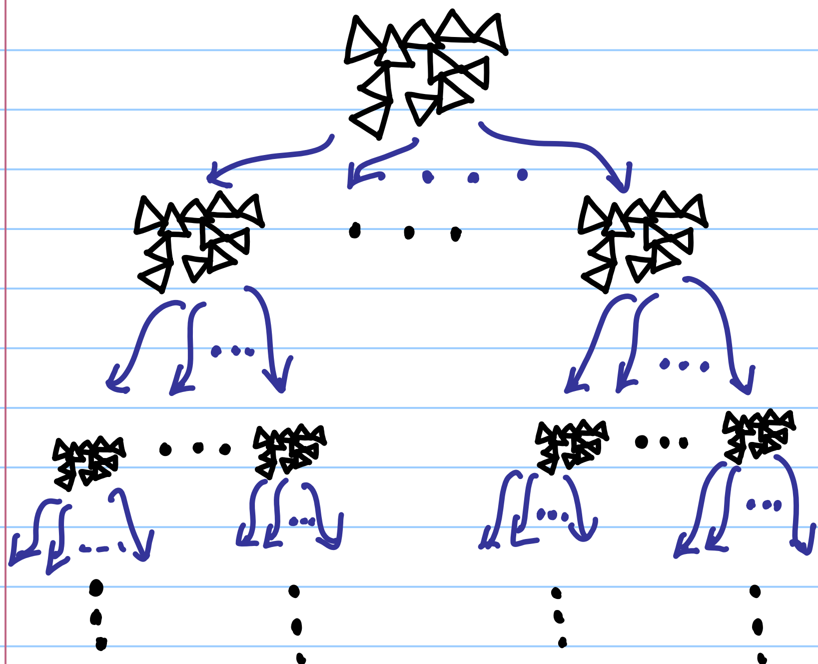
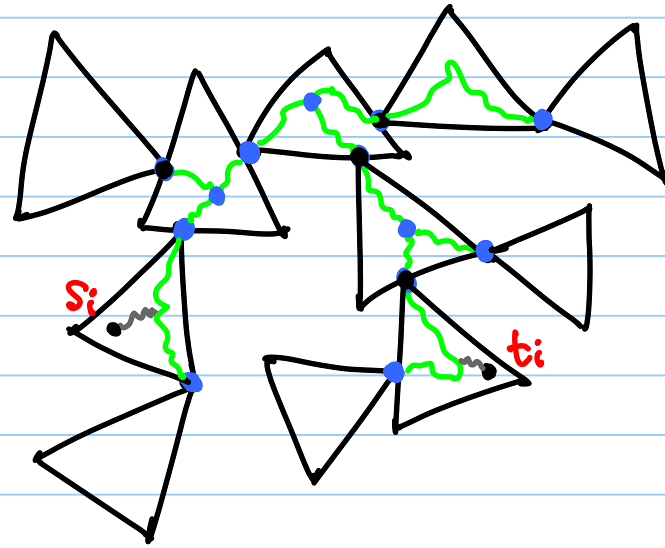


QUICK RECAP



$O\left(\frac{\log m}{\log\log m}\right)$



outside skeleton
along the skeleton $\Big\}\ O(1)$

can be derandomized

OPEN QUESTIONS

- improved factor, even for highway problem ?
- additional applications ?

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- improved factor, even for highway problem ?

- additional applications ?

THANK YOU