

A Sublogarithmic Approximation

for

Highway and Tollbooth Pricing

Iftah Gamzu

Joint work with Danny Segev

OUTLINE

- introduction
- new algorithm – main ideas
- conclusions

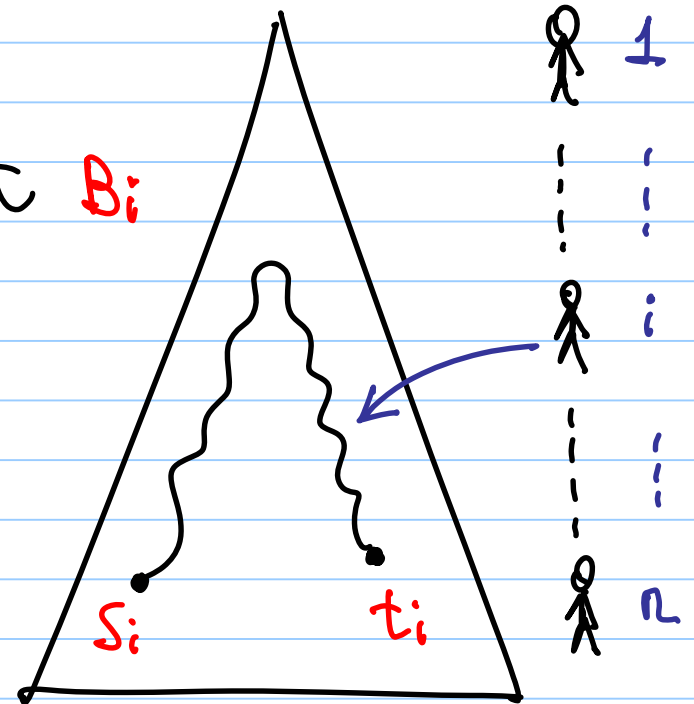
INTRODUCTION

TOLLBOOTH PRICING ON TREES

• Input: tree $T = (V, E)$ on m edges

n customers

customer i : path P_i + budget B_i



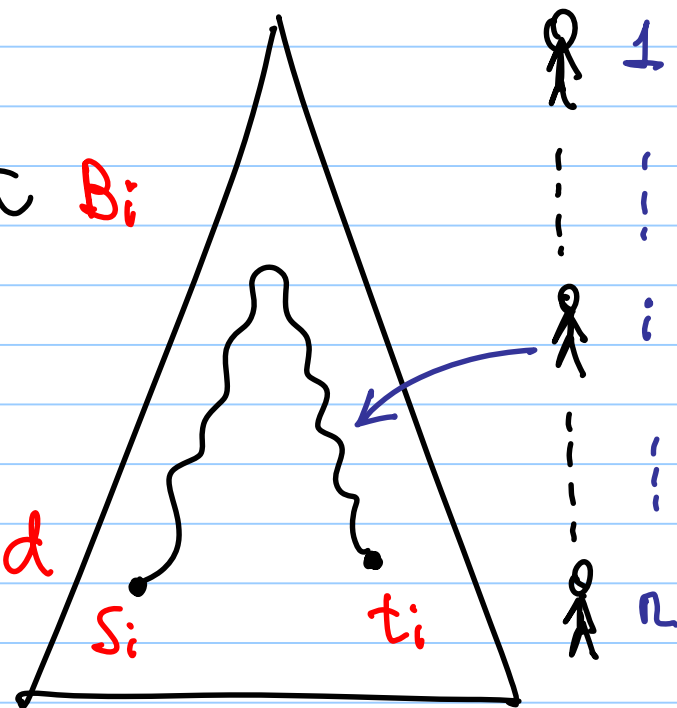
TOLLBOOTH PRICING ON TREES

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TOLLBOOTH PRICING ON TREES

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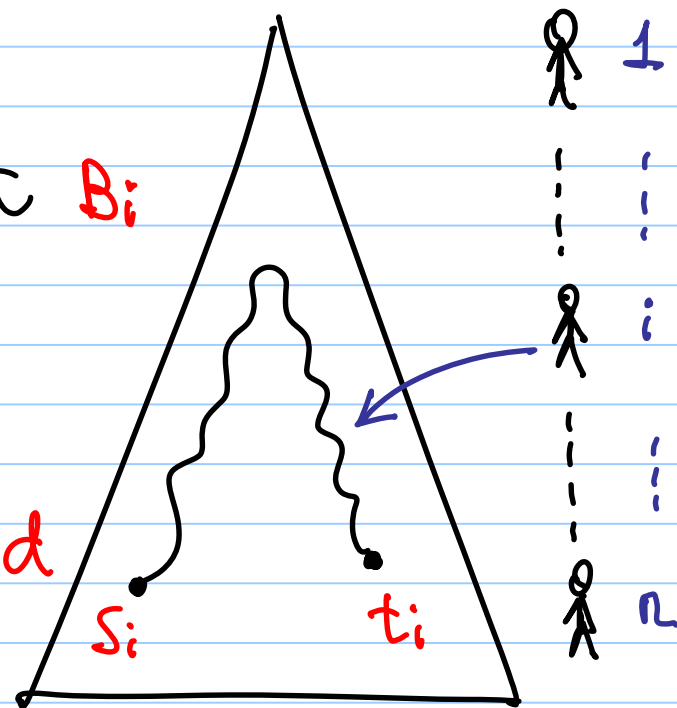
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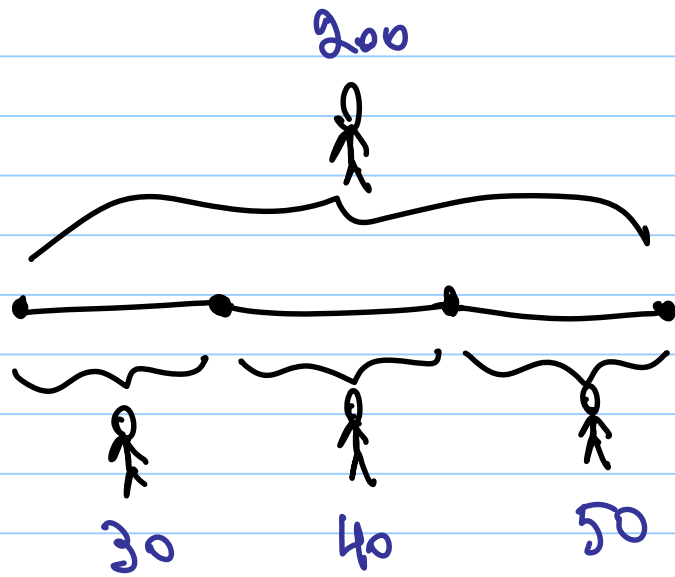
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- revenue due to customer i :

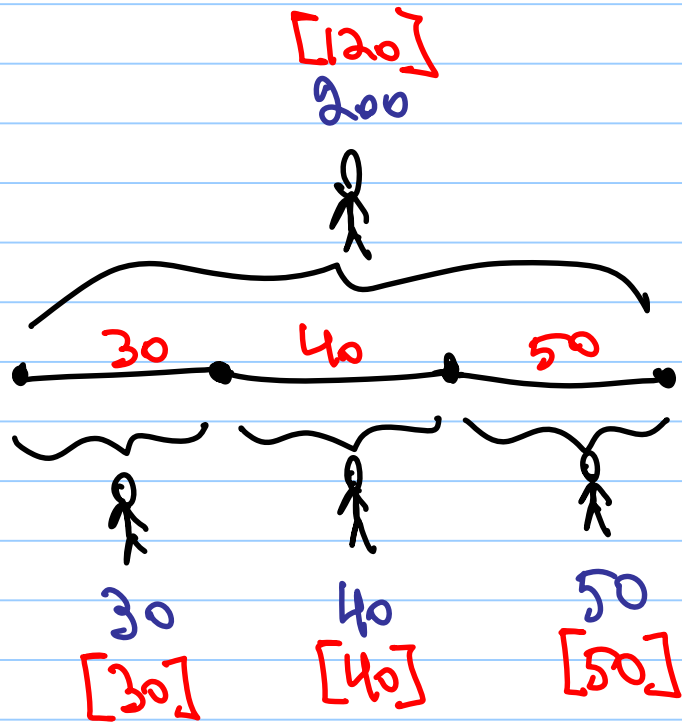
- total price of P_i , when it does not exceed B_i
- no revenue at all, otherwise



EXAMPLE

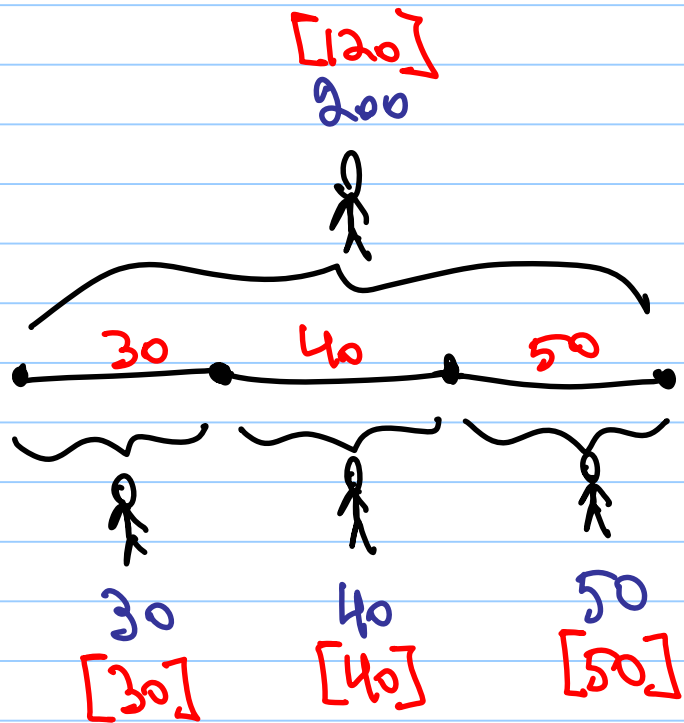


EXAMPLE

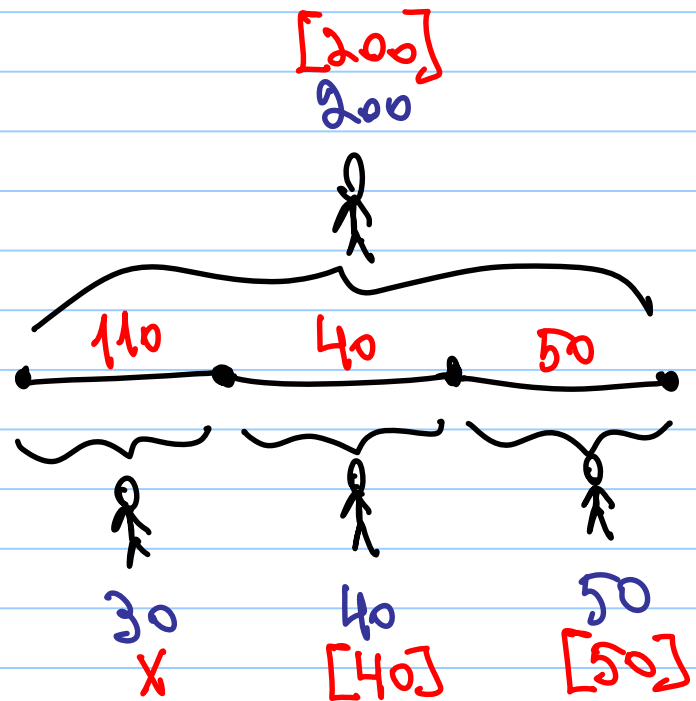


total: 240

EXAMPLE



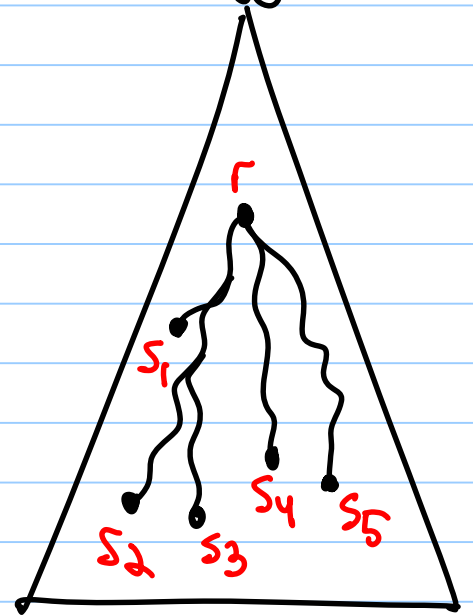
total: 240



total: 290

PREVIOUS WORK

- Tollbooth problem (on trees):
 - **APX-hard**: Guruswami et al [SODA '05]
 - **exact algorithm for single-source case**
 - **$o(\log n)$ -approx**: Elbassioni et al. [SAGT '09]



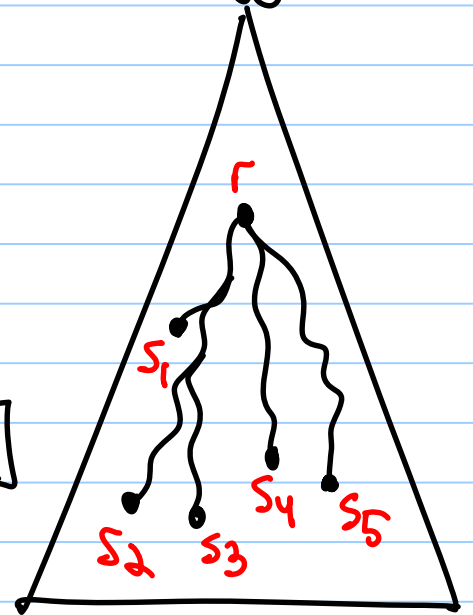
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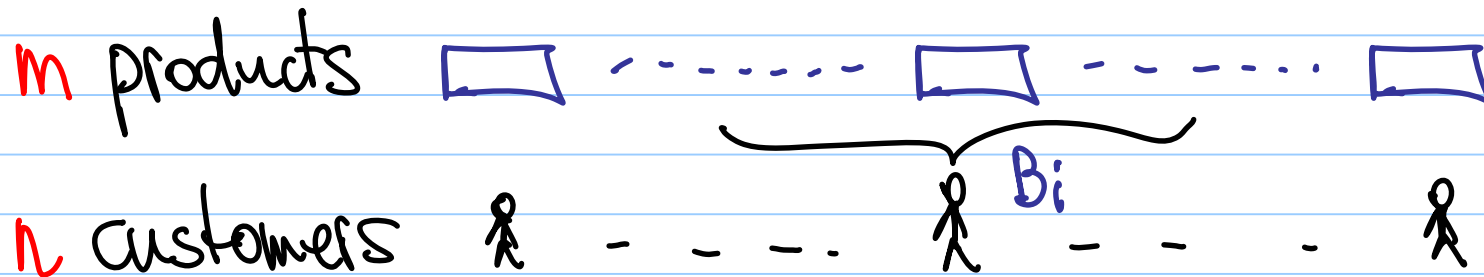
- Highway Problem (on simple paths):

- **NP-hard**: Briest and Krysta [SODA '06]
- **$o(\log m)$ -approx**: Balcan and Blum [EC '06]



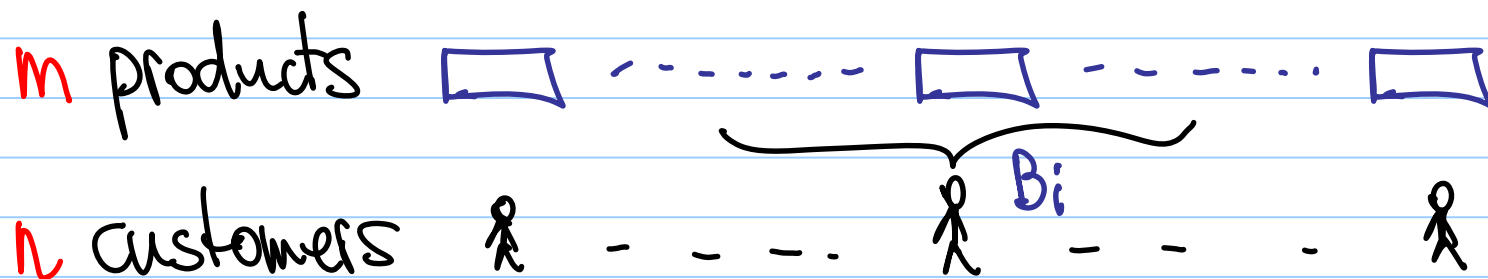
PREVIOUS WORK

- single-minded unlimited supply pricing



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- lots of work; will mention only:

- $\Theta(\log m + \log n)$ -approx: Guruswami et al. [SODA '05]

- $\Omega(\log m)$ inapproximability: Demaine et al. [SODA '06]

↳ plausible hardness assumption

MAIN RESULT

deterministic algorithm with ratio

$$O(\log m / \log \log m)$$

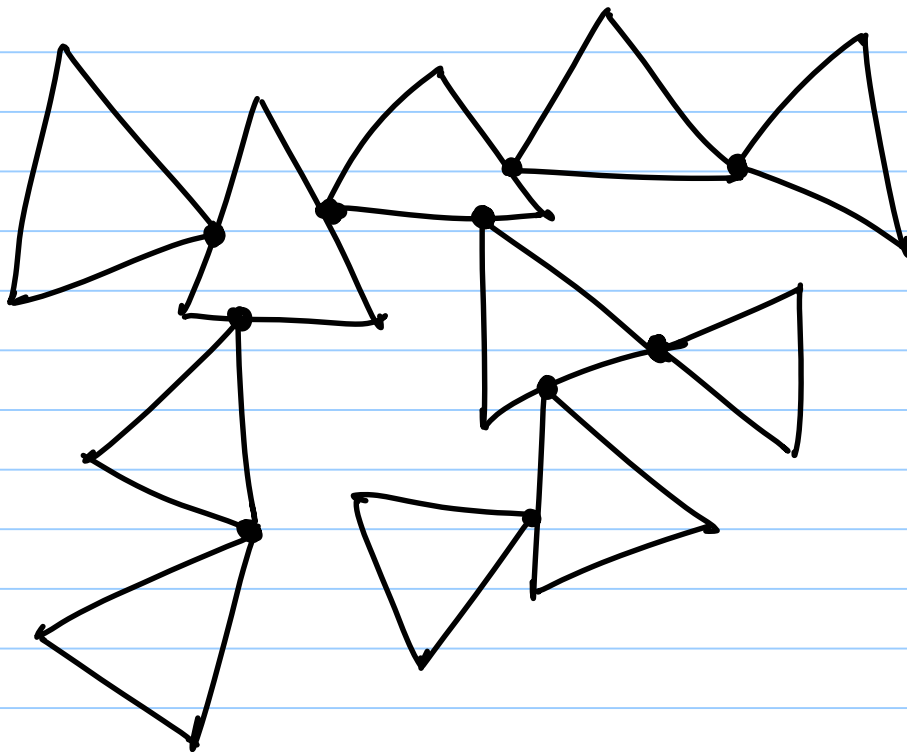
for fullbush pricing on trees

THE ALGORITHM:

GENERAL IDEA

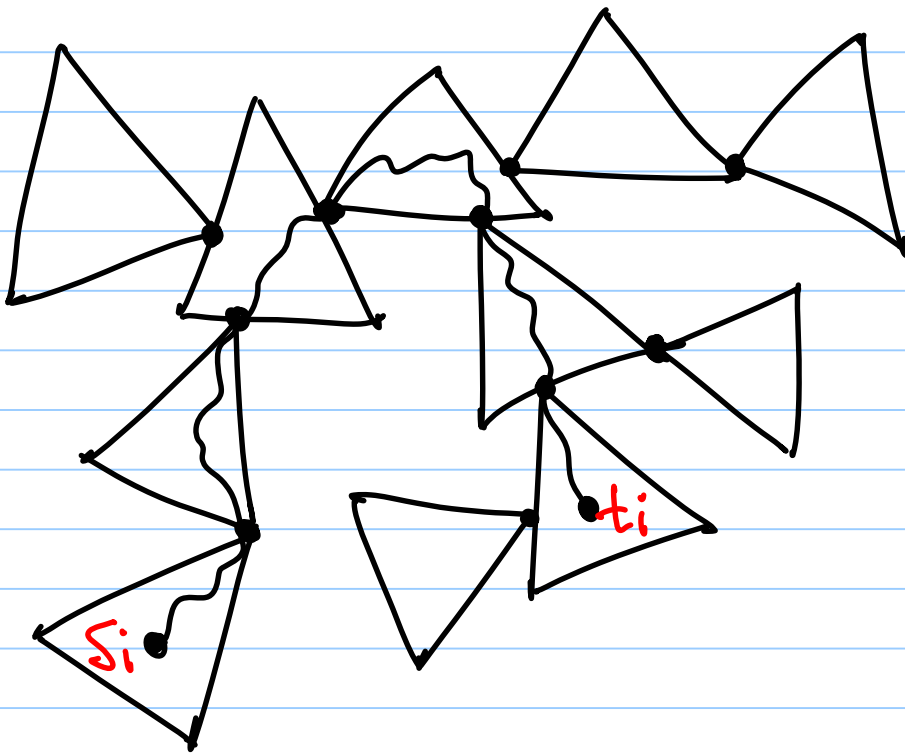
ALMOST BALANCED k -DECOMPOSITIONS

- any tree $T = (V, E)$ can be decomposed into k edge-disjoint subtrees, each containing $O(|E|/k)$ edges



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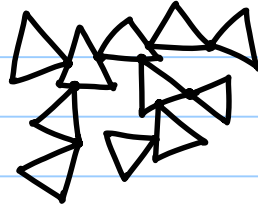
- any tree $T = (V, E)$ can be decomposed into k edge-disjoint subtrees, each containing $\Theta(|E|/k)$ edges



- customer i is separated when s_i and t_i reside in different subtrees

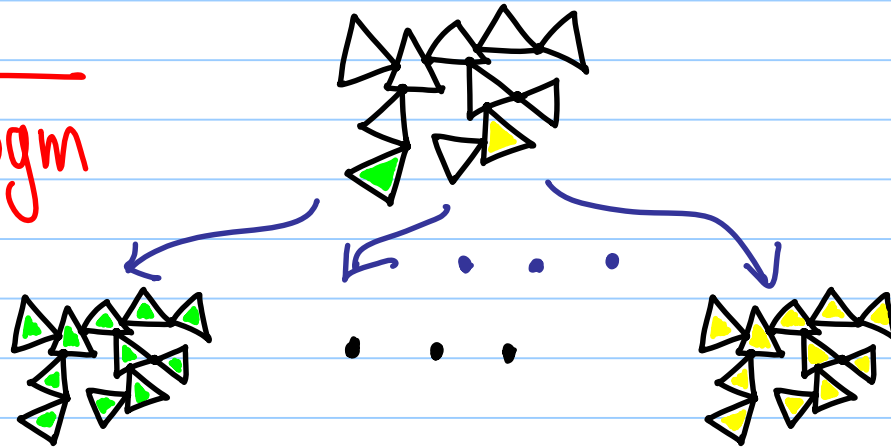
REPEATED k -DECOMPOSITIONS

pick $k = \sqrt{\log m}$



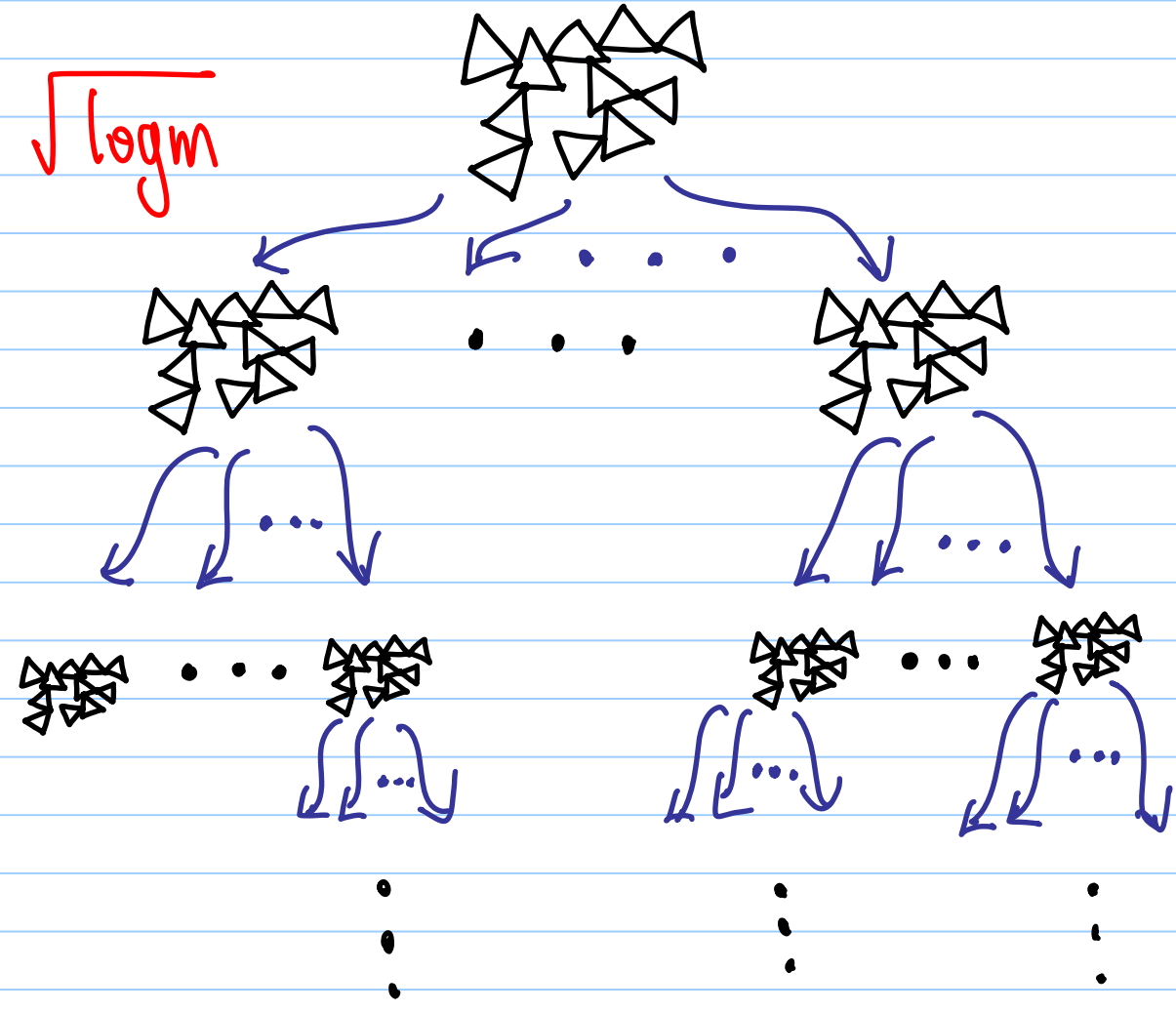
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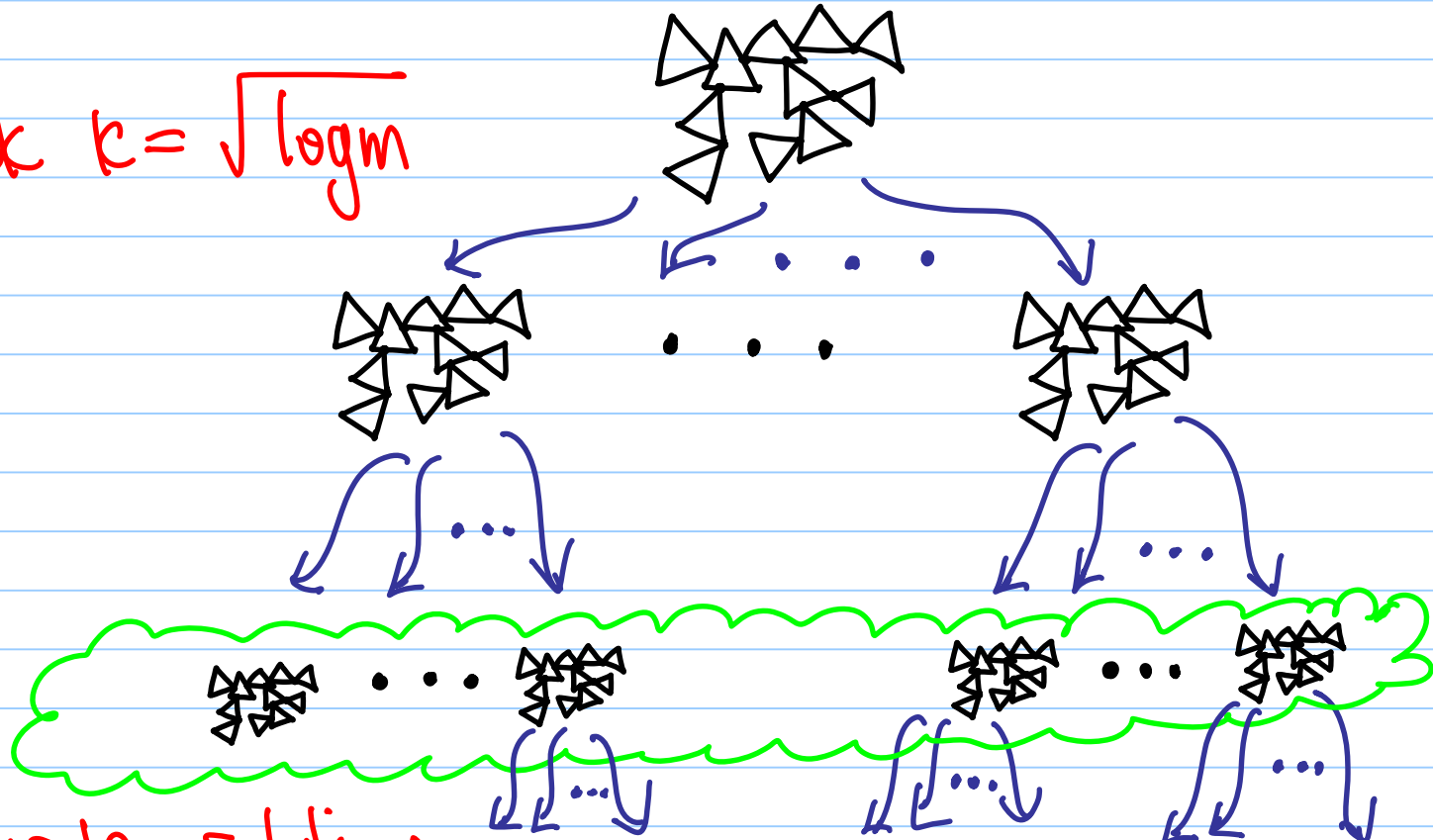
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$O\left(\frac{\log m}{\log \log m}\right)$
levels

REPEATED K-DECOMPOSITIONS

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separate solution
for each subtree

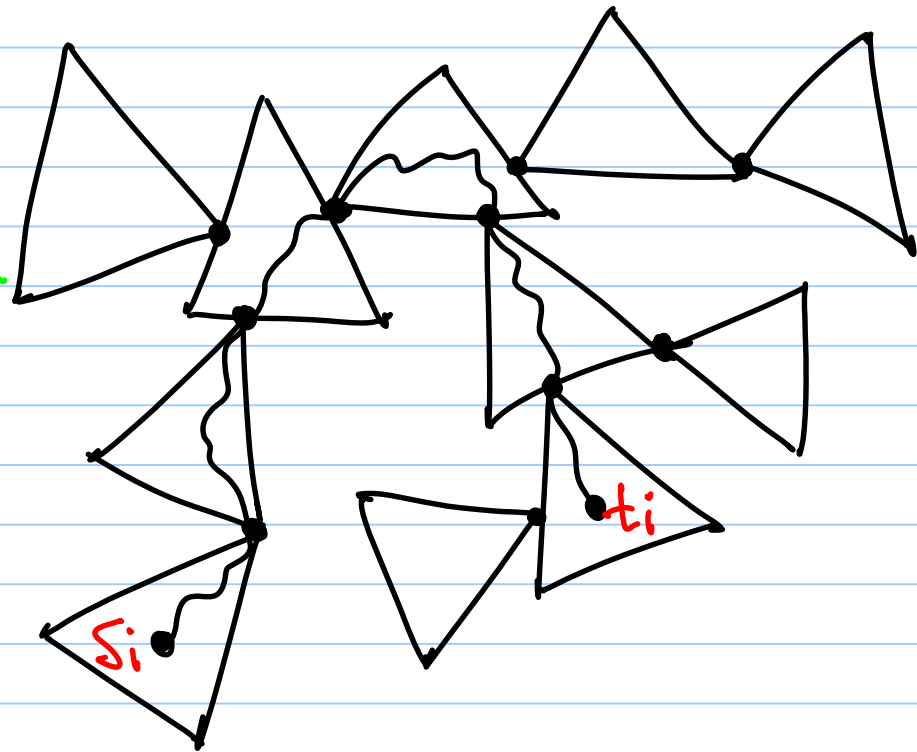
$O\left(\frac{\log m}{\log \log m}\right)$
levels

NEED $O(1)$ -APPROX FOR...

- Input: tree $T=(V,E)$, decomposed into T_1, \dots, T_k
for each customer i : path P_i + budget B_i

- additional structure:
customer paths are separated

- goal: find pricing scheme
 $p:E \rightarrow \mathbb{R}_+$ that maximizes
overall revenue



CONSTANT-FACTOR APPROXIMATION

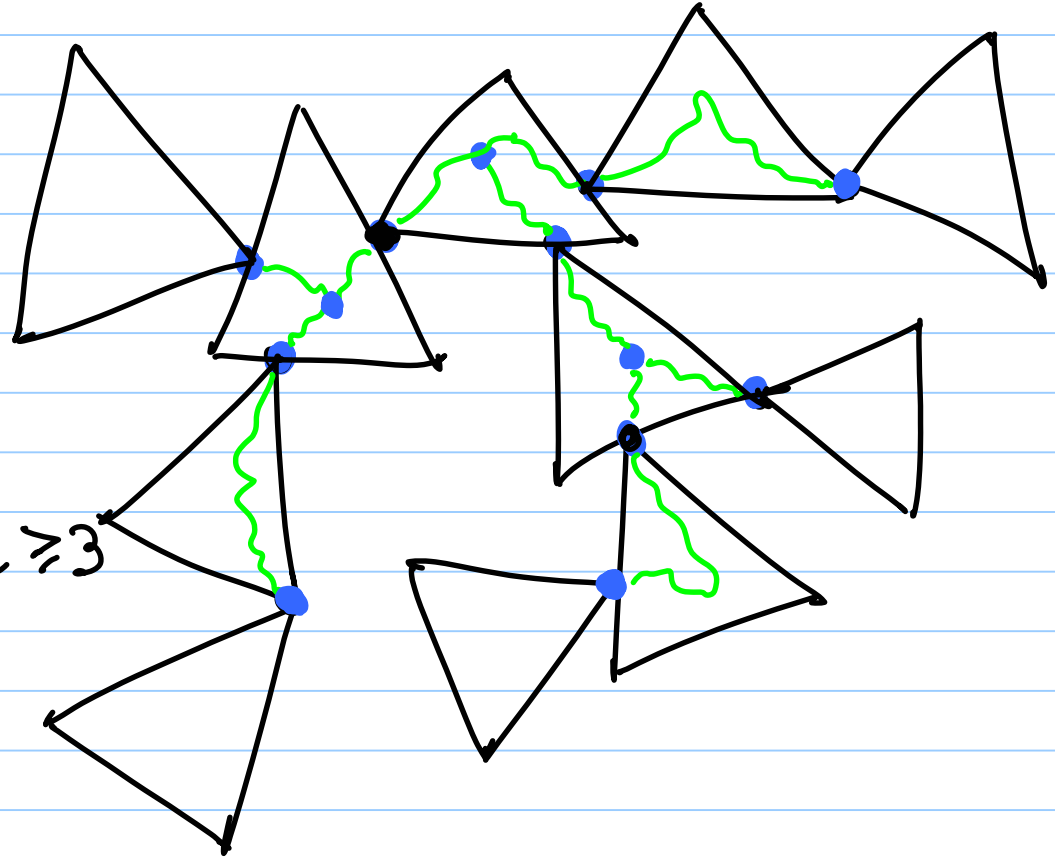
FOR

K-DECOMPOSITIONS

NOTATION AND TERMINOLOGY

- **skeleton \mathcal{S}** = minimal subtree that spans all intersection vertices

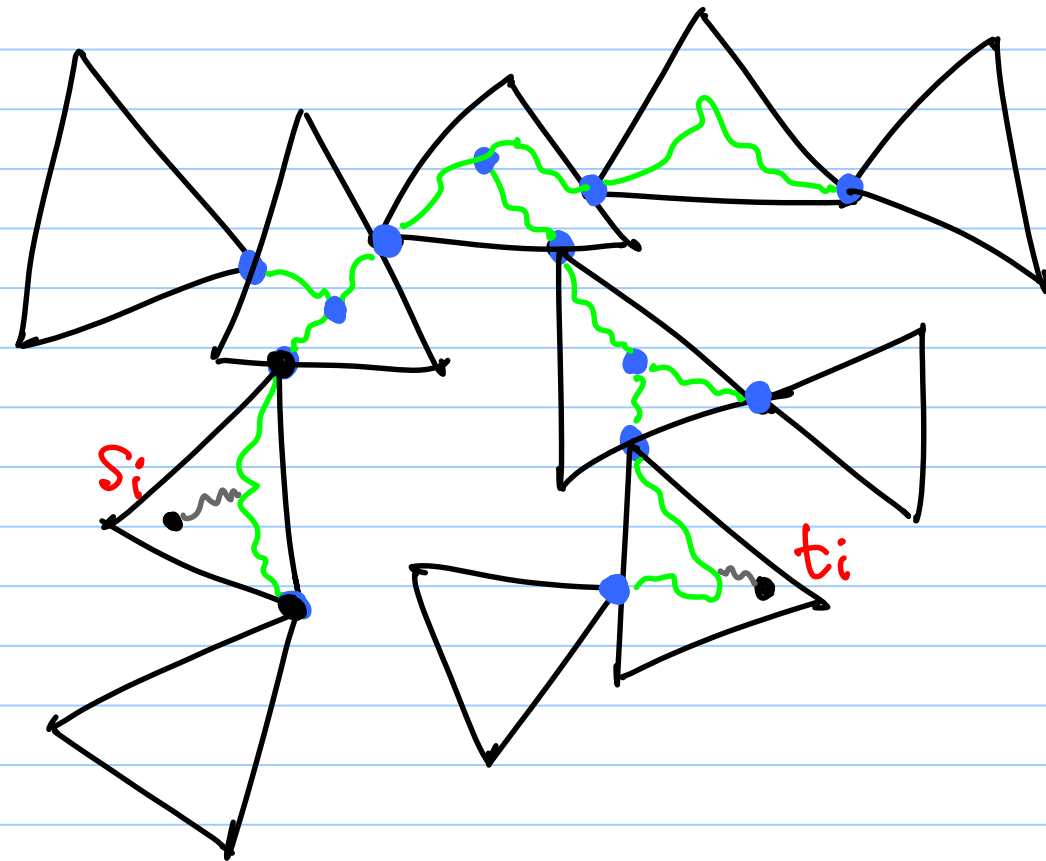
- **special vertices** =
intersection vertices +
vertices in \mathcal{S} with degree ≥ 3



CASE ANALYSIS

- revenue from customer i :

outside skeleton + along skeleton



CASE ANALYSIS

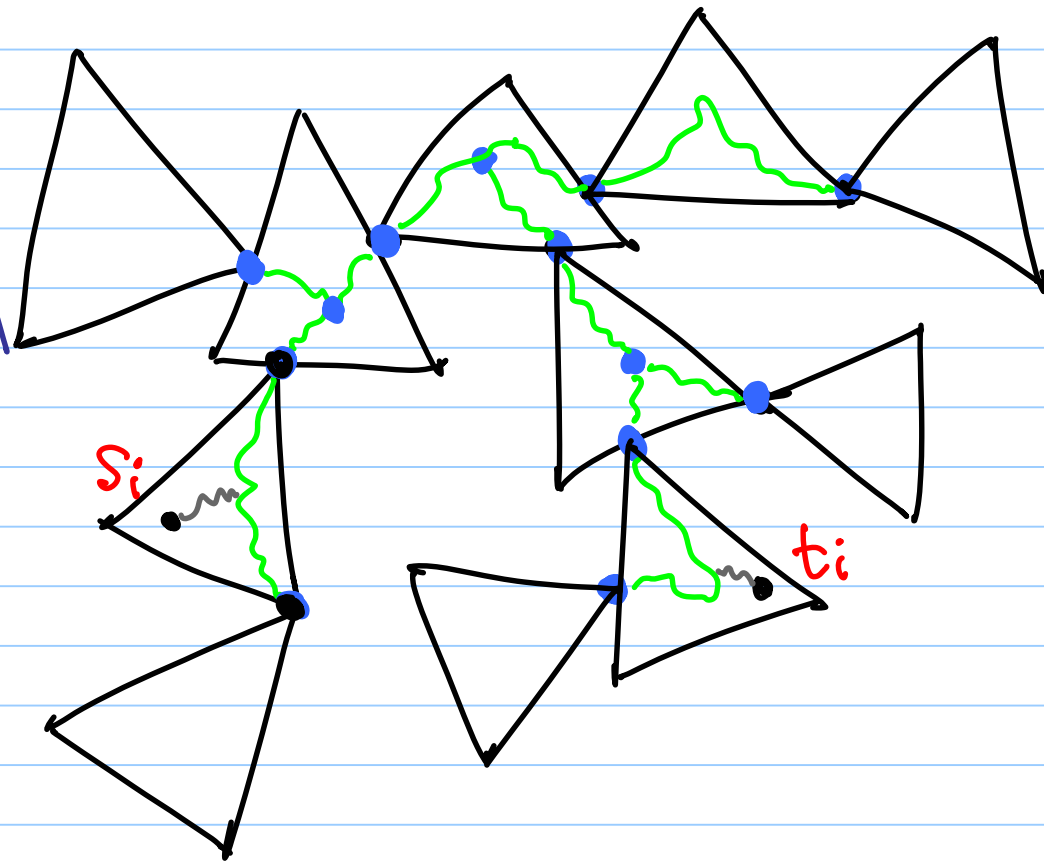
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- case I:

total revenue
outside skeleton $\geq \frac{OPT}{2}$

EASY: randomized reduction
to single-source case



CASE ANALYSIS

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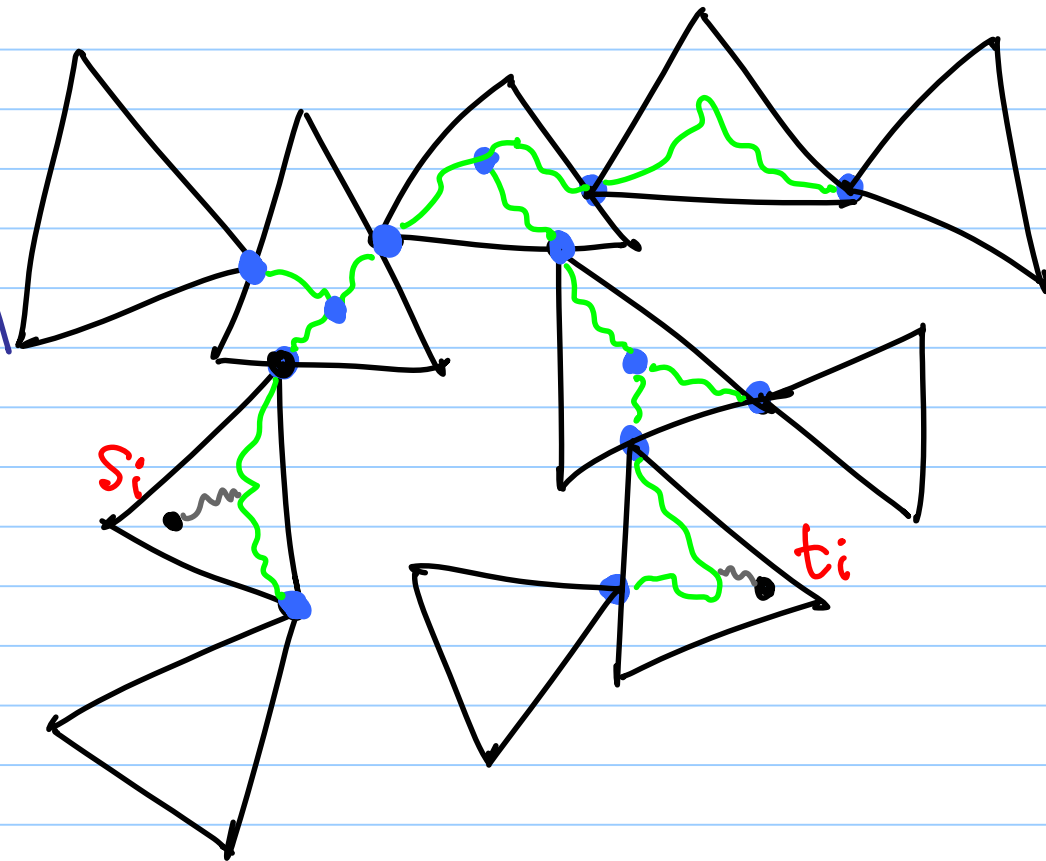
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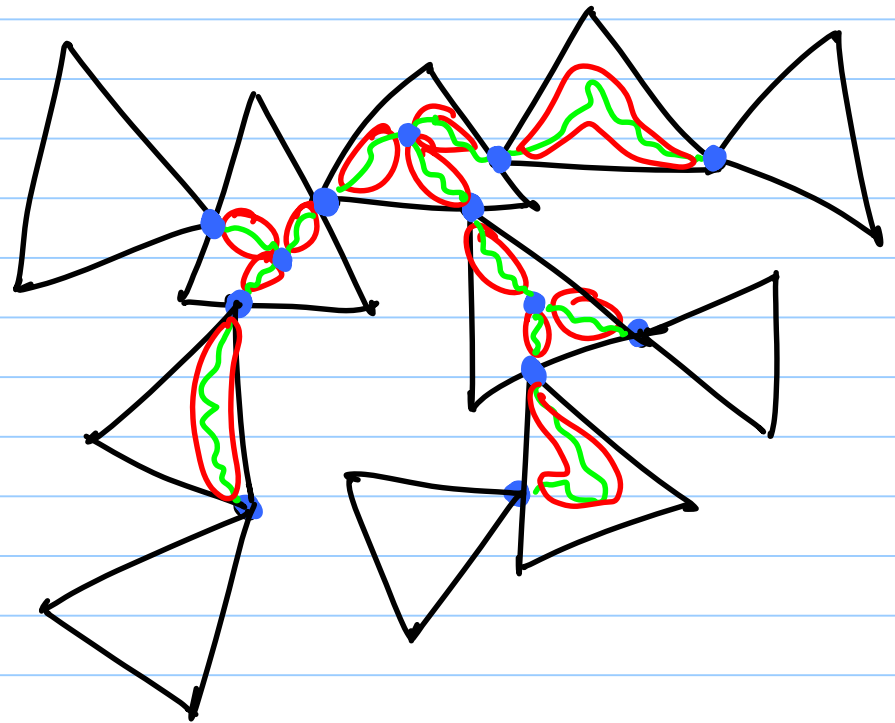
- case II:

total revenue
along skeleton $\geq \frac{OPT}{2}$
???



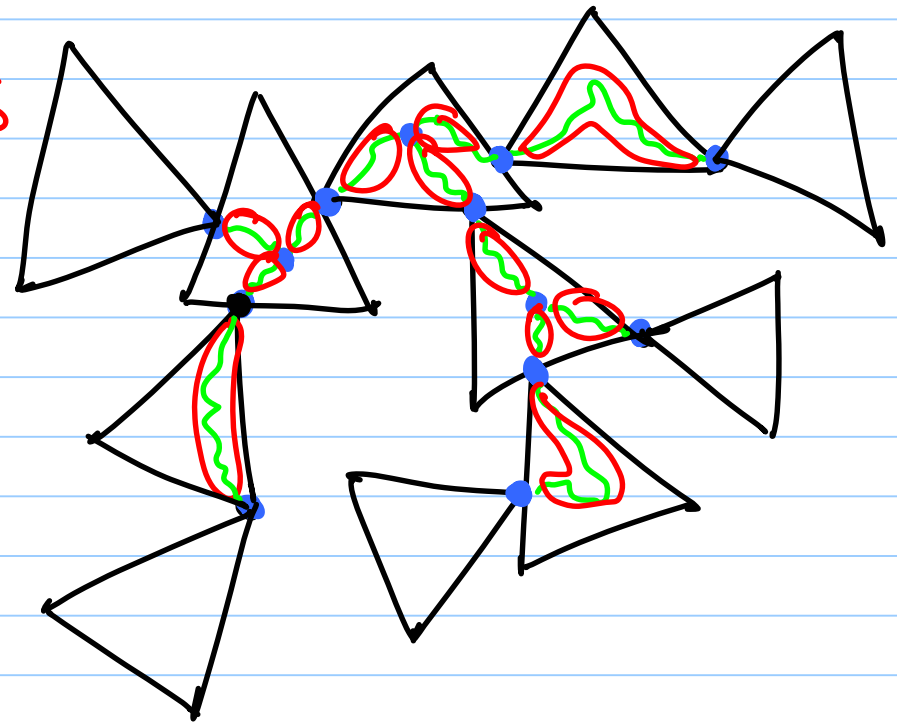
CASE II: HIGH REVENUE ALONG SKELETON

- all non-skeleton edges are sold for free



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- all non-skeleton edges are sold for free

- $\Sigma(\mathcal{S}) =$ set of segments

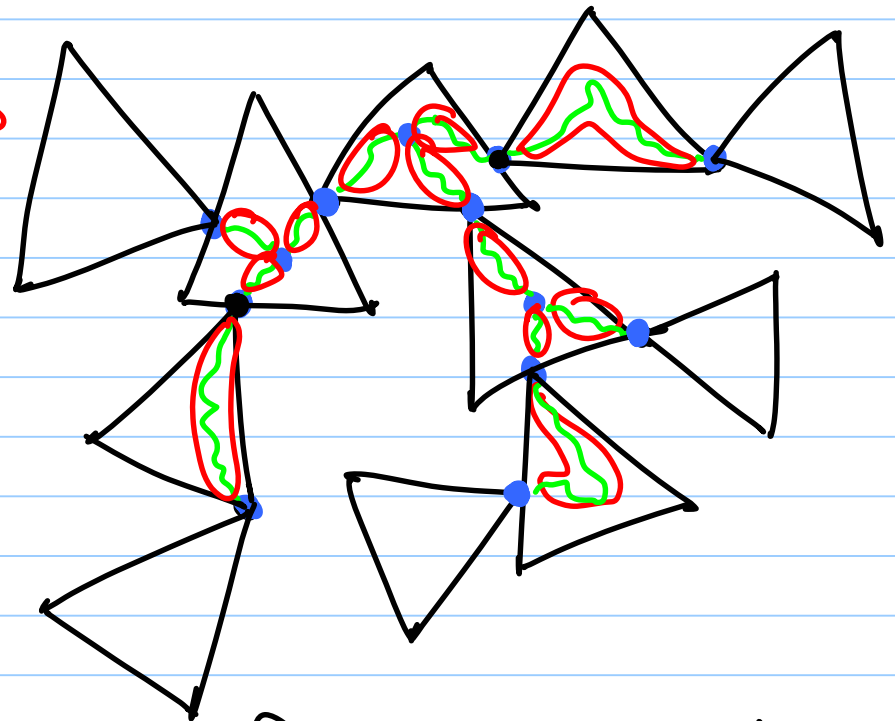
- Claim (very informal):

there exists a "near-optimal" pricing scheme

$\tilde{p}: E \rightarrow \mathbb{R}_+$ and a set of

prices Δ , where: 1. $\tilde{p}(\sigma) \in \Delta$ for each segment σ

2. $|\Delta| = o(\log(nm))$



SEGMENT GUESSING

- assume that $\tilde{p}(\sigma)$ is known, for all $\sigma \in \Sigma(\mathcal{S}^k)$
- number of guesses is:

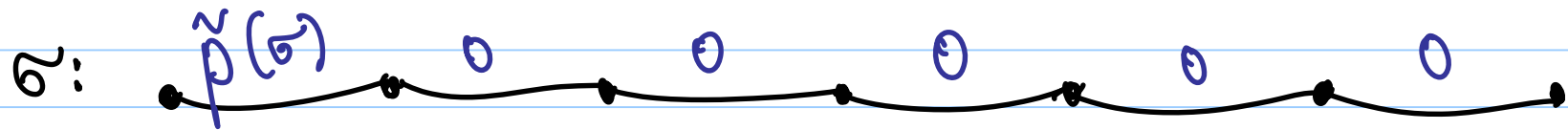
$$\begin{aligned} |\Delta|^{|\Sigma(\mathcal{S}^k)|} &= [o(\log(nm))]^{o(k)} \\ &= [o(\log(nm))]^{o(\sqrt{\log m})} \\ &= o(nm) \end{aligned}$$

- remaining task: for each segment, spread $\tilde{p}(\sigma)$ over its edges

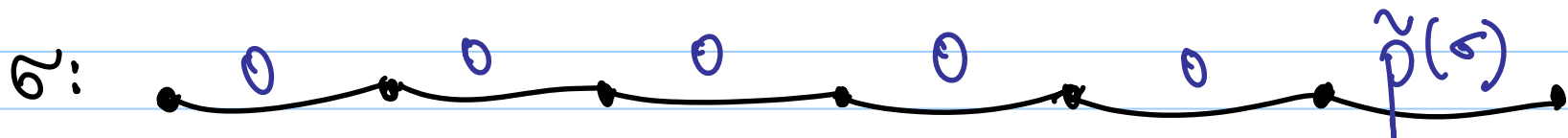
RANDOMIZED ASSIGNMENT

4 options, with equal probabilities:

(1) concentrate price on leftmost edge

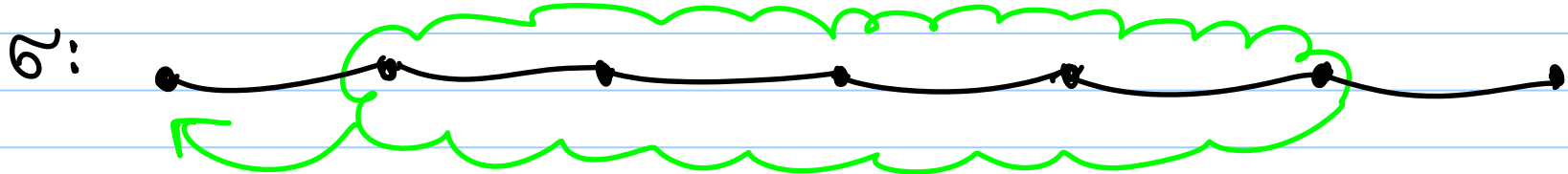


(2) concentrate price on rightmost edge

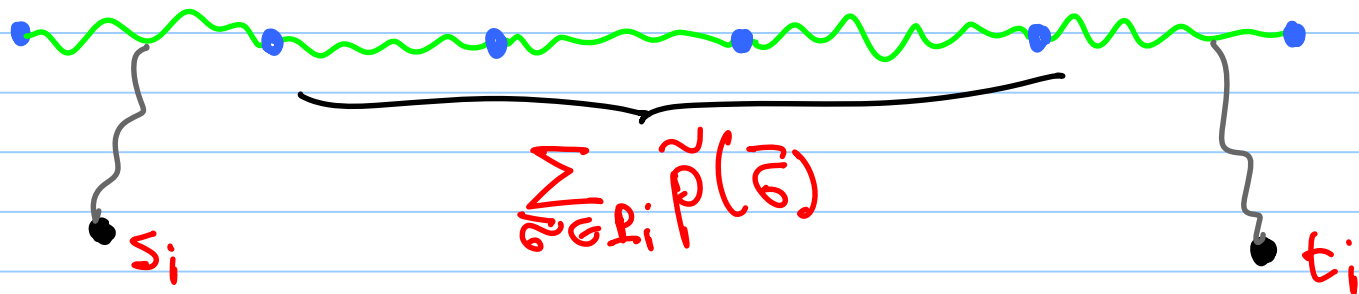


RANDOMIZED ASSIGNMENT - CONTINUED

(3) solution to left-side single-source instance

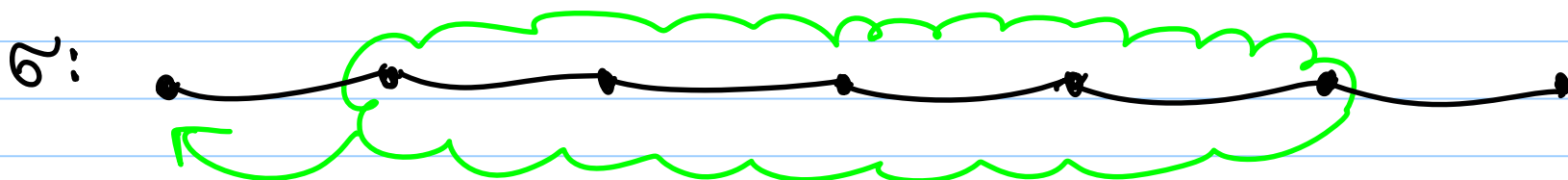


new budget of customer i : $\min \{ \tilde{p}(\sigma), B_i - \sum_{\sigma \in L_i} \tilde{p}(\sigma) \}$



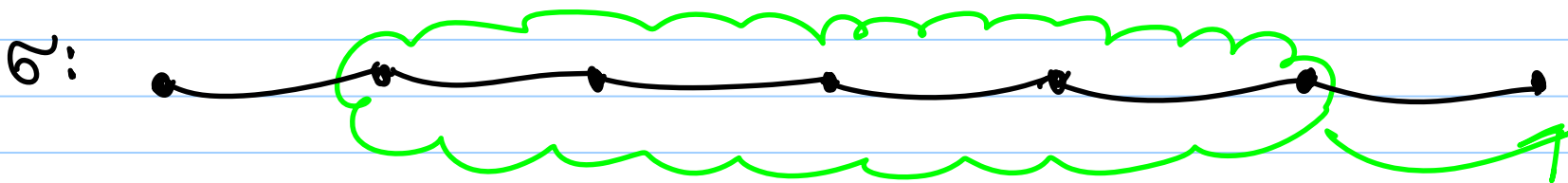
RANDOMIZED ASSIGNMENT - CONTINUED

(3) solution to **left-side single-source** instance



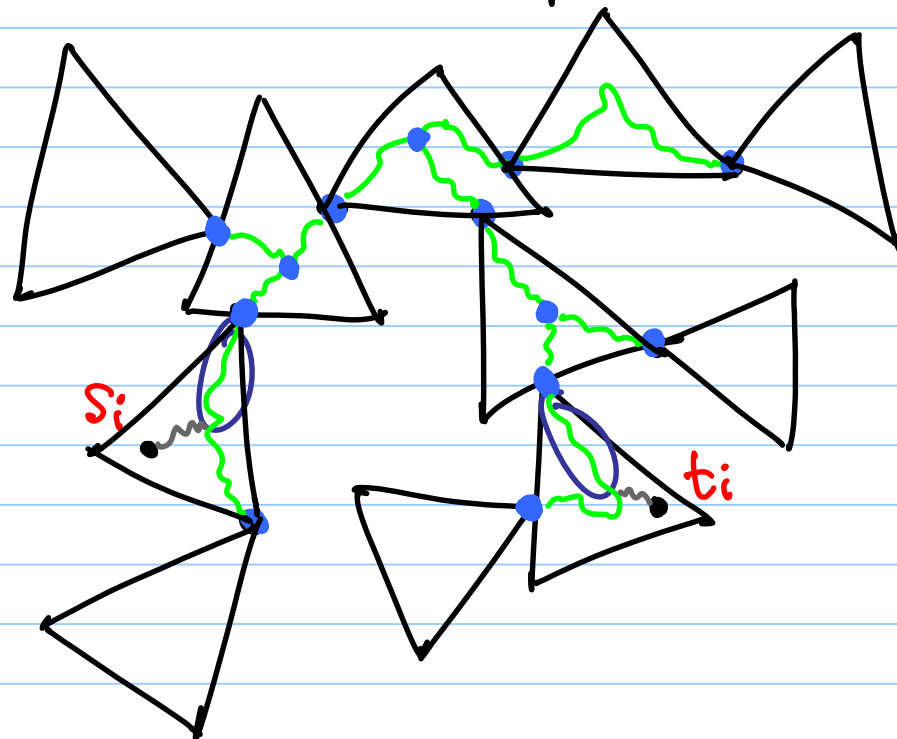
new budget of customer i : $\min \{ \tilde{p}(\sigma), B_i - \sum_{\sigma \in R_i} \tilde{p}(\sigma) \}$

(4) solution to **right-side single-source** instance

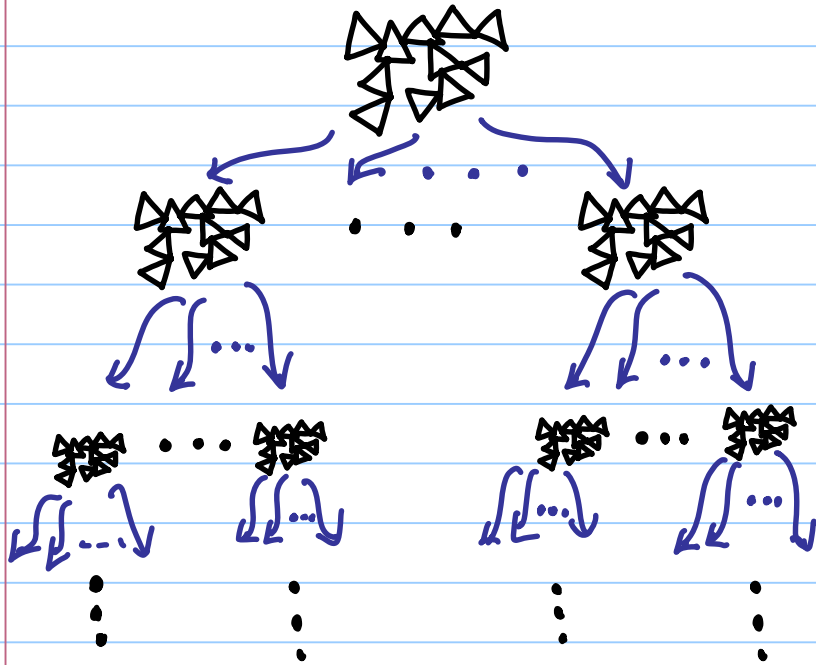


CASE II: BOTTOM LINE

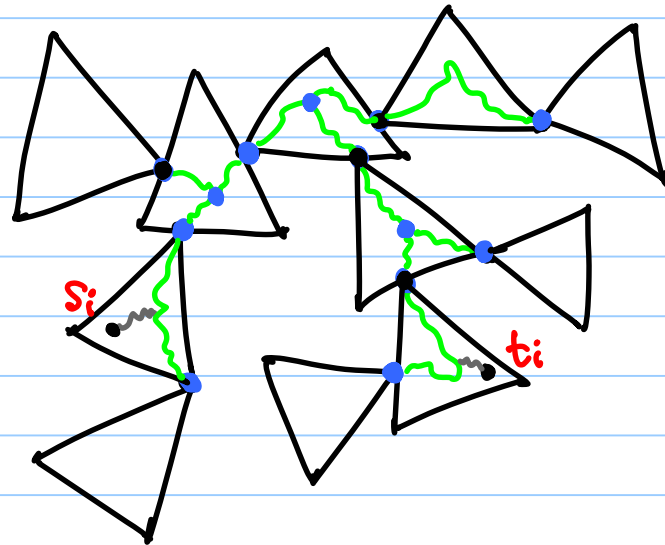
Claim: the randomized price assignment guarantees a constant fraction of the optimal revenue along the skeleton in expectation



QUICK RECAP



$$O\left(\frac{\log m}{\log \log m}\right)$$



outside skeleton } $O(n)$
along the skeleton }

can be derandomized ↪

OPEN QUESTIONS

- improved factor, even for highway problem ?
- additional applications ?

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THANK YOU