Peer to Peer Networks

Lecture notes – 4

**Rehearsal and introduction**

2

1

3

Discrete Markov Chain

p – transition **prob.**

2

1

3

Continuous Markov Chain

λ – transition **rate**

 Equilibrium:  Equilibrium: 

At this course we remain in the Markovian space. We’ll focus mostly on continuous Markov chains from now on.

 is the transition rate, and not the transition probability, and .

We defined Q to be the transition rate matrix such that .

**Birth-Death Chains:**

1

0

2

k

……..

$$λ\_{1}$$

$$λ\_{0}$$

$$λ\_{2}$$

$$λ\_{k-1}$$

$$λ\_{k}$$

$$µ\_{k}$$

$$µ\_{k+1}$$

$$µ\_{3}$$

$$µ\_{2}$$

$$µ\_{1}$$

The assumptions we need on a birth-death process are that in homogeneous Markov chain X(t) where births and deaths are independent and



While is an expression describing a term that converges to 0 ‘faster’ than .

Look at the time Axis at . If we want to be at , what are all the possible states to be at -at time t?

states

Time

t+Δt

t

$$E\_{k}$$

$$E\_{k+1}$$

$$E\_{k}$$

$$E\_{k-1}$$

D1

B1

**Note**: when t is infinitesimally small, only one death/birth can happen at t+Δt.

Denote  the probability to be at state k at time t+Δt:



The above equation is for any k>0. If we open the above terms we get:



Changing sides and dividing by Δt would result:



Adding *lim* to both sides results:



**Note**: when k=0, we need to zero the *k-1* terms in the above equation.

The above describes the probability growth rate to be at state k.

We see that the probability growth rate to be at any state is the sum of all ‘incoming’ rates minus the sum of all ‘outgoing’ rates.

We can ‘wrap’ any number of states and find out what the incoming and outgoing rates are, and retrieve the probability to be at those states.

1

0

2

k

……..

$$λ\_{1}$$

$$λ\_{0}$$

$$λ\_{2}$$

$$λ\_{k-1}$$

$$λ\_{k}$$

$$µ\_{k}$$

$$µ\_{k+1}$$

$$µ\_{3}$$

$$µ\_{2}$$

$$µ\_{1}$$

**Birth Chains:**

Let’s examine a simpler process where- meaning birth-only process:



We can make the process even simpler by assuming  meaning the rates are **constant.** We now get:



where the above is a simple solution of differential equation.

From that we can deduce:



The global formula for would then be:



This describes a **Poisson distribution function**: the probability of k arrivals during interval t.

**Mean value E[k]:**

Consider the random variable K: the number of arrivals at interval t (what used to be α(t)).

K’s mean value:



Remember the following holds: . It’s easy to understand why if we think of each *i* as the probability for exactly i arrivals to happen at interval t. the summation of all probabilities of possible arrival# must give 1.

Therefore, for every t:

 and the variance: 

**Moment generating function G(z):**

Let’s examine the Z-Transform of the Poisson probability function:

Define: 



Finally resulting G(Z) – Pk’s generating function:



By differentiating m times and assigning z=1, we can calculate moment m of K specifically its mean and variance.

**Interarrival-times distribution**

We’d like to break (0,t) to 2k+1 time frames as shown in the figure below:

0

$$β\_{1}$$

$$β\_{1}$$

$$α\_{2}$$

$$α\_{2}$$

$$α\_{1}$$

$$α\_{1}$$

$$β\_{2}$$

……..

Time

$$α\_{k+1}$$

$$α\_{k+13}$$

$$β\_{k}$$

Consider the joint distribution of arrivals when it’s known beforehand that exactly k arrivals have occurred during (0,t).

**Define**: $A\_{k}$- [exactly 1 arrival in each of the $β\_{i}$’s **and** no arrivals at in any of $α\_{i}$’s].

We’d like to calculate the probability of $A\_{k}$ given exactly k arrivals occurred in (0,t):



Note that the α’s and β’s events are independent events, thus the probability can be calculated as a product of the individual probabilities:



And



Giving:

SAME AS IF EACH OF THE POINTS WAS PLACED UNIFORMILY ON (0,t) !!!

\*the last equation is since t=$α\_{1}$+$α\_{2}+…+α\_{k}+β\_{1}+β\_{2}+…+β\_{k+1}$

Note that it doesn’t matter if the points on the time axis were picked randomly with uniform distribution, or picked in advance like in the above analysis.

Furthermore, it is easy to prove that the interval’s starting time doesn’t matter- only its length:



 **interarrival time = exponential distribution:**

**Define**  - the timebetween adjacent arrivals.



Time

arrival

arrival

Accordingly,

**Define the PDF and pdf** and 

Calculation A(t) and a(t) is easy:



This is the *PDF* and *pdf* of the exponential distribution – concluding:

Poisson process implies interarrival process with exponential distribution!

Specifically, this is a *memoryless* process whose properties were largely discussed at lecture-2:



mean value:





**- a constant birth rate**

Let’s finally examine the rate at an infinitesimal period Δt. Recall that:



Similarly (using Taylor):



Moreover – the probability to have two or more arrivals in (t0,t0+Δt) is 0:



And so –

Exponentially distribution implies **constant** birth rate.

**Summary:**

We can summarize by concluding the relations between Poisson’s arrival process, the exponential interarrival times and the constant birth rate as depicted in the following figure.

Each circle implies the other one!

Poisson arrival process

Exponential interarrival times

Constant birth rate

For further usage we’ll calculate A(t) Laplace transform:



**M/M/1 Birth-Death process:**

Consider a model where  and  - Poisson arrival process and Poisson service time:



1

0

2

k

……..

$$λ\_{1}$$

$$λ\_{0}$$

$$λ\_{2}$$

$$λ\_{k-1}$$

$$λ\_{k}$$

$$µ\_{k}$$

$$µ\_{k+1}$$

$$µ\_{3}$$

$$µ\_{2}$$

$$µ\_{1}$$

Note that the service time is the same as the death rate, since when finishing service the customer is gone.

From now and during chapter 3 we’re going to talk about the system after reaching equilibrium – the changes ‘rate’ is 0. Recall that is the probability to have k ‘customers’ at time t, the mathematical equivalent would be:



This is the stationary state 🡪 t has no meaning since no changes occur in time anymore. For each state, the ‘enter-rate’ equals the ‘exit-rate’.

consider the following diagram, focusing on the dashed cut between states k and k-1.

1

0

2

k

……..

$$λ$$

$$λ$$

$$λ$$

$$λ$$

$$λ$$

$$µ$$

$$µ$$

$$µ$$

$$µ$$

$$µ$$

k+1

The following holds:

 and therefore:

 🡪 .

**Define** 

And we get that 

As always, the following holds: 

Giving: 

Note that is the probability for the queue to be empty – meaning that 1- is the probability for the queue to be full/active – which is why  is the utilization of the queue/server.



**Chapter 3: Birth Death queueing Systems in Euqilibrium**

What would change in case  and ?

The equation should be changed to represent the dashed cut in the above figure:





At some cases, this phrase converges to something we can calculate (for example when special relations between the different’s and ’s apply).

**Convergence note**: The above term converges iff starting some for every : .

**Note**: the probabilities converge – the system does not converge to a certain state.

Calculating the queue size’s mean value N:





Recall Little Law: 

Note that the average service time is and that is 1-(utility) 🡪 if the customer will be in the system exactly the average service time as described in the following graph for M/M/1:





T

1

Upto here 3/5/2012