Seminar on Probabilistic Graphical Models - Homework 3

Exact Inference: Variable Elimination

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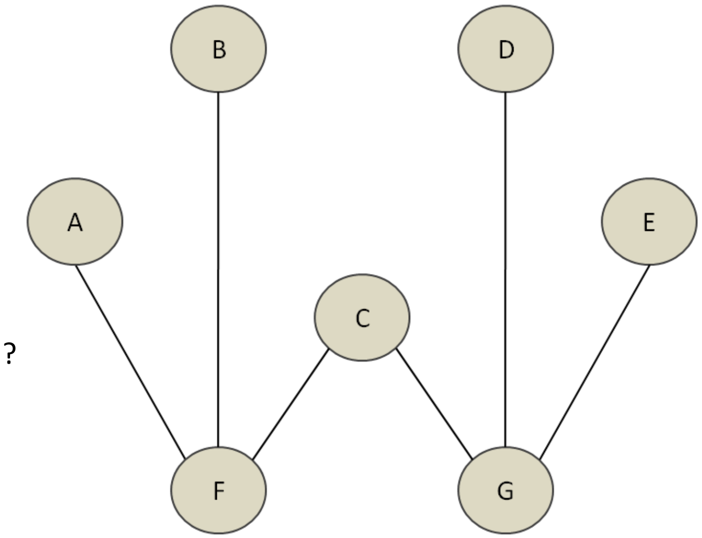
Consider the Markov network (H, Φ) where H is the graph on the right,
and $\Phi = \{\phi_1(A,F), \phi_2(B,F), \phi_3(C,F), \phi_4(C,G), \phi_5(D,G), \phi_6(E,G)\}$.



If we were to run the Variable Elimination (VE) algorithm (algorithm 9.1 in book)
for calculating $P_\Phi(E)$, i.e. invoke *Sum-Product-VE* with
$\Phi = \{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6\}, \boldsymbol{Z} = \{A, B, C, D, F, G\}$:

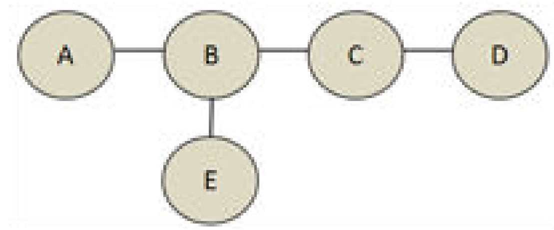
1. When using the elimination ordering $\prec = (A, B, F, C, D, G)$,
   what would be the largest scope size of an intermediate factor ψ ever encountered ?
   (by "scope size" we mean the number of variables a factor operates over,
   its number of inputs).

2. What would it be for the elimination ordering $\prec = (F, G, C, A, B, D)$?

An example run of VE is given below.

An example run of Variable Elimination

For a modeled distribution $P_\Phi(A,B,C,D,E) = (1/Z_\Phi)\phi_1(A,B)\phi_2(B,C)\phi_3(C,D)\phi_4(B,E)$, and the calculation of the marginal distribution $P_\Phi(D)$:



We follow the invocations of the sub-procedure *Sum-Product-Eliminate-Var,* when *Sum-Product-VE* is invoked with $\Phi = \{\phi_1, \phi_2, \phi_3, \phi_4\}, \boldsymbol{Z} = \{A, B, C, E\},\ \prec = (A, B, E, C)$.

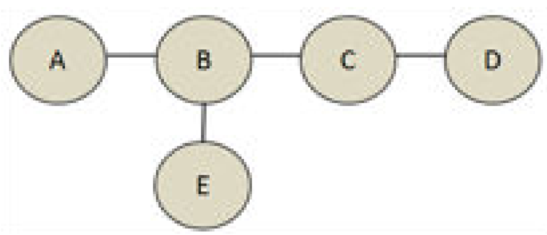
We ultimately get $\phi^*(D) = \Sigma_{A,B,C,E}\phi_1(A,B)\phi_2(B,C)\phi_3(C,D)\phi_4(B,E)$,
which equals $P_\Phi(D) * Z_\Phi$ (D's unnormalized marginal distribution), which can then be normalized to get $P_\Phi(D)$.

Algorithm 9.1 Sum-product variable elimination algorithm

Procedure Sum-Product-VE (
  Φ, // Set of factors
  $\boldsymbol{Z}$, // Set of variables to be eliminated
  $\prec$ // Ordering on $\boldsymbol{Z}$
)
Let $Z_1, \ldots, Z_k$ be an ordering of $\boldsymbol{Z}$ such that
  $Z_i \prec Z_j$ if and only if $i < j$
for $i = 1, \ldots, k$
  $\Phi \leftarrow$ Sum-Product-Eliminate-Var(Φ, Z_i)
$\phi^* \leftarrow \prod_{\phi\in\Phi} \phi$
return ϕ^*

Procedure Sum-Product-Eliminate-Var (
  Φ, // Set of factors
  Z // Variable to be eliminated
)
$\Phi' \leftarrow \{\phi \in \Phi \ :\ Z \in Scope[\phi]\}$
$\Phi'' \leftarrow \Phi - \Phi'$
$\psi \leftarrow \prod_{\phi\in\Phi'} \phi$
$\tau \leftarrow \sum_Z \psi$
return $\Phi'' \cup \{\tau\}$

<u>*Sum-Product-Eliminate-Var(* $\Phi = \{\phi_1(A,B), \phi_2(B,C), \phi_3(C,D), \phi_4(B,E)\}$, $Z = A$):</u>



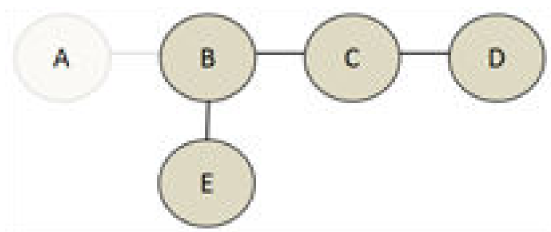
initial graph before elimination

select factors $\phi \in \Phi$ for which $A \in scope(\phi)$:
$\Phi' \leftarrow \{\phi_1\}$

construct intermediate factor:
$\psi(A,B) \leftarrow \prod_{\phi\in\Phi'} \phi = \phi_1(A,B)$

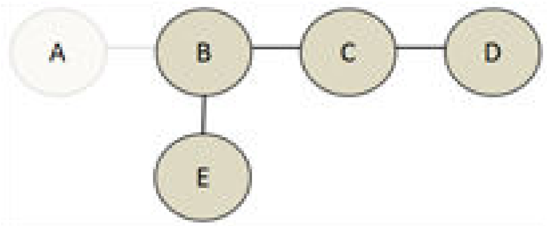
create new factor by summing-out A:
$\tau_1(B) \leftarrow \Sigma_Z \psi = \Sigma_A \psi(A,B)$

update factor set:
$\Phi \leftarrow (\Phi - \Phi') \cup \{\tau_1\} = \{\phi_2, \phi_3, \phi_4, \tau_1\}$



graph after elimination, created by: adding edges between all variables in scope of ψ (no new edges added this time), removing eliminated variable A.

Sum-Product-Eliminate-Var($\Phi = \{\phi_2(B,C), \phi_3(C,D), \phi_4(B,E), \tau_1(B)\}$, $Z = B$):



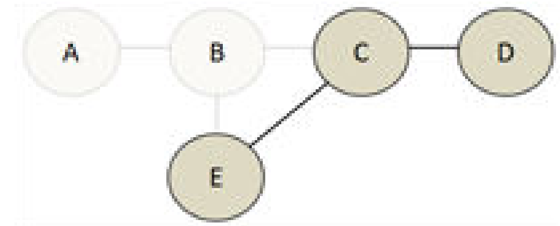
graph before elimination

select factors $\phi \in \Phi$ for which $B \in scope(\phi)$:
$\Phi' \leftarrow \{\phi_2, \phi_4, \tau_1\}$

construct intermediate factor:
$\psi(B,C,E) \leftarrow \prod_{\phi\in\Phi'} \phi = \phi_2(B,C)\phi_4(B,E)\tau_1(B)$

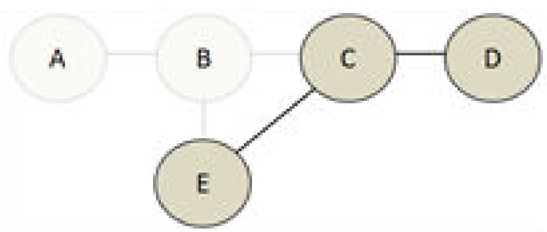
create new factor by summing-out B:
$\tau_2(C,E) \leftarrow \Sigma_Z \psi = \Sigma_B \psi(B,C,E)$

update factor set:
$\Phi \leftarrow (\Phi - \Phi') \cup \{\tau_2\} = \{\phi_3, \tau_2\}$



graph after elimination, created by: adding edges between all variables in scope of ψ (we added $C - E$), removing eliminated variable B.

Sum-Product-Eliminate-Var($\Phi = \{\phi_3(C,D), \tau_2(C,E)\}$, $Z = E$):



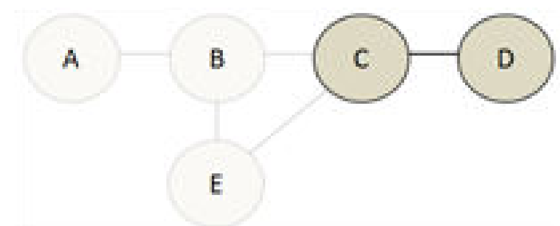
graph before elimination

select factors $\phi \in \Phi$ for which $E \in scope(\phi)$:
$\Phi' \leftarrow \{\tau_2\}$

construct intermediate factor:
$\psi(C,E) \leftarrow \prod_{\phi\in\Phi'} \phi = \tau_2(C,E)$

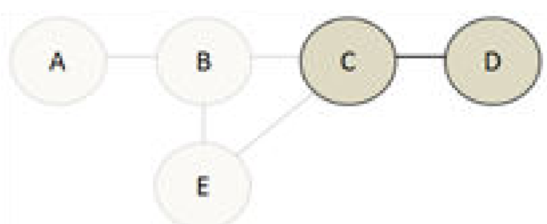
create new factor by summing-out E:
$\tau_3(C) \leftarrow \Sigma_Z \psi = \Sigma_E \psi(C,E)$

update factor set:
$\Phi \leftarrow (\Phi - \Phi') \cup \{\tau_2\} = \{\phi_3, \tau_3\}$



graph after elimination

Sum-Product-Eliminate-Var($\Phi = \{\phi_3(C,D), \tau_3(C)\}$, $Z = C$):



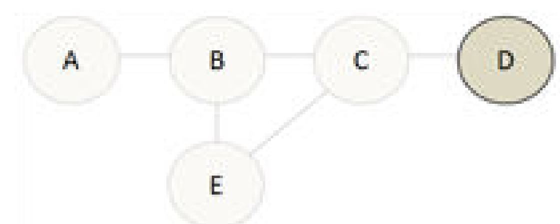
graph before elimination

select factors $\phi \in \Phi$ for which $C \in scope(\phi)$:
$\Phi' \leftarrow \{\phi_3, \tau_3\}$

construct intermediate factor:
$\psi(C,D) \leftarrow \prod_{\phi\in\Phi'} \phi = \phi_3(C,D)\tau_3(C)$

create new factor by summing-out C:
$\tau_4(D) \leftarrow \Sigma_Z \psi = \Sigma_C \psi(C,D)$

update factor set:
$\Phi \leftarrow (\Phi - \Phi') \cup \{\tau_4\} = \{\tau_4\}$



graph after elimination