Trimmed Mobius Inversion And Graphs of Bounded Degree

Seminar on Exponential Algorithms
TAU
In previous episodes...

Lecture 2 (Guy)
Combinatorial techniques

Lecture 3 (Keren)
Inclusion Exclusion (k-coloring)

Previous Lecture (Ido)
Mobius inversion.
Plan

• Fast Mobius inversion on graphs of bounded degree
• Applications to Coloring
Zeta and Möbius transform
Zeta Transform

• For a universe $U$, we consider functions from $P(U)$ to $\mathbb{R}$. For such a function $f : P(U) \to \mathbb{R}$, we define the Zeta and Mobius transforms

\[
\zeta f(S) = \sum_{X \subseteq S} f(X)
\]

\[
\mu f(S) = \sum_{X \subseteq S} (-1)^{|S \setminus X|} f(X)
\]
Key Properties

• Zeta and Mobius invert each other.
• Namely,

\[ \zeta \mu f(S) = \mu \zeta f(S) = f(S). \]

• Proof was shown last lecture
  • main idea - close relation of these transforms to the principle of inclusion exclusion.
Running Example

What is $\zeta f(CD)$?

Based on “Invitation to Algorithmic Uses of Inclusion–Exclusion” (Thore Husfeldt)
Example (Continued)
How to compute Zeta and Mobius Transforms?

• We would like to calculate Zeta (or Mobius) for every subset.
• How fast can we do it?

• Trivial - $O(2^n \cdot 2^n) = O(4^n)$
• Still trivial - $\sum_{k=0}^{n} \binom{n}{k} 2^k = O(3^n)$ (Newton)
• Yates - $O^*(2^n)$ (Last Lecture)
Fast Zeta Transform

- We may assume that $N = \{1, 2, \ldots, n\}$
- To compute $\zeta f$, let initially
  
  $$g_0(X) = f(X)$$
  
  $$g_j(X) = [j \in X]g_{j-1}(X\{j\}) + g_{j-1}(X)$$
FZT - Invariant

\[ g_i(S) = \sum_{R \subseteq S} B(R) f(R) \]

Where:

\[ B(R) = [S \cap \{i + 1, i + 2, \ldots n\} = R \cap \{i + 1, i + 2, \ldots n\}] \]
Example

$$g_j(X) = \left[ j \in X \right] g_{j-1}(X \setminus \{j\}) + g_{j-1}(X)$$
Example

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Example

\[ g_j(X) = [j \in X]g_{j-1}(X\{j\}) + g_{j-1}(X) \]
\[ \{D, \{BD\}, \{CD\}, \{BCD\}\} \]

\[ g_i(S) = \sum_{R \subseteq S} B(R) f(R) \]

\[ B(R) = [S \cap \{i + 1, i + 2, \ldots n\} = R \cap \{i + 1, i + 2, \ldots n\}] \]

\[ B(R) = [S \cap \{D\} = R \cap \{D\}] \]
Example

\[ g_j(X) = [j \in X]g_{j-1}(X \setminus \{j\}) + g_{j-1}(X) \]
Fast Mobius Transform

• We may assume that \( N = \{1, 2, \ldots, n\} \)
• To compute \( \mu f \), let initially

\[
g_0(X) = f(X)
\]

And then:

\[
g_j(X) = -[j \in X] g_{j-1}(X \{j\}) + g_{j-1}(X)
\]
Trimmed Evaluation
Closure

- If $F$ is a family of sets in a universe $U$ we’ll denote the outer closure of $F$ by

$$\overline{F} = \{S \in P(U) \mid \exists A \in F, A \subseteq S\}$$

$$U = \{1,2,3\}$$

$$F = \{\{1,2\}, \{3\}\}$$

$$\overline{F} = \{\{1,2\}, \{1,2,3\}, \{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$
Closure of Zeta and Mobius

• We will try and develop an algorithm that runs in time

\[ O^*\left(|\text{supp}(f)|\right) \]
Closure of Zeta and Mobius

• Claim:

\[ \text{supp}(\zeta f), \text{supp}(\mu f) \subseteq \text{supp}(f) \]

Why?

If \( \zeta f (S) \neq 0 \) it means that \( \sum_{R \subseteq S} f(R) \neq 0 \).

Hence, there is some \( R \subseteq S \) with \( f(R) \neq 0 \) and

\[ S = R \cup S \setminus R \in \text{supp}(f) \]
Towards a trimmed algorithm

• Note that this algorithm allows us to compute elements according the their rank (size of the subset).

• Also, we have that $g_j(X) = 0$ if $X \not\in \text{supp}(f)$.
  • Recall the invariant $g_i(S) = \sum_{R \subseteq S} B(R)f(R)$.

\[ g_0(X) = f(X) \]
\[ g_j(X) = \lfloor j \in X \rfloor g_{j-1}(X\setminus\{j\}) + g_{j-1}(X) \]
Trimmed Zeta Evaluation

1. Create n lists (for each subset size) $L_i$
2. Add every $x \in supp(f)$ into $L_{|x|}$. Let $r$ be the first index for which $L_r$ is not empty.
3. Loop until $r = n$ and $L_n = \emptyset$:
4. $X = \text{POP}(L_r)$
5. Set $g_0(X) = f(X)$ for each $j = 1, 2, \ldots, n$, compute
   $$g_j(X) = g_{j-1} + [j \in X]g_{j-1}(X \setminus \{j\})$$
6. If $g_n(X) \neq 0$ then:
   output $X, g_n(X)$.
   For each $j \notin X$ insert $X \cup \{j\}$ into $L_{r+1}$. 
\[ g_0(ABC) = f(ABC) = 4 \]
\[ g_1(ABC) = g_0(ABC) + g_0(BC) = 5 \]
\[ g_2(ABC) = g_1(ABC) + g_1(AC) = 7 \]
\[ g_3(ABC) = g_2(ABC) + g_2(AB) = g_2(ABC) \]
\[ g_4(ABC) = g_3(ABC) = 7 \]
Colouring and Zeta
Covers - Introduced in Keren’s Lecture (2)

• Let $F$ be a family of sets in a universe $U$.

• The cover number $c(X)$ of a set $X$ is the number of $k$-tuples $(S_1, ... S_k)$ such that $S_i \in F$ and $S_1 \cup S_2 \cup S_3 ... S_k = X$.

$$c(X) = \sum_{S_1 \cup S_2 \cup S_3 ... S_k = X} f(S_1)f(S_2) ... f(S_k)$$
Covering and Coloring

Let $G = (V, E)$ be a graph. Then $G$ is $k$-colorable if and only if $c(V) > 0$ where, $c(V)$ is the cover number with $F = \{ A \subseteq V, A \text{ is an independent set} \}$
Cover with zeta

Claim: Let $f$ be the indicator function of the family $F$. Then

$$
\zeta c(X) = [\zeta f(X)]^k
$$

$$
\sum_{R \subseteq X} c(R) = \sum_{R \subseteq X} \sum_{R_1 \cup R_2 \ldots \cup R_k = R} f(R_1) \ldots f(R_k) = \sum_{R_1, \ldots, R_k \subseteq X} f(R_1)f(R_2) \ldots f(R_k)
$$

Both count the number of k-tuples with $S_i \in F$ and $S_i \subseteq X$

We have actually used this property with different notations in lecture 2!
Running Example
1. Create $n$ lists (for each subset size) $L_i$
2. Add every $x \in \text{supp}(f)$ into $L_{|x|}$. Let $r$ be the first index for which $L_r$ is not empty.
3. Loop until $r = n$ and $L_n = \emptyset$:
4. $X = \text{POP}(L_r)$
5. Set $g_0(X) = f(X)$ for each $j = 1, 2, \ldots, n$, compute
   \[ g_j(X) = g_{j-1} + [j \in X]g_{j-1}(X \setminus \{j\}) \]
6. If $g_n(X) \neq 0$ then start inverting:
7. Set $h_0(X) = g_n(X)^k$ for each $j = 1, 2, \ldots, n$, compute:
   \[ h_j(X) = h_{j-1} - [j \in X]h_{j-1}(X \setminus \{j\}) \]
8. If $h_n(X) \neq 0$ then:
   For each $j \not\in X$ insert $X \cup \{j\}$ into $L_{r+1}$.
   Output $(X, h_n(X))$
Bounds for graphs of Bounded Degree
Main Combinatoric Theorem

Let $G = (V, E)$ be a graph with maximum degree $\Delta$. Let $D$ be the family of dominating sets of $G$.

Then:

$$|D| \leq (2^{\Delta+1} - 1)^{\frac{n}{\Delta+1}}$$
Bounds for coloring

• A graph can be covered with \( k \) independent sets if and only if it can be covered by \( k \) maximal independent sets.
• Every maximal independent set is a dominating set.
• We may decide for a graph whether it is k-colorable using cover number algorithm with the family $F$ of maximal independent sets.

• Hence $|\text{supp}(f)| \leq \#\{\text{dominating sets}\} \leq (2^{\Delta+1} - 1)^{\frac{n}{\Delta+1}}$

• Can list all maximal independent sets in time $O^*(1.5^n)$ [Moon and Moser ; Tsukiyama et al.]
Shearer’s Lemma - Guy’s Lecture (2)

- Let $U$ be a finite set with a collection $S = \{S_1, S_2, \ldots, S_r\}$ of subsets of $U$ such that every element $u \in U$ is contained in at least $\delta$ subsets of $S$.
- Let $T$ be another family of subsets of $U$.
- For each $1 \leq i \leq r$, we define the projection
  $$T_i = \{t \cap S_i : t \in T\}$$
- Then
  $$|T|^\delta \leq \prod_{i=1}^{r} |T_i|$$
Proof of main theorem

- Define for each vertex $A_v = v \cup N(v)$.

- For each vertex $u$ with degree $d < \Delta$ add $u$ to $\Delta - d$ sets not already containing $u$.

- Let $a_v = |A_v|$. What is $\sum a_v$?

  $$\sum a_v = (\Delta + 1)n$$

$A_d = \{A, D\}$
• By construction, every vertex $u$ is exactly in $\Delta + 1$ subsets $A_v$.

• Let’s look at the projections

$$D_v = \{A_v \cap S, |S \in D\}$$

• Every subset in $D_v$ is a dominating set, thus $\emptyset \notin D_v$. Hence:

$$|D_v| \leq 2^{a_v} - 1$$
From Shearer we have that:

\[ |D|^{\Delta+1} \leq \prod_{v}(2^{a_v} - 1) \]

If we expand the right term we have

\[
\left[ \prod_{v}(2^{a_v} - 1) \right]^\frac{1}{n} = \frac{1}{n} \log[\prod_{v}(2^{a_v} - 1)] = \frac{1}{n} \sum \log[2^{a_v} - 1]
\]

Since the function \( \log(2^x - 1) \) is concave Jensen gives us:

\[
\frac{1}{n} \sum_{v} \log(2^{a_v} - 1) \leq \log \left( 2 \frac{\sum a_v}{n} - 1 \right) = \log(2^{\Delta+1} - 1)
\]

We finally have that \( |D|^{\Delta+1} \leq \left( 2^{\Delta+1} - 1 \right)^n \) which yields the required result.
Summary

- We have shown how to decide if a graph is $k$-colorable in time $O^*\left[\left(2^{\Delta+1} - 1\right)^{\frac{n}{\Delta+1}}\right]$.
- For example, for $\Delta = 3$ we have a bound of $\approx 1.96^n$. 
Bipartite Subgraphs

- Let $B$ be the family of vertex sets that induce bipartite subgraphs.
- Covering by $k$ maximal bipartite sets is equivalent to coloring with $2k$ colors.
Odd case

- If the graph is $k$-colorable for odd $k$ we may cover as set $V' \subseteq V$ with $\frac{k-1}{2}$ bipartite sets such that $V \setminus V'$ is an independent set.
Let $\mathcal{B}$ denote the family of independent sets of a graph $G$.

**Theorem:**

$$\left| \max \mathcal{B} \right| \leq (2^{\Delta+1} - \Delta - 1)^{\frac{n}{\Delta+1}}$$

**Proof** essentially the same as before, noting that if we look at $\max \mathcal{B} \cap A_v$, then it doesn’t contain $\emptyset, u_1, \ldots, u_\Delta$ where the $u_i$’s are neighbors of $v$.

This allows us to improve the bound to $\approx 1.86^n$
Bipartite subgraphs and coloring

- A graph can be covered by $k$ independent sets if and only if it can be covered by $k$ maximal bipartite sets.
- If $k$ is odd we will use $\frac{k-1}{2}$ maximal bipartite sets and check whether we may find output $X$ such that $c(X) > 0$ and $V \setminus X$ is independent.
- Hence, we may use algorithm C using $\text{maxB}$ and get a better running time of
  \[
  O^* \left[ \left( 2^{\Delta+1} - \Delta - 1 \right)^\frac{n}{\Delta+1} \right]
  \]
- Remark: Assume that we can list maximal bipartite sets efficiently.
Proof of theorem

- Define for each vertex $A_v = v \cup N(v)$. Assume first the $G$ is regular. Let $A_v = \{v, u_1, u_2, \ldots u_\Delta\}$.
- Now, to use shearer we should consider $B_v = A_v \cap \text{maxB}$.
- Note that $\emptyset, \{u_1\}, \{u_1\}, \ldots, \{u_\Delta\} \notin B_v$.
- Why? Follows easily from maximality.
- Hence, $B_v \leq 2^{\Delta+1} - \Delta - 1$
Non Regular Case

For each vertex $u$ with degree $d < \Delta$ add $u$ to $\Delta - d$ sets $A_v$ not already containing $u$.

- If we added $y = \Delta - d(v)$ new vertices to a set $B_v$ that contained $x = d(v) + 1$ vertices

$$|B_v| \leq 2^y (2^x - x - 1) \leq 2^{y-x+y} 2^{y+x} - (y + x) - 1 = 2^{\Delta+1} - \Delta - 2$$

So again we have that

$$|B_v| \leq 2^{\Delta+1} - \Delta - 1$$

We can proceed by Shearer and Jensen.
Questions?