Strongly Connected Components with DFS

- **Tarjan 1972** ([wikipedia](https://en.wikipedia.org/wiki/Strongly_connected_component)
- Performs a single DFS with an additional Stack
Strongly connected components (SCC)

Def: Equivalence classes of the relation \((u, v)\) iff there is a path from \(u\) to \(v\) and a path from \(v\) to \(u\)

Goal: Compute for each vertex the number of its component
Strongly connected components (SCC)

Observation: Let \( u \) and \( v \) be in a SC \( C \). Let \( P \) be a path from \( u \) to \( v \) then every vertex on \( P \) is also in \( C \).
DFS(G)
1. For each vertex $u \in G.V$
2. $u.\text{color} = \text{WHITE}$
3. $u.\pi = \text{NIL}$
4. $\text{time} = 0$
5. For each vertex $u \in G.V$
6. if $u.\text{color} == \text{WHITE}$
7. DFS-Visit(G,u)

DFS-Visit(G,u)
1. $\text{time} = \text{time} + 1$
2. $u.d = \text{time}$
3. $u.\text{color} = \text{GRAY}$
4. For each vertex $v \in G.\text{Adj}[u]$
5. if $v.\text{color} == \text{WHITE}$
6. $v.\pi = u$
7. DFS-Visit(G,v)
8. $u.\text{color} = \text{BLACK}$
9. $\text{time} = \text{time} + 1$
10. $u.f = \text{time}$

Thm 2: $[v.d, v.f] \subset [u.d, u.f]$ iff $v$ is a descendant of $u$ in the DFS-forest,
$[v.d, v.f] \cap [u.d, u.f] = \emptyset$ iff $u$ and $v$ are unrelated in the DFS-forest.
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10. $u.f = \text{time}$

**Thm 3:** $v$ is a descendant of $u$ in the DFS-forest iff when we invoke DFS-Visit(u) there is a white path from $u$ to $v$
Edge classification

• Tree edges: \((u,v)\), DFS-Visit\(v\) was called from DFS-visit\(u\)
• Back edges: \((u,v)\) such that \(v\) is an ancestor of \(u\) in the DFS-forest
• Forward edges: nontree edges \((u,v)\) such that \(v\) is a descendant of \(u\)
• Cross edges: \((u,v)\) such that \(v.f < u.d\)

Observe, among non-tree edges:
1. (Only) backward edges go to a vertex with a later finish time
2. (Only) forward edges go to a vertex with later discovery time
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Tarjan’s algorithm

• We will find SCCs using only one DFS

Thm 9: If $u$ and $v$ are in the same SC then they are in same tree of the DFS-forest and $LCA(u, v)$ is also in this component

Corr: Each SC form a **connected** subtree of the DFS-forest
Identify the roots of these subtrees
Use an additional stack ($S$)
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