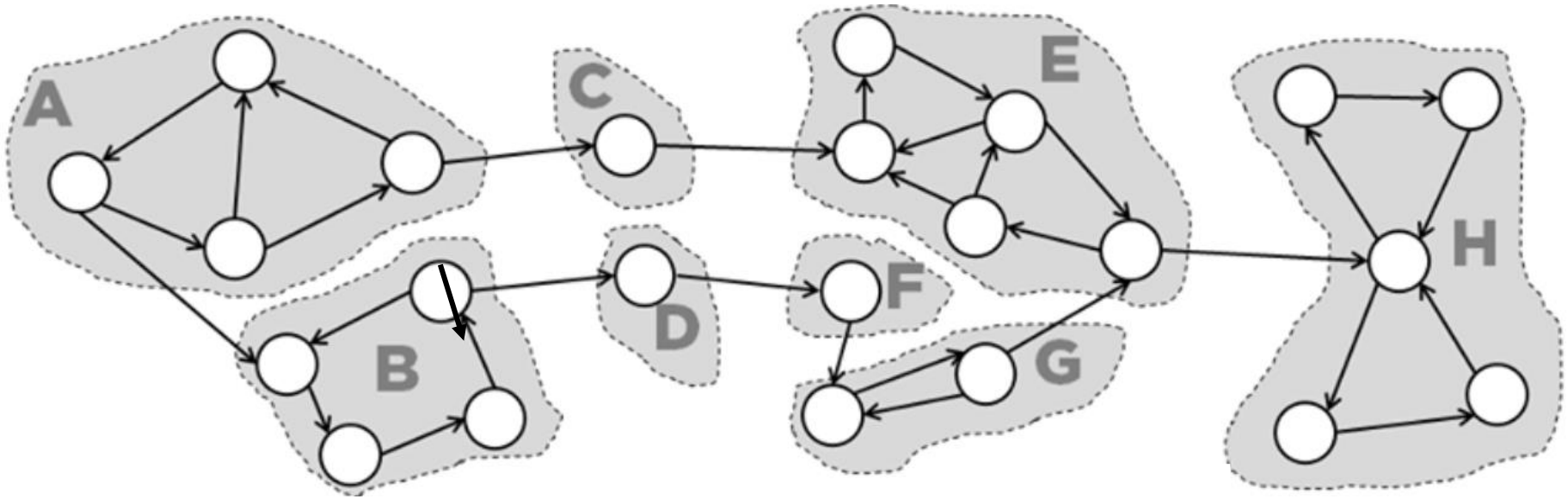
Strongly Connected Components with DFS

- Tarjan 1972 (wikipedia)
- Performs a single DFS with an additional **S**tack

Strongly connected components (SCC)

Def: Equivalence classes of the relation (u, v) iff there is a path from u to v and a path from v to u



Goal: Compute for each vertex the number of its component

Strongly connected components (SCC)

Obs: Let u and v be in a SC C. Let P be a path from u to v then every vertex on P is also in C

```
DFS(G)
For each vertex u ∈ G.V
    u.color = WHITE
    u.π = NIL
time = 0
For each vertex u ∈ G.V
    if u.color == WHITE
        DFS-Visit(G,u)
```

```
DFS-Visit(G,u)
time = time + 1
u.d = time
u.color = GRAY
For each vertex v ∈ G.Adj[u]
    if v.color == WHITE
        v.π = u
        DFS-Visit(G,v)
u.color = BLACK
time = time + 1
u.f = time
```

Thm 2: $[v.d, v.f] \subset [u.d, u.f]$ iff v is a descendant of u in the DFS-forest, $[v.d, v.f] \cap [u.d, u.f] = \emptyset$ iff u and v are unrelated in the DFS-forest

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u.color = BLACK
time = time + 1
u.f = time
```

Thm 3: v is a descendant of u in the DFS-forest iff when we invoke DFS-Visit(u) there is a white path from u to v

Edge classification

- Tree edges: (u,v), DFS-Visit(v) was called from DFS-visit(u)
- Back edges: (u,v) such that v is an ancestor of u in the DFS-forest
- Forward edges: nontree edges (u,v) such that v is a descendant of u
- Cross edges: (u,v) such that $v.f < u.d$

Observe, among non-tree edges:

1. (Only) backward edges go to a vertex with a later finish time
2. (Only) forward edges go to a vertex with later discovery time

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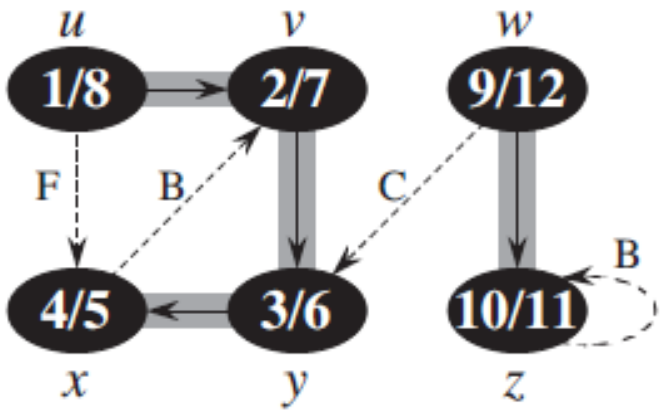
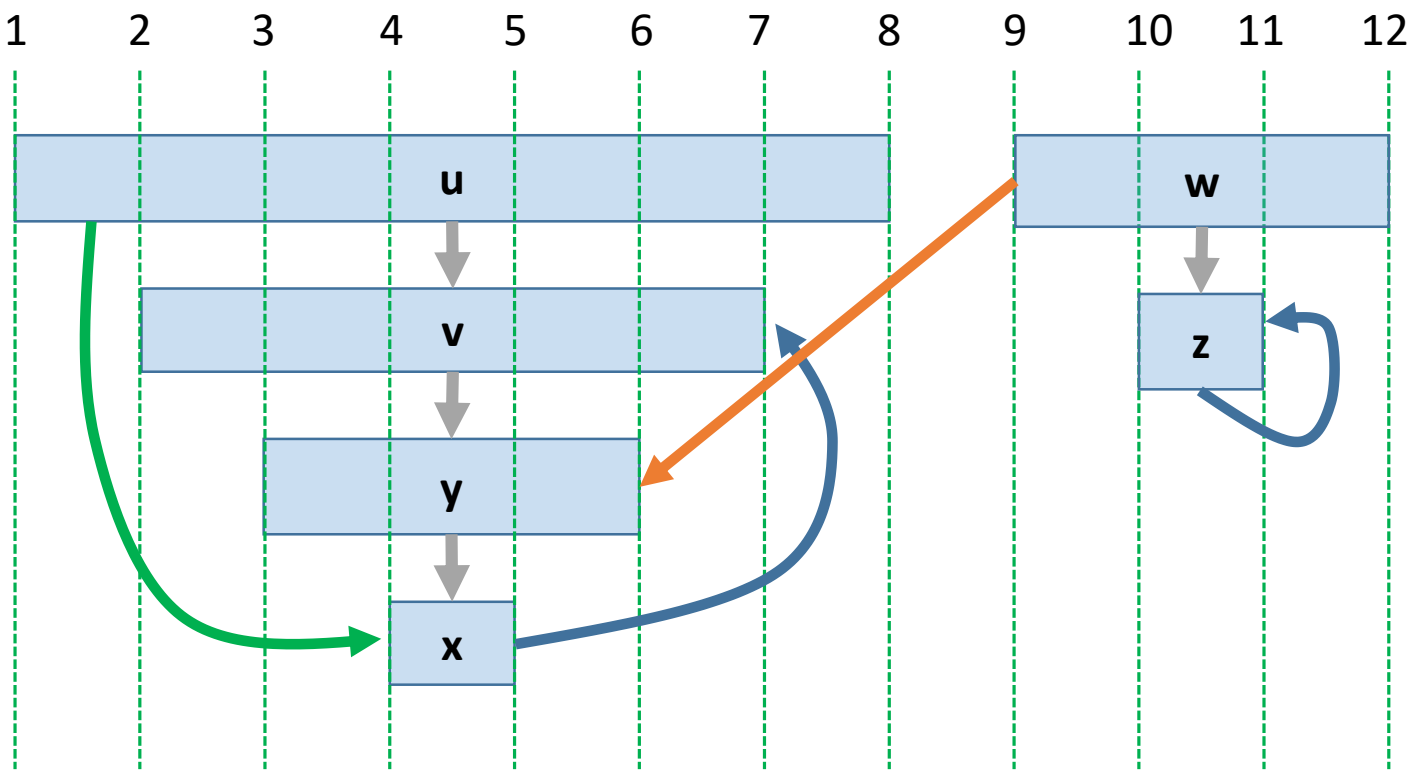
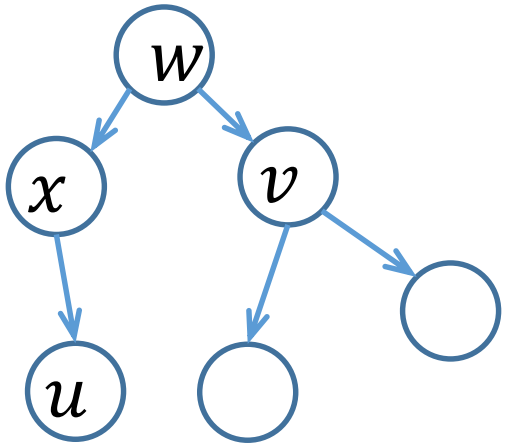

u
v
w
1/8
2/7
9/12
F
B
C
B
4/5
3/6
10/11
x
y
z


1
2
3
4
5
6
7
8
9
10
11
12
u
w
v
z
y
x

Tarjan's algorithm

- We will find SCCs using only one DFS

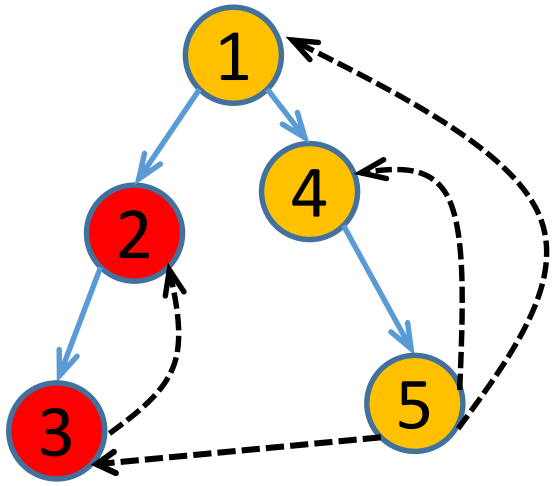
Thm 9: If u and v are in the same SC then they are in same tree of the DFS-forest and $LCA(u, v)$ is also in this component

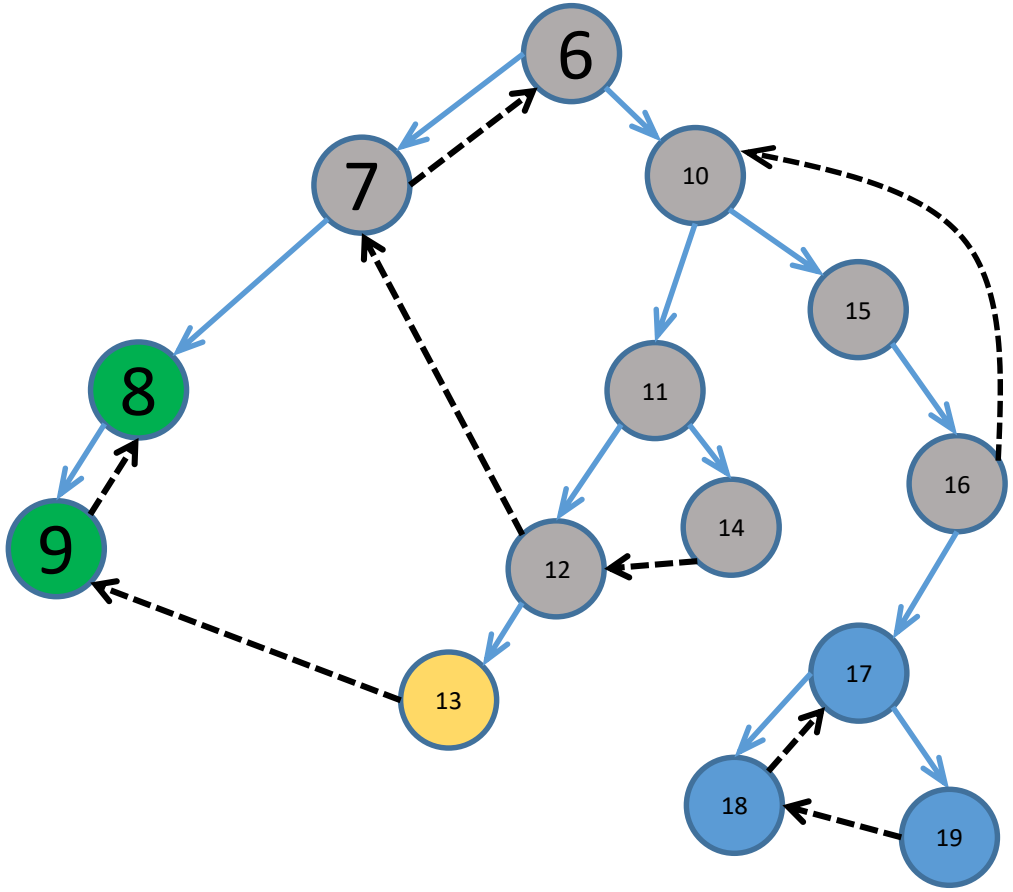


Proof: …

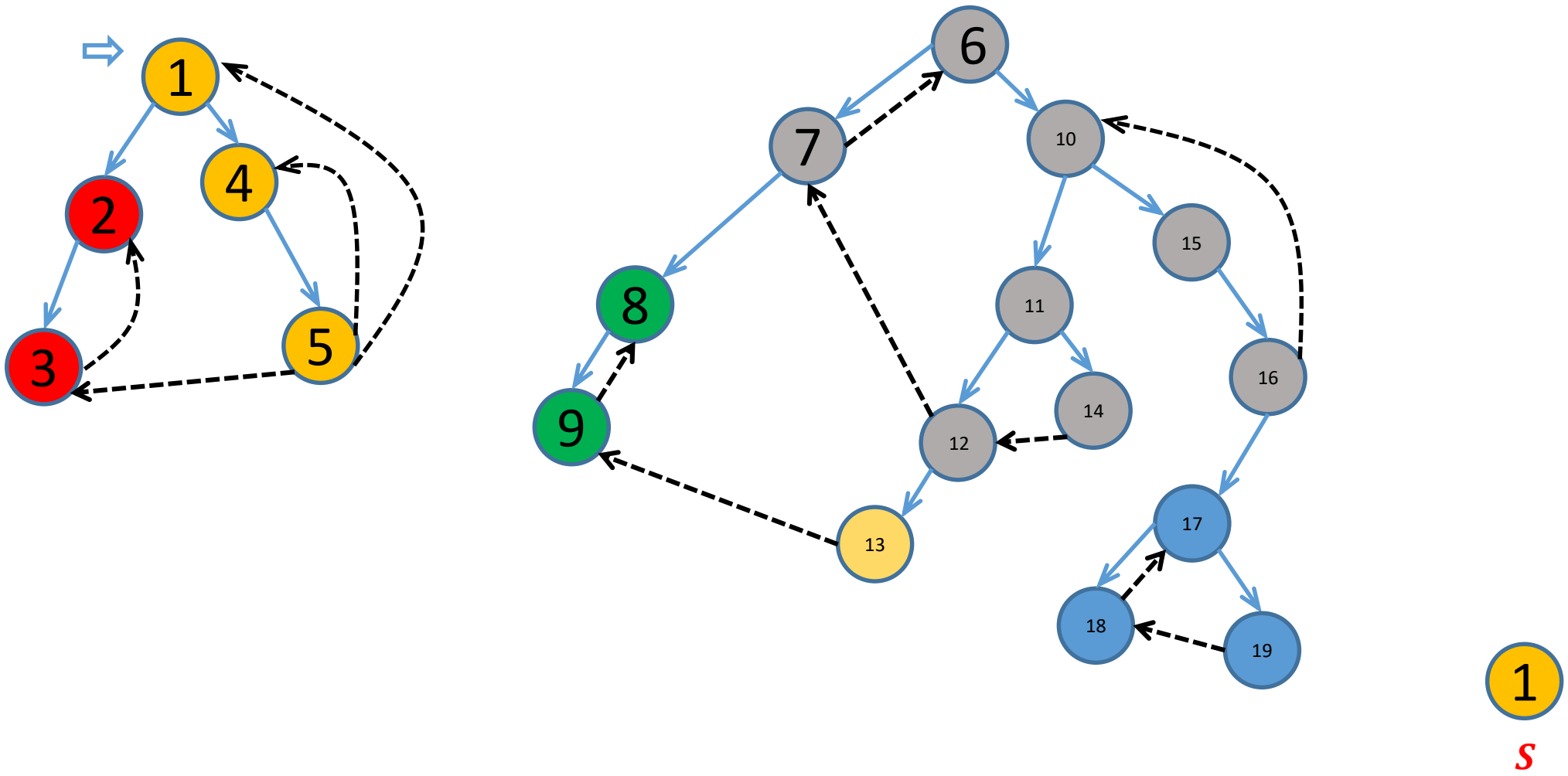
Corr: Each SC form a **connected** subtree of the DFS-forest

Identify the roots of these subtrees





Use an additional stack (S)



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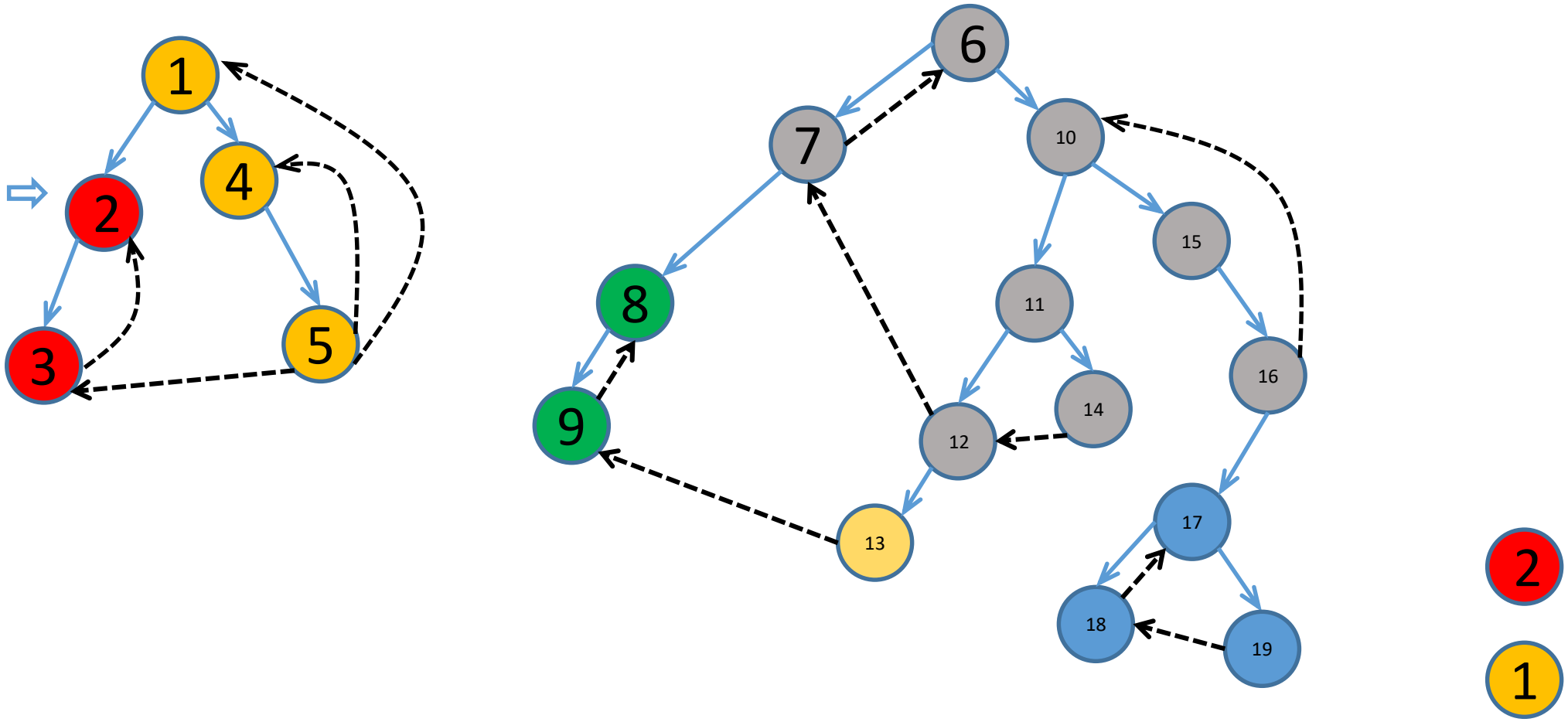

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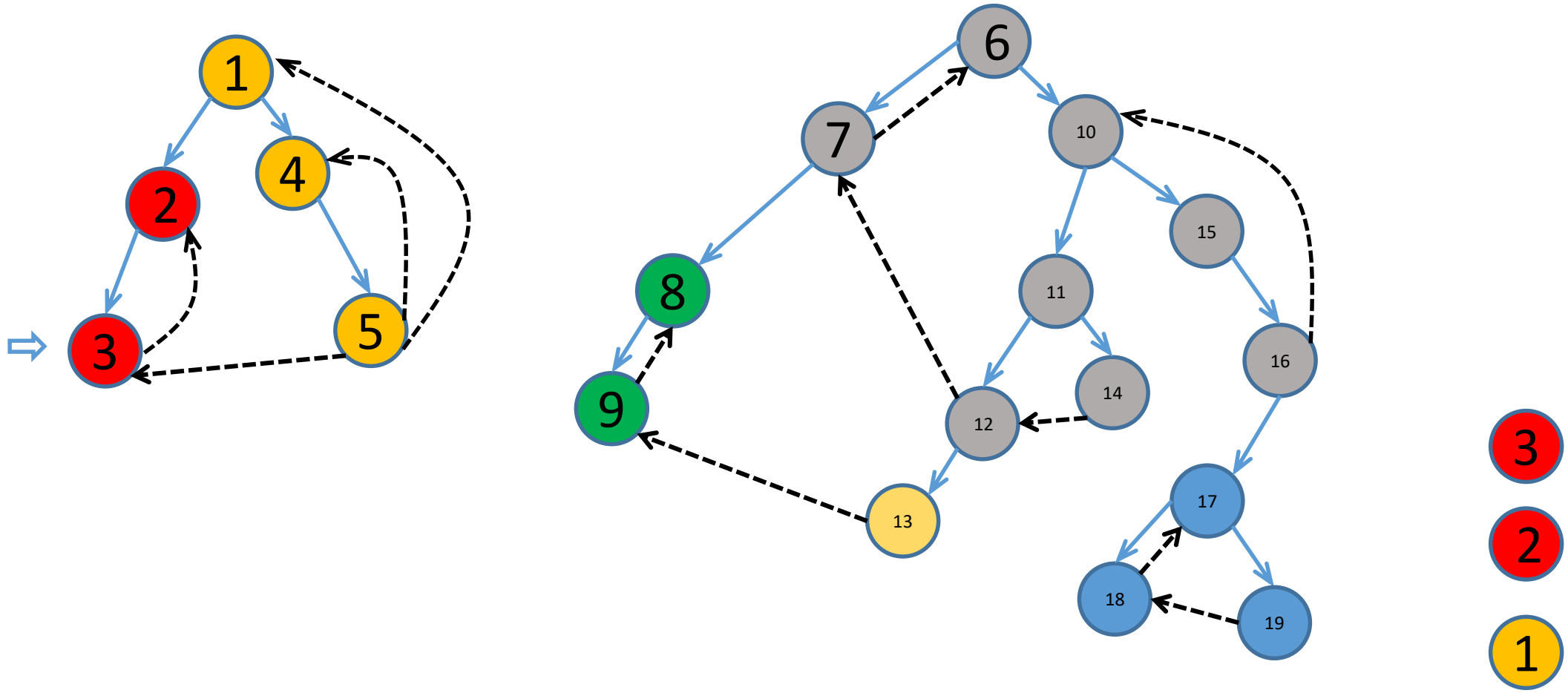

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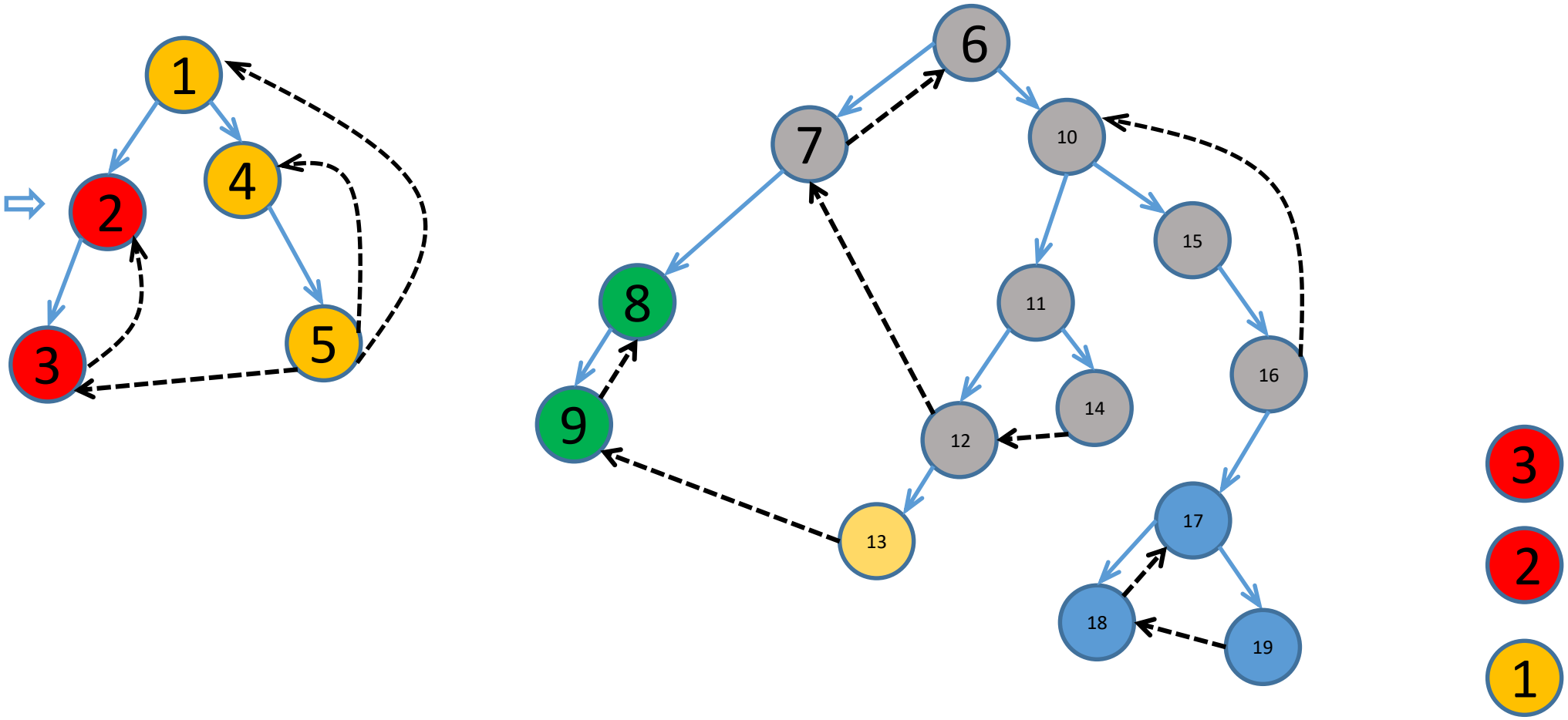

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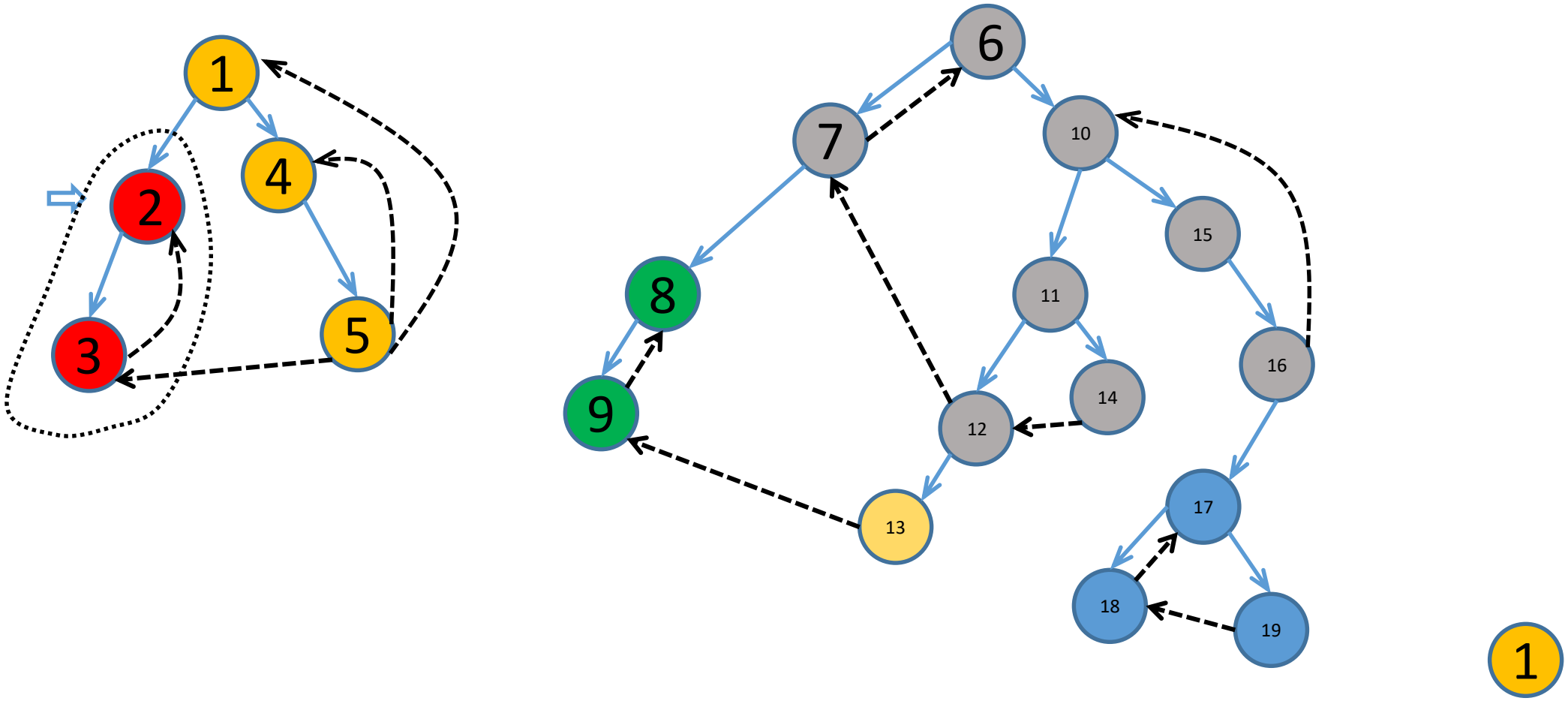

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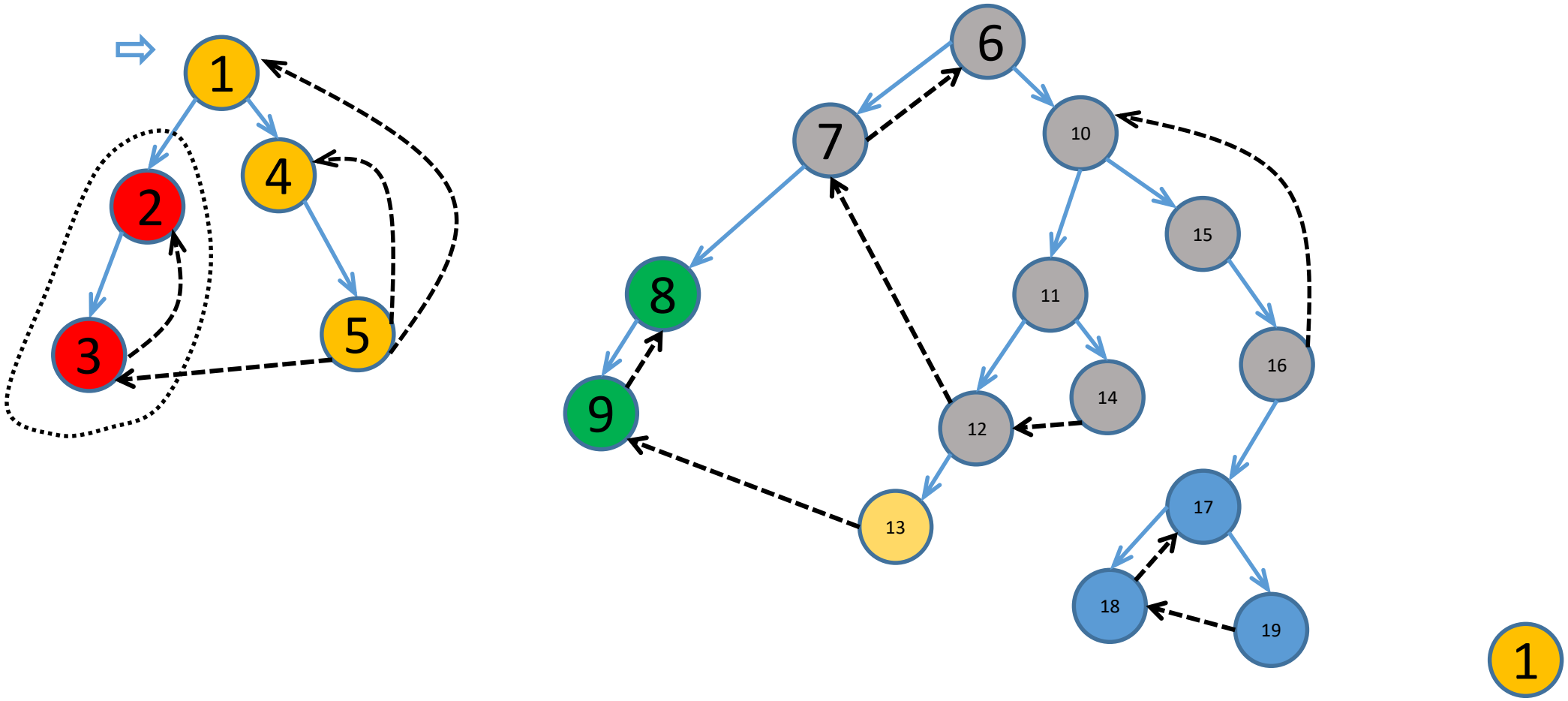

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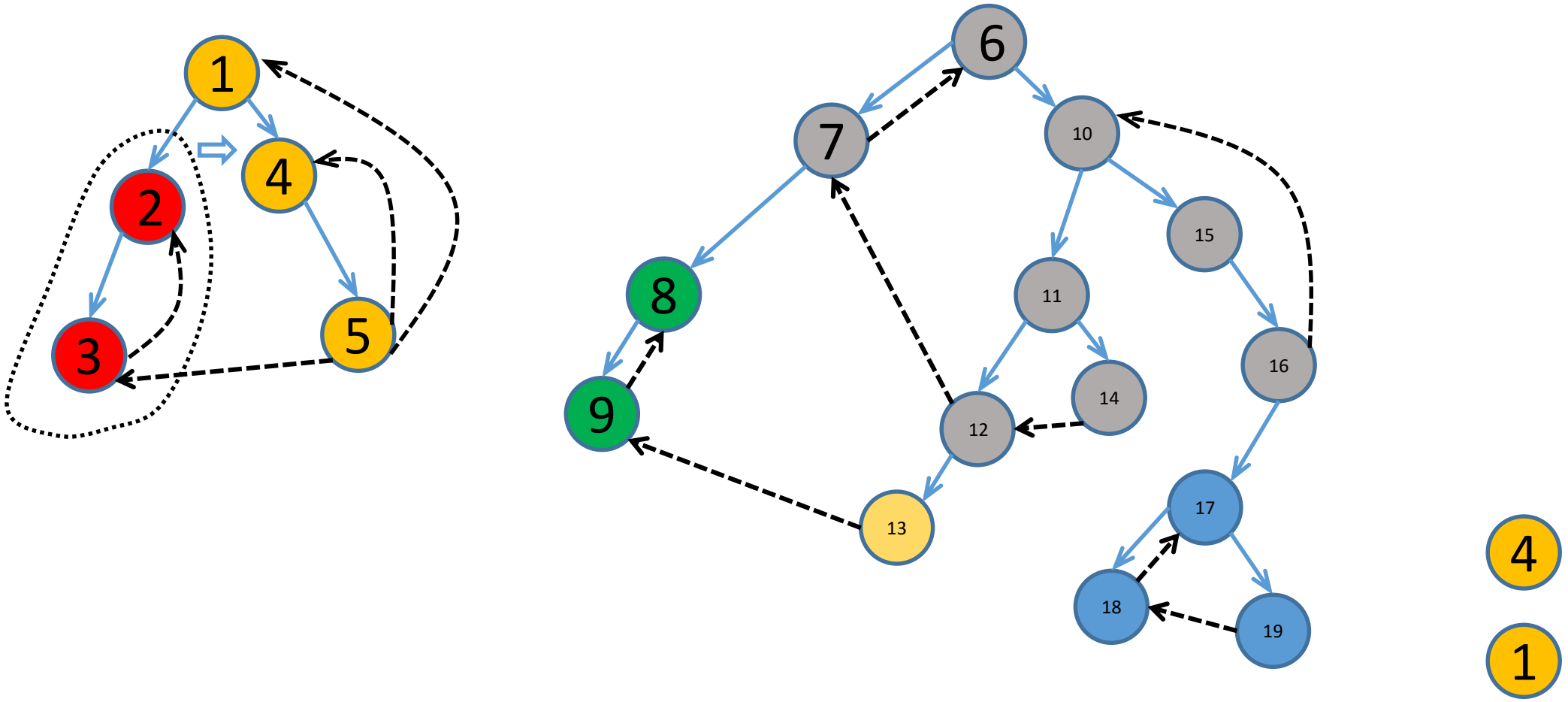

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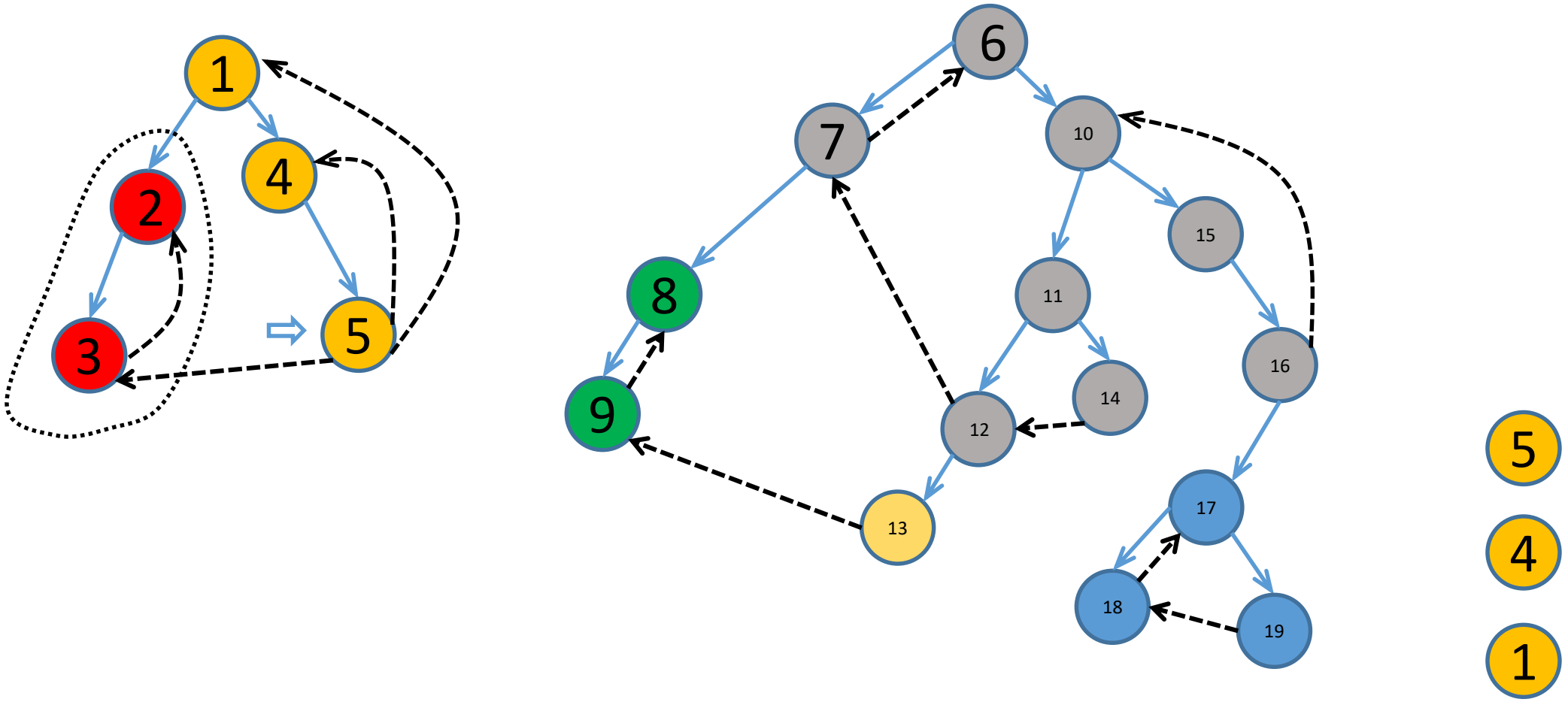

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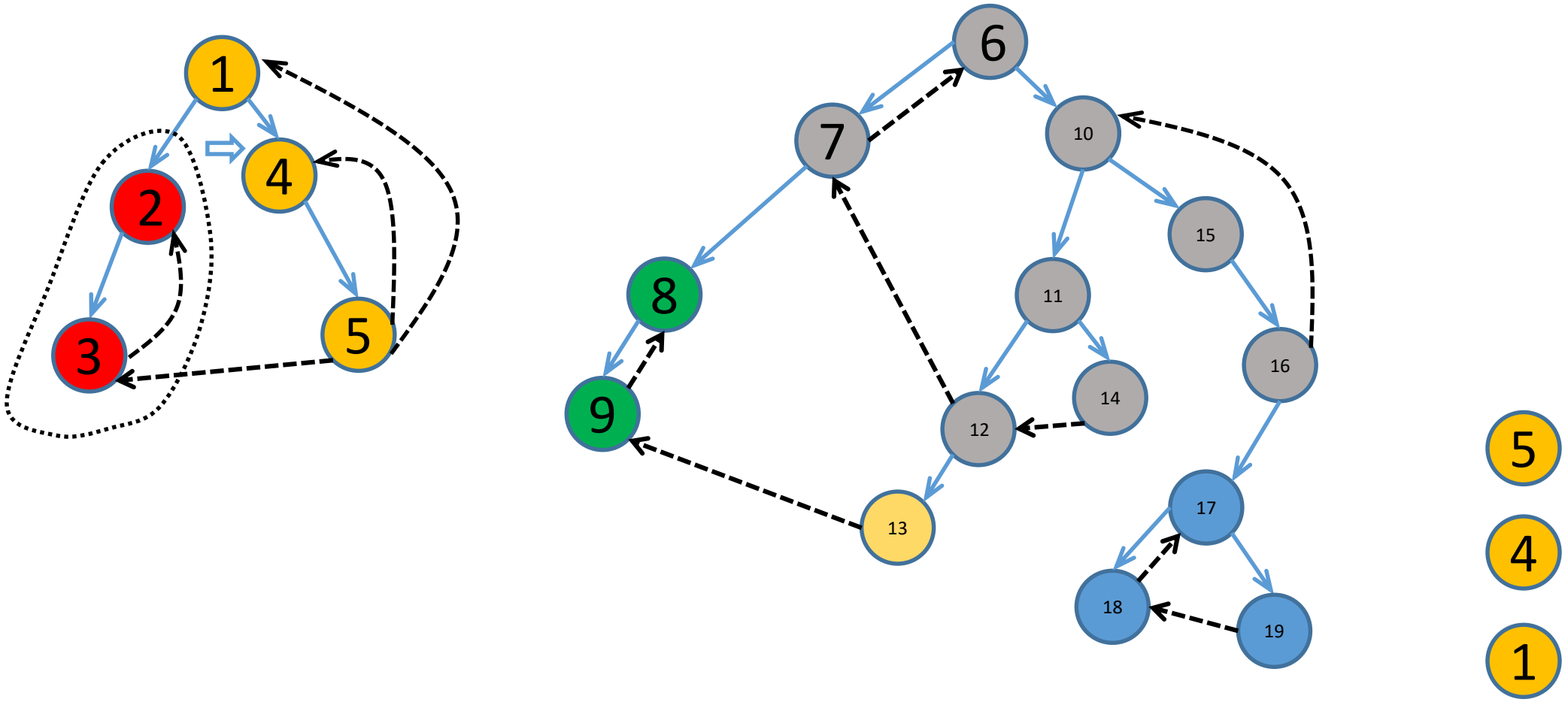

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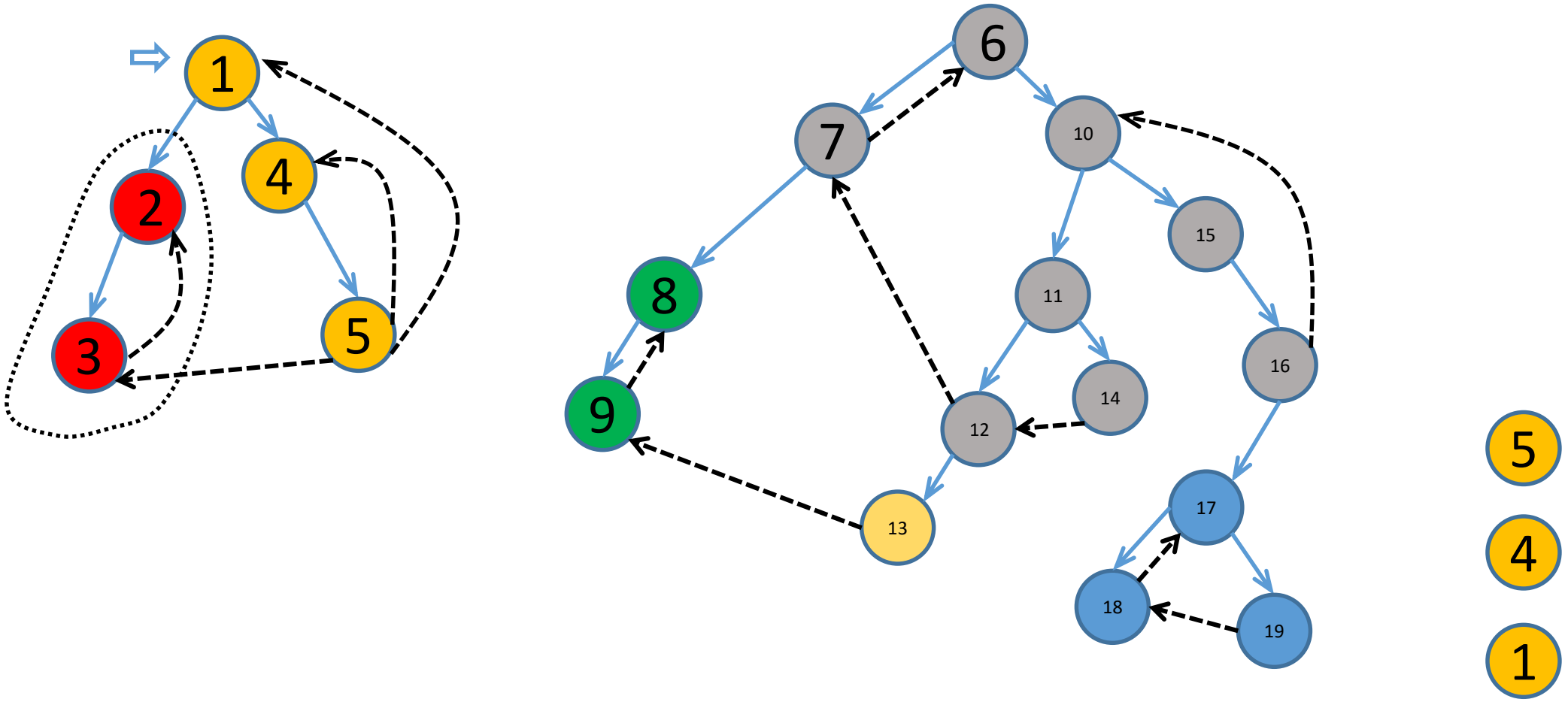

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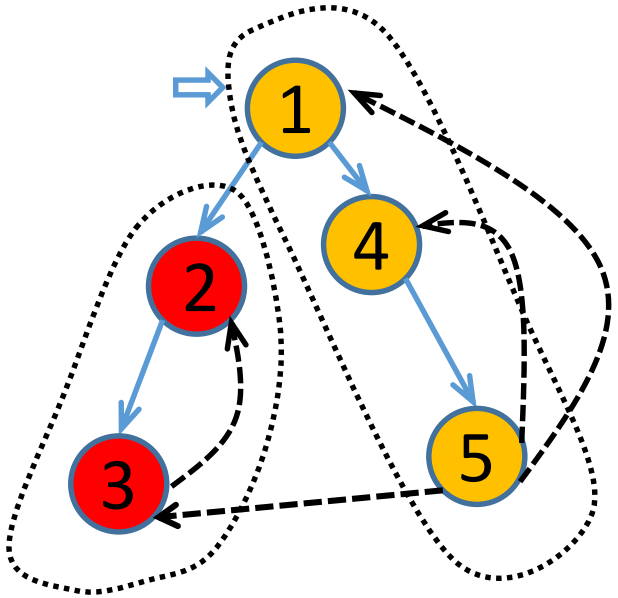

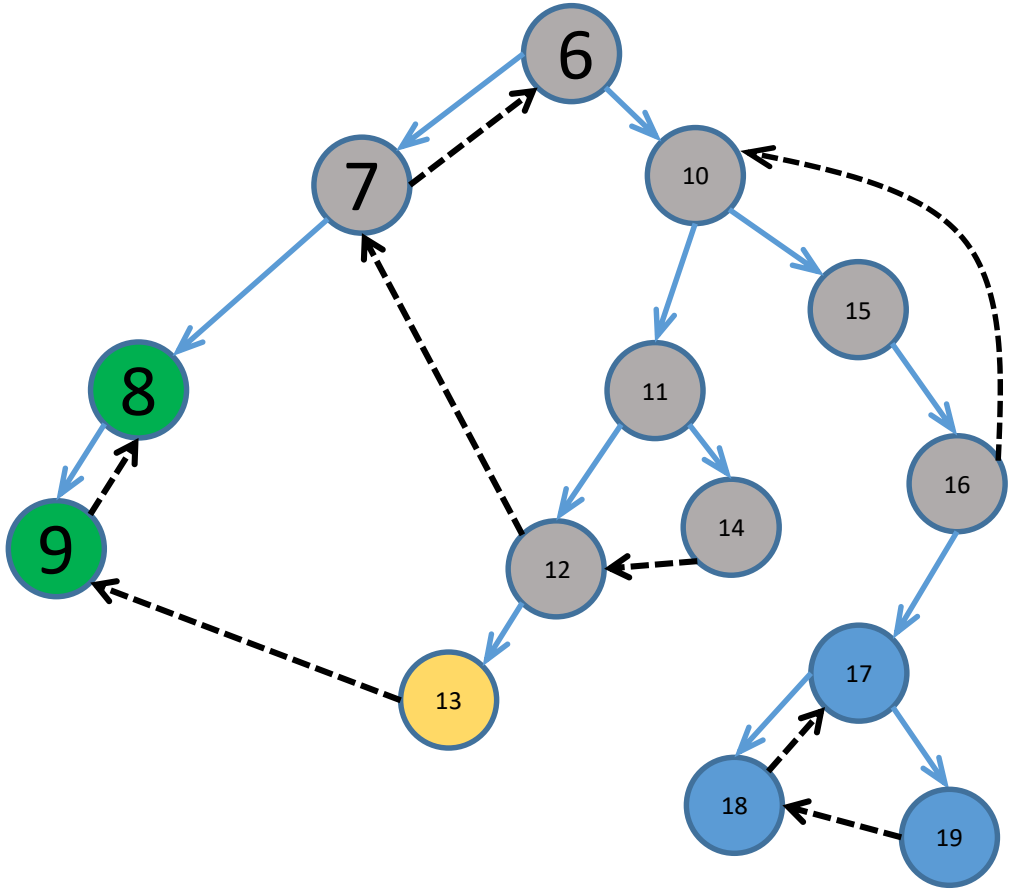

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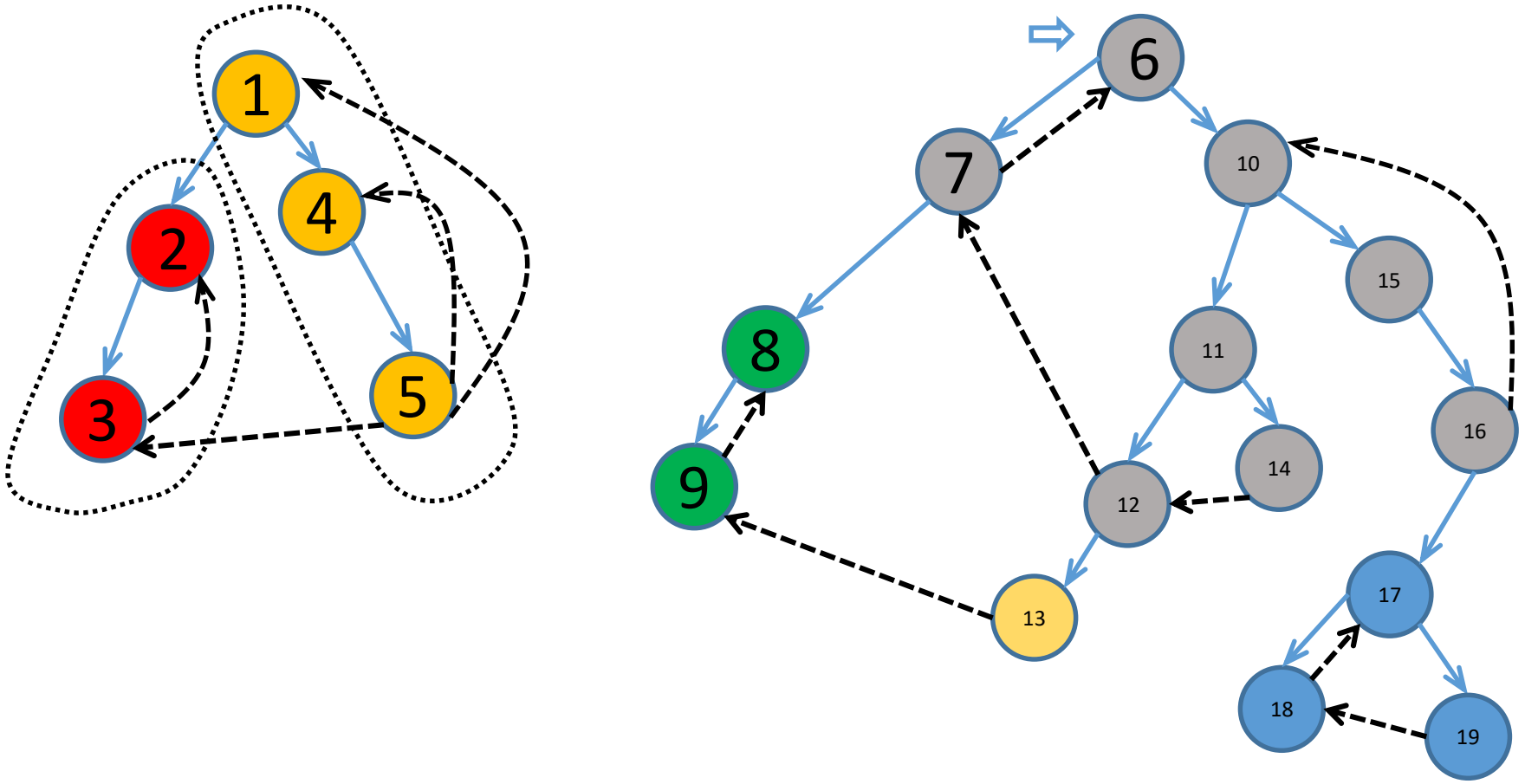

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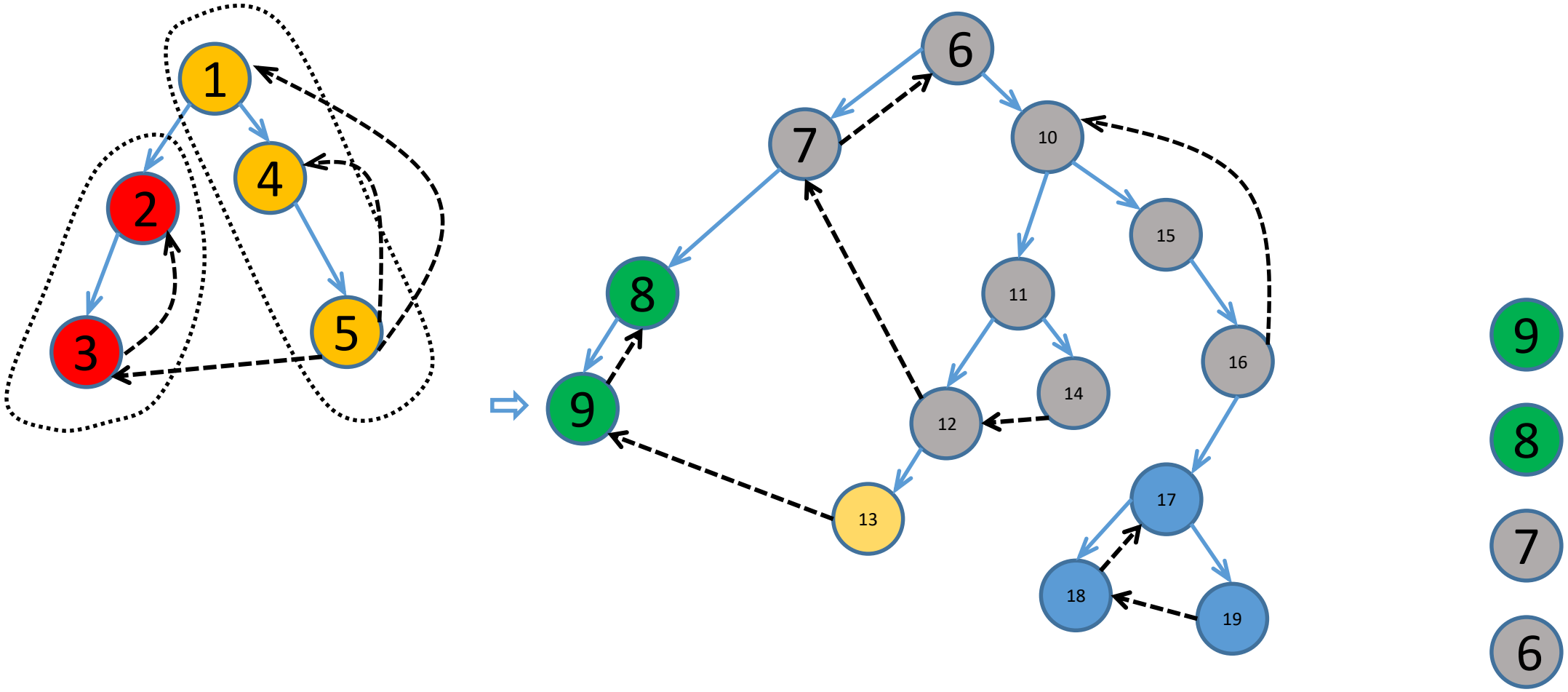

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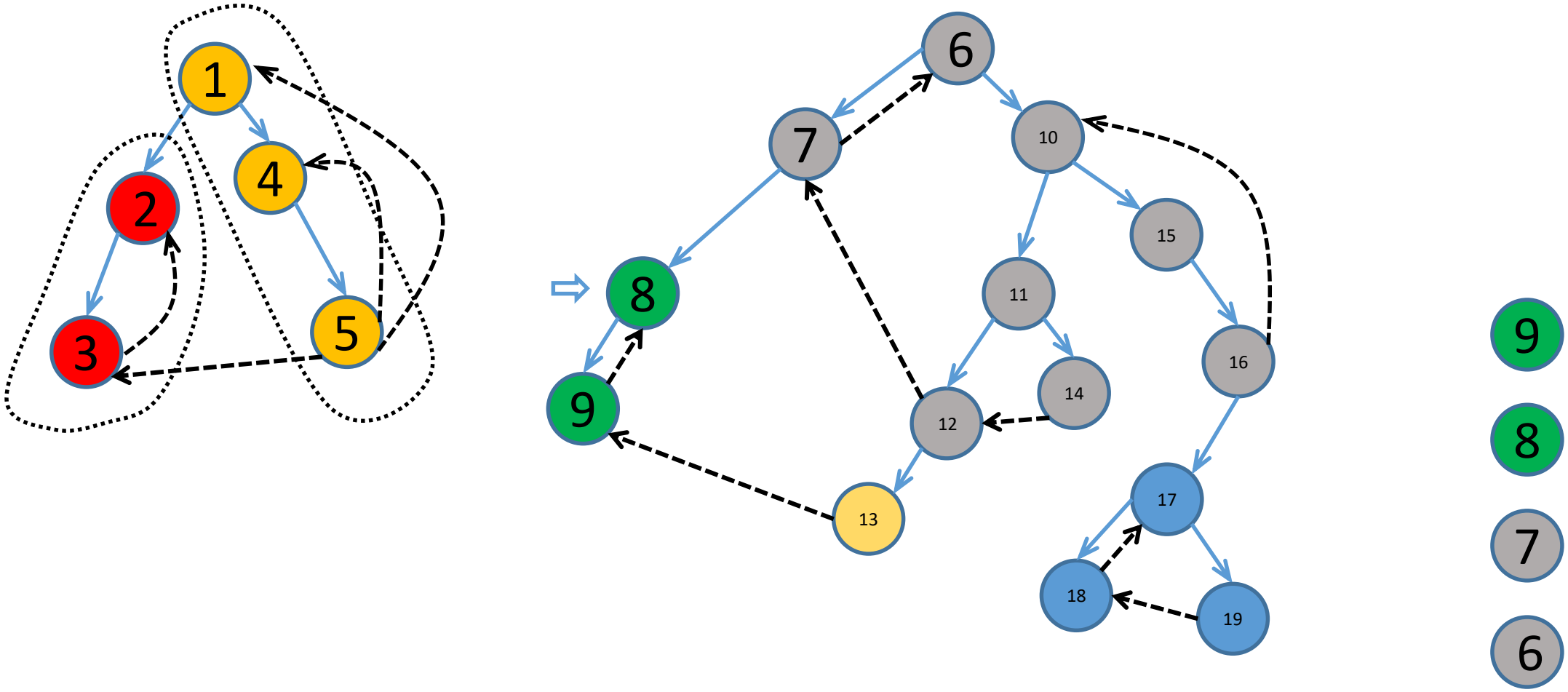

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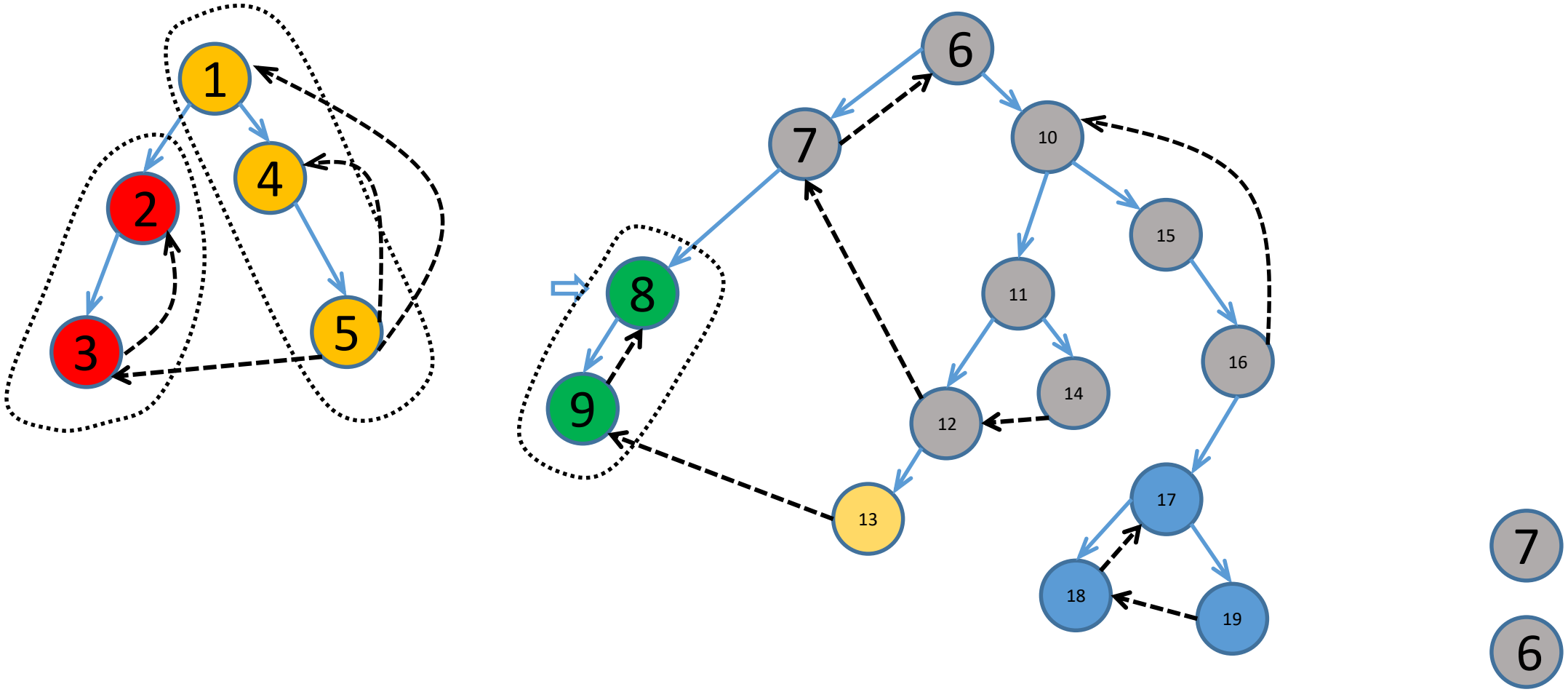

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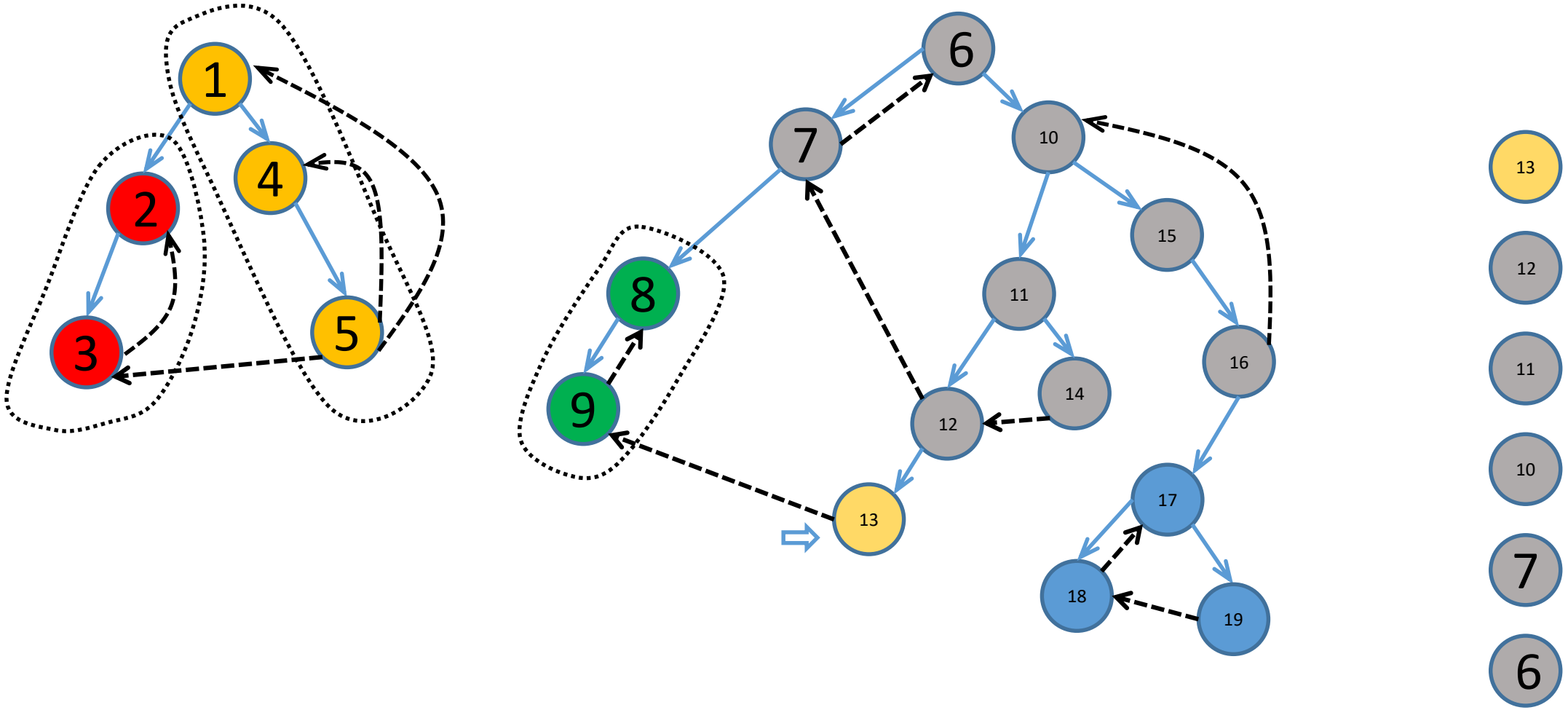

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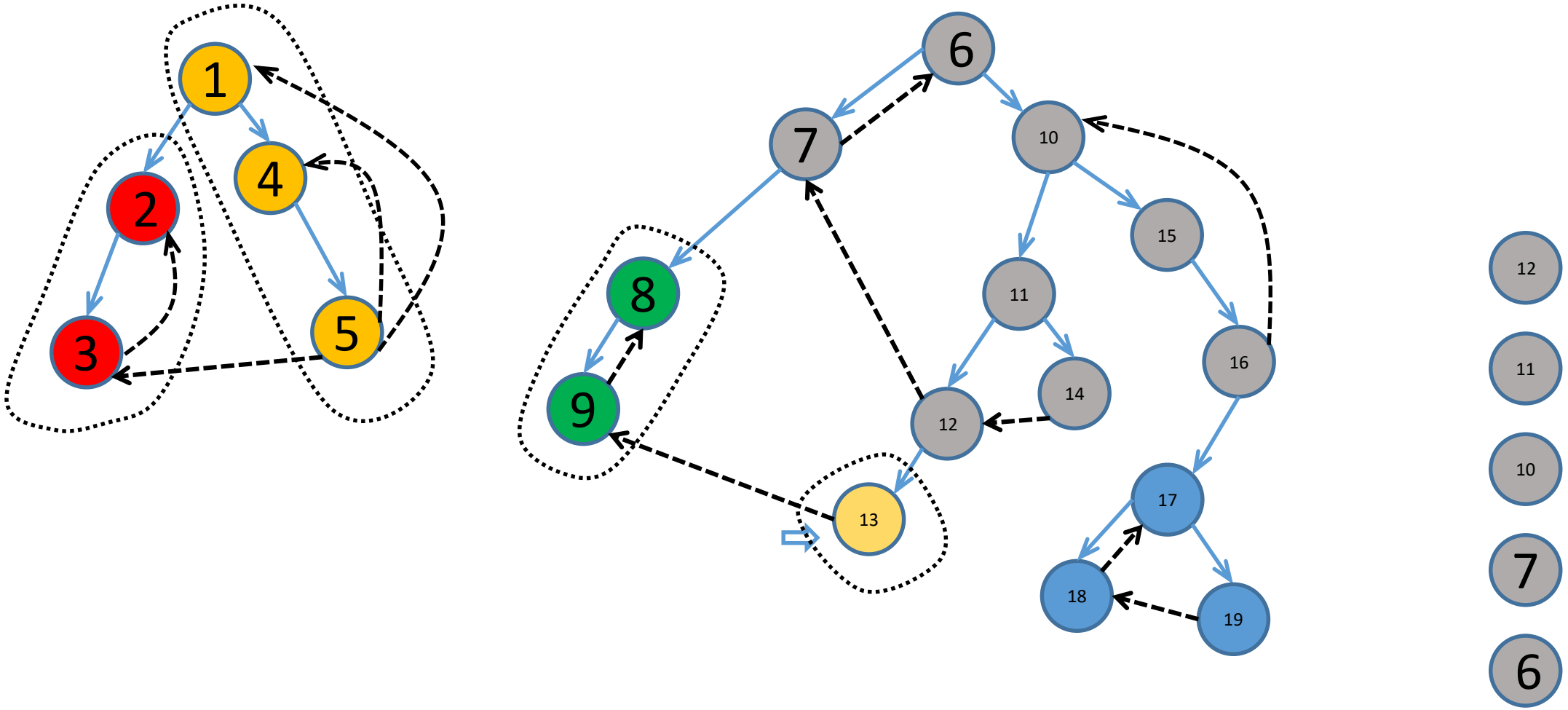

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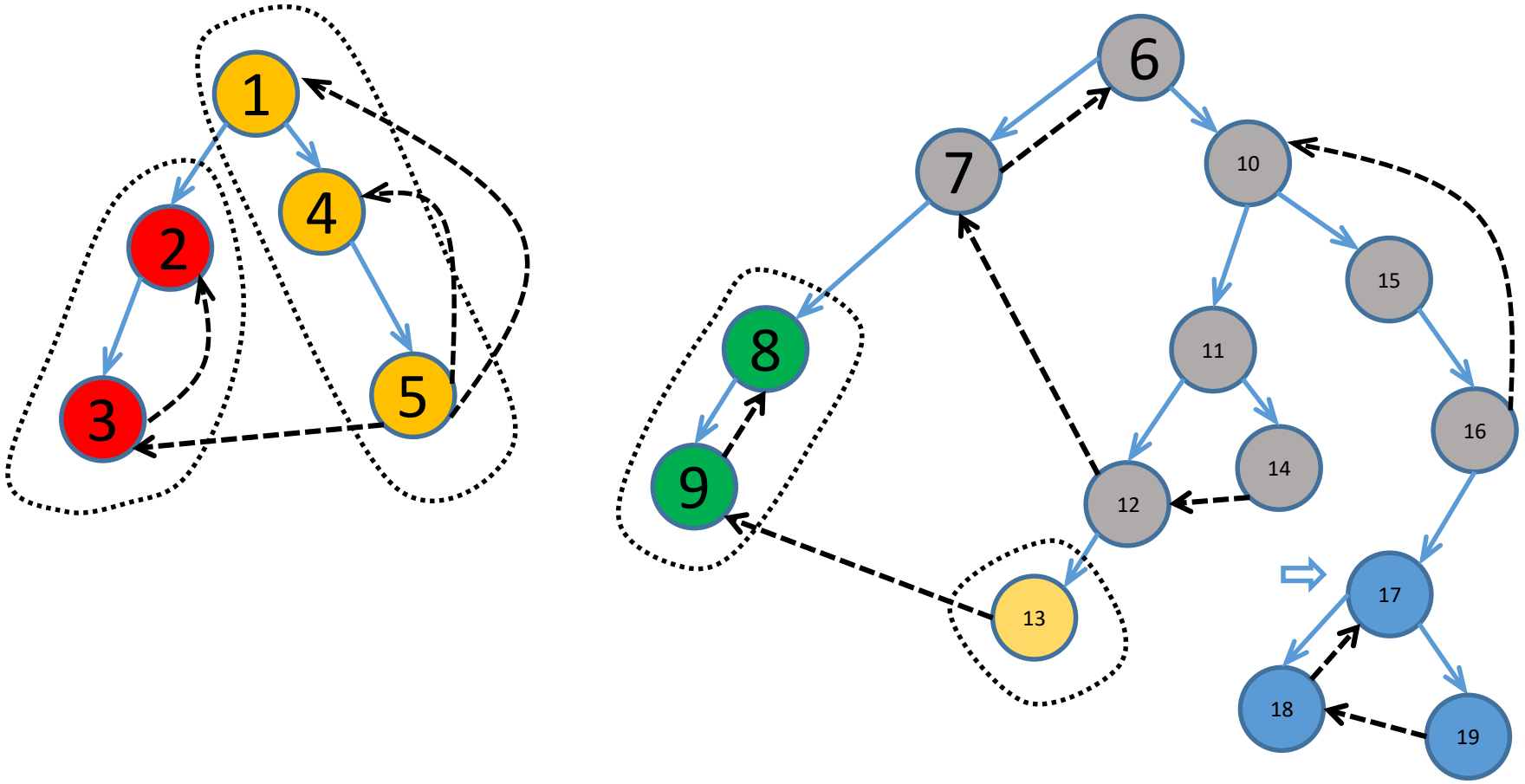

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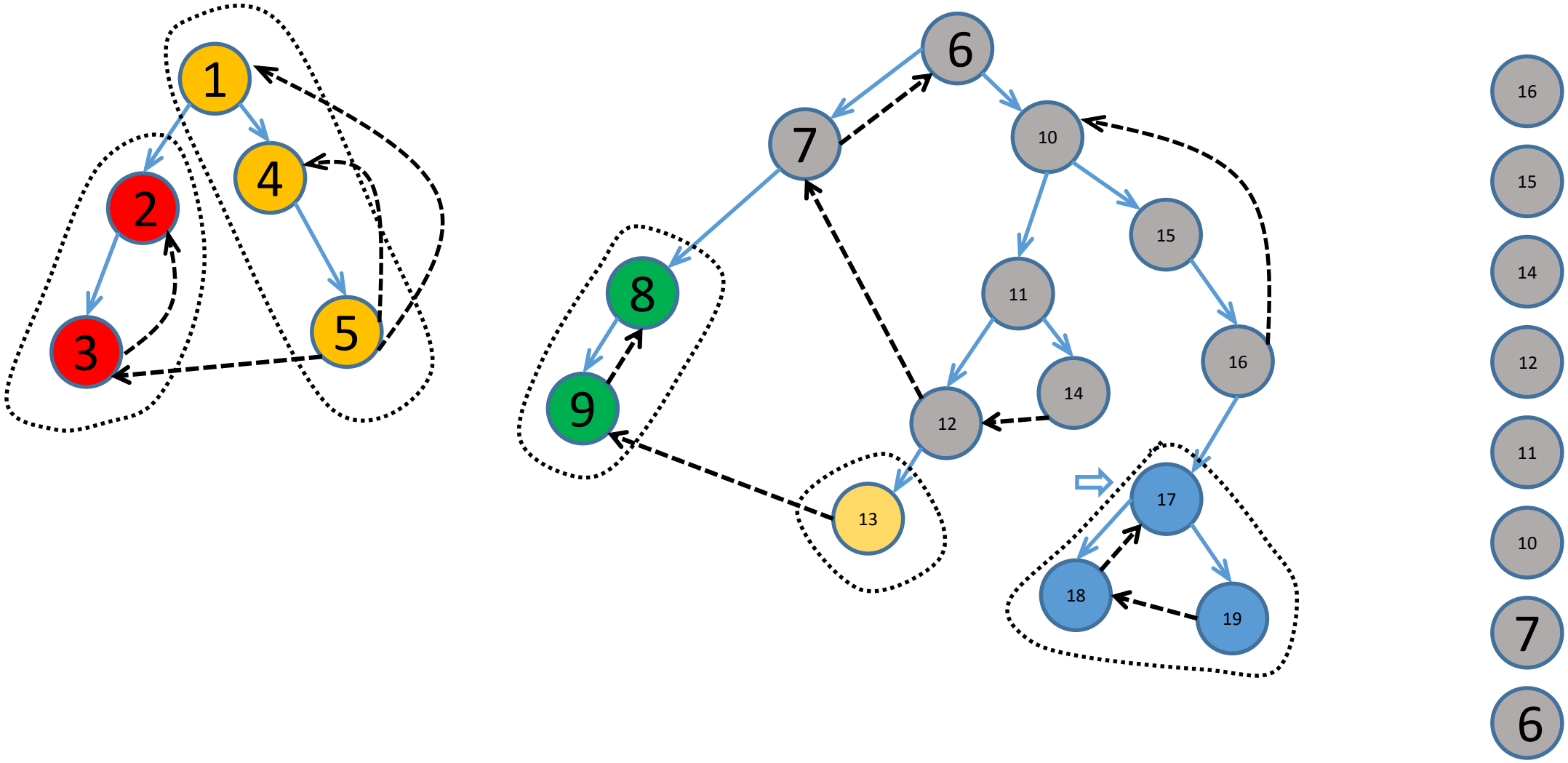

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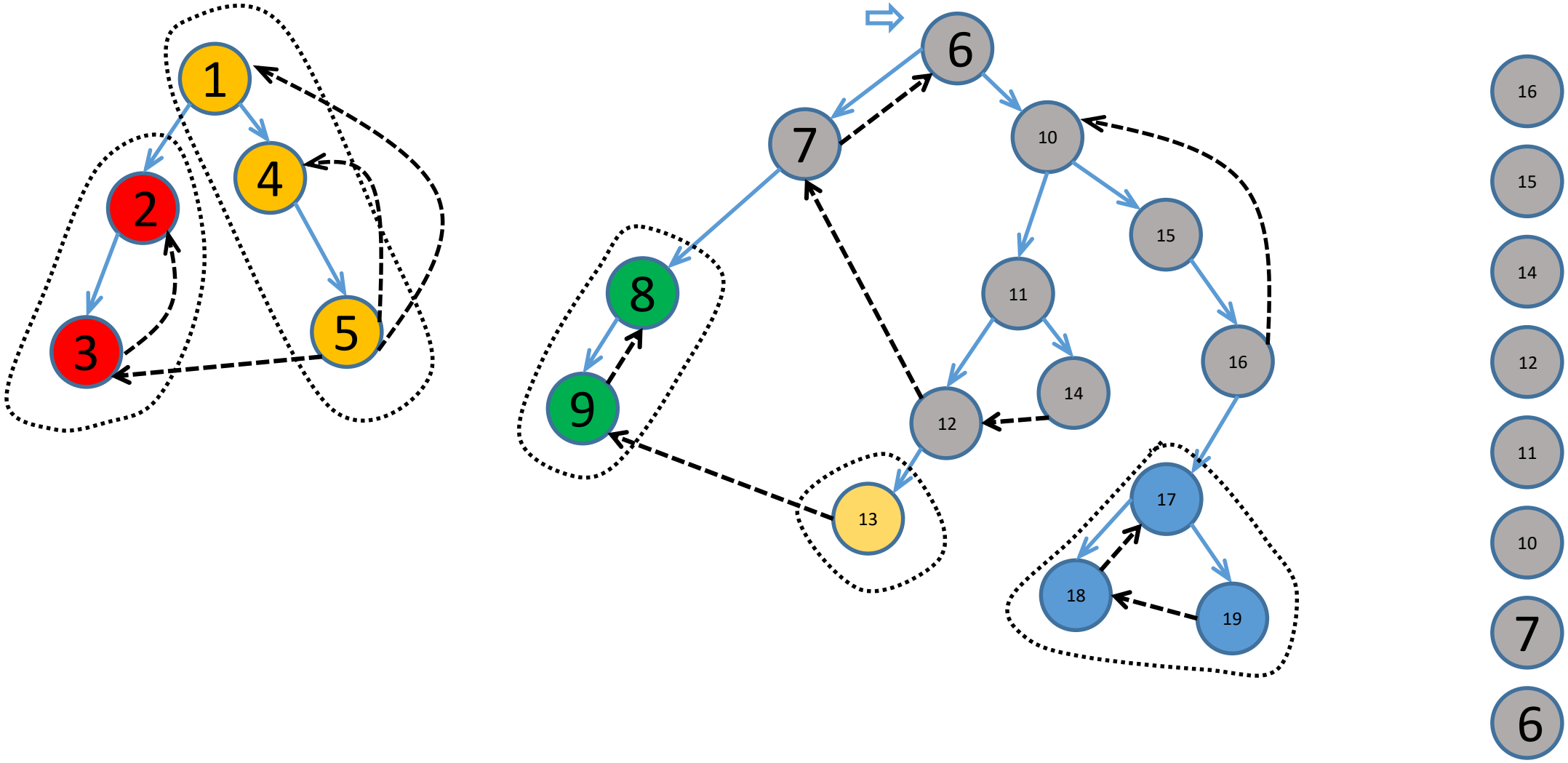

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