Exercise 3.1 Recall that a group is a set $G$ with an associative operation $\times$ and an unit element $id \in G$ such that $id \times g = g \times id = g$ for all $g \in G$. Furthermore, for each element $g \in G$ there exists an inverse $g^{-1} \in G$ such that $g^{-1} \times g = g \times g^{-1} = id$. Assume $G$ is finite.

Given a distribution $\mu$ on $G$ we define a random walk $P_\mu$ on $G$ as follows. It is a Markov chain with state space $G$ which moves by multiplying the current state on the left by a random element of $G$, selected according to $\mu$. That is $P_\mu(g, h \times g) = \mu(h)$ for all $g, h \in G$.

a) Prove that the uniform distribution on $G$ is a stationary distribution of $P_\mu$.

b) Prove that $P_\mu$ is irreducible if $S = \{g \in G \mid \mu(g) > 0\}$ generates $G$.

c) Suppose that $G = S_n$ is the set of permutations of $1, 2, \ldots, n$, and $\times$ is the composition of permutations. Assume $\mu$ is a uniform distribution on the transpositions. (A transposition is a permutation that swaps 2 different elements and maps all other elements to themselves.) Is $P_\mu$ irreducible? Is $P_\mu$ aperiodic?

Exercise 3.2 Let $P$ be an irreducible Markov chain with $n$ states.

a) For two states $x$ and $y$, let $\tau_{xy}$ be number of steps that we do starting from $x$ until the first time we get to $y$. Prove that $E(\tau_{xy})$ is finite for any two states $x$ and $y$.

b) Prove that the stationary distribution of $P$ is unique. (You do not need to prove that $P$ has a stationary distribution.)

(Hint: One way to do this is by proving that the rank of $P - I$ is $n - 1$.)

Exercise 3.3 Let $P$ be a Markov chain obtained from an undirected, non-bipartite, $d$-regular (all vertices are of the same degree $d$) and connected graph. (i.e. $P$ picks a neighbor uniformly at random from the $d$ neighbors of $v$)

a) Prove that $P$ is irreducible and aperiodic.

b) Prove that for any probability distribution $x^0$, $||x^0 P^t - \pi||_2 \leq |\lambda_2|^t$, where $\pi$ is the stationary distribution of $P$ and $\lambda_2$ is the second largest eigenvalue of $P$ in absolute value. ($|| \cdot ||_2$ is the Euclidean $L_2$ norm).

c) Prove that the mixing time of $P$ is at most $[\log(4\sqrt{n})/ \log(1/|\lambda_2|)]$.

Exercise 3.4 In the Traveling Salesman Problem (TSP) we are given a set $\{1, \ldots, n\}$ of $n$ cities and the distances $d(i, j)$ between any pair $i, j$ of cities. Our goal is to find a permutation $\pi_1, \ldots, \pi_n$ of the cities that minimized $\sum_{i=1}^n d(\pi_i, \pi_{i+1})$ (where we define $\pi_{n+1} = \pi_1$). A popular local search algorithm for TSP, called 2OPT, defines two permutations $\pi^1$ and $\pi^2$ as neighbors if $\pi^2$ can be obtained from $\pi^1$ by reversing an interval. I.e. if there exist two indices $k$ and $\ell$, $1 \leq k < \ell \leq n$, such that $\pi^2_j = \pi^1_{j+\ell-k}$ for $k \leq j \leq \ell$ and $\pi^2_j = \pi^1_j$ for $j < k$ and $j > \ell$. Describe a simulated annealing algorithm for TSP which is based on a random walk on the graph which is defined by this local search scheme. Write down the transition matrix of the underlying chain at a fixed temperature $T$. Prove that this chain is irreducible.
Exercise 3.5 Let $S = \{s_1, s_2, s_3, s_4\}$ and let $f : S \rightarrow \mathbb{R}$ be given by $f(s_1) = 1$, $f(s_2) = 2$, $f(s_3) = 0$, $f(s_4) = 2$. Suppose we want to find the minimum of $f(s_i)$ using simulated annealing.

a) Construct the Metropolis chain for the Boltzmann distribution with respect to $f$ with parameter $T$ using an underlying chain which is a random walk on the cycle $(s_1, s_2), (s_2, s_3), (s_3, s_4), (s_4, s_1)$ (when at $s_i$ you choose each of your two neighbors with the same probability). Write the transition probabilities for this Metropolis chain.

b) Suppose we set the temperature at step $k$ to be $T_k$, what is the probability, $P_n$, that if we start at state $s_1$, we never leave $s_1$ during the first $n$ steps.

c) Suppose that $T_k = 1/(2 \ln(k + 1))$ for $k = 1, 2, \ldots$, what is $\lim_{n \rightarrow \infty} P_n$? Is it good or bad?