Dynamic trees (Steator and Tarjan 83)
Operations that we do on the trees

maketree(v)

w = findroot(v)

(w,c) = mincost(v)  (can do maxcost(v) instead)

addcost(v,c)

link(v,w,c(v,w))

cut(v)

evert(v)
Applications
Incremental Minimum Spanning Forest

Maintain a minimum spanning forest of a graph to which we insert edges
Add an edge \((v,w)\)

Discover if \(v\) and \(w\) are in the same component by comparing
\(\text{findroot}(v)\) and \(\text{findroot}(w)\)
Add an edge \((v,w)\)

If \(v\) and \(w\) are in different components then add \((v,w)\) to the forest by \(\text{link}(v,w,c(v,w))\)
Add an edge \((v,w)\)

If \(v\) and \(w\) are in different components then add \((v,w)\) to the forest by \(\text{evert}(v)\), and \(\text{link}(v,w,c(v,w))\)
Add an edge \((v,w)\)

What if \(v\) and \(w\) are in same component?
Add an edge \((v,w)\)

We have to figure out if \(c(v,w)\) is smaller than the largest cost of an edge along the tree path between \(v\) and \(w\).
Add an edge $(v,w)$

Find the largest edge along the tree path from $v$ to $w$ by $\text{evert}(v)$ follows by $\text{maxcost}(w)$
Add an edge \((v,w)\)

If \(c(v,w) < c(x,y)\)
then \(\text{cut}(x)\) and \(\text{link}(v,w,c(v,w))\)
Add an edge \((v, w)\)

If \(c(v, w) < c(x, y)\) then
\(\text{cut}(x)\) and
\(\text{link}(v, w, c(v, w))\)
Application (2)

- Minimum spanning forest with a particular number of blue edges
Suppose we add $\lambda$ to the weights of the blue edges and compute the MSF.

If $\lambda = -\infty$ we will get a MSF with as many blue edges as possible.
The blue edges excluded cannot be in any MSF. Black edges included will be in any MSF

Let M be the maximum # of blue edges in a spanning forest
Imagine we increase $\lambda$
At some point we would trade a black edge for a more expensive blue edge
At some point we would trade a black edge for a more expensive blue edge
At some point we would trade a black edge for a more expensive blue edge

Let $T$ be the forest we get with $M-1$ blue edges
The cost of this forest is \((M-1)\lambda + c(T)\)

The cost of any other forest \(T'\) with \((M-1)\) blue edges is \((M-1)\lambda + c(T')\)

\(\Rightarrow c(T) < c(T')\)
Keep increasing $\lambda$ we will find the lightest MSF with $M-2$ edges and so on..
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Keep increasing $\lambda$ we will find the lightest MSF with M-2 edges and so on..
The key observation

• We can find the critical values of $\lambda$ efficiently
Start with a spanning forest of the blue subgraph
Process the black edges in increasing order of their weight
Process the black edges in increasing order of their weight

Let e be the current **black** edge
If $e$ closes a cycle find the **blue** edge $e'$ of maximum cost on the cycle

$\lambda_e = c(e) - c(e')$ is a critical value

Consider the cut defined by removing $e'$ from the forest
If e closes a cycle find the **blue** edge $e'$ of maximum cost on the cycle

$$\lambda_e = c(e) - c(e')$$

is a critical value

For $\lambda < \lambda_e$ the edge $e'$ is the smallest crossing it and for $\lambda > \lambda_e$ the edge $e$ is the smallest crossing it.
Replace $e'$ by $e$ and continue
Invariant: each blue edge of the forest is the smallest blue edge crossing the cut that it defines
Why is this invariant true?

It is clear if the cut does not change.
Why is this invariant true?

But the cut may change…
Why is this invariant true?

We have $c(e'') \leq c(e')$
Why is this invariant true?

And $c(e'') \leq c(e') \leq c(e''')$
Invariant: each blue edge of the forest is the smallest blue edge crossing the cut that it defines
Consider the next largest black edge \( e \)

Find the largest \textcolor{blue}{blue} edge \( e' \) on the cycle that \( e \) closes with the current forest
Consider the next largest black edge $e$

Find the largest blue edge $e'$ on the cycle that $e$ closes with the current forest

$\lambda_e = c(e) - c(e')$ is a critical value
Replace $e'$ by $e$
and continue
Replace $e'$ by $e$
and continue
Replace $e'$ by $e$ and continue.

If the black edge connects two components then it appears in any spanning forest.

Add it and continue.
• If a black edge closes a cycle of black edges just discard it
Summary

• We identify a set of blue edges that are never in the tree

• We identify a set of black edges always in the tree (say b many)

• Other edges are partitioned into black-blue pairs each with an associated critical $\lambda$

• Sort the pairs by $\lambda$

• If you want $b + z$ black edges then take the black edges of the first $z$ pairs and the blue edges of the rest

• $O(m\log(n))$ total time
Application (3)
1D Range reporting

Given a set of intervals $S$ on the line, preprocess them to build a structure that allows efficient queries of the form:

Given a point $x$ find all intervals containing it.
Dynamic range reporting + priorities

Given a set of intervals $S$ on the line, each with priority assigned to it, build a structure that allows efficient queries of the from:

Given a point $x$ find interval with minimum priority containing it.

Updates – insert or delete an interval
Motivation – Packet classification

1. Block & report to Bill
2. Block
3. Forward to Interface A

IP address

190.0.1.0 190.0.2.0 190.0.3.0 190.0.4.0 190.0.5.0 190.0.6.0 190.0.7.0 190.0.8.0 190.0.9.0 190.0.10.0 190.0.11.0 190.0.12.0 190.0.13.0 190.0.14.0

Forward to Interface B
Nested intervals, IP prefixes

IP address

190.0.0.0  190.0.1.0  190.0.2.255  190.0.255.255  190.1.0.0  190.1.255.255

Forward to Interface A

Forward to Interface B

Block

190.0.*.*

190.0.1.*

190.1.*.*
Extension to 2D

- Query = point in $\mathbb{R}^2$
  - (Sender IP, receiver IP)
- interval = rectangle with priority

![Diagram showing rectangles with numbers 5, 7, and 9]
One dimensional data structure for nested intervals
Nested Intervals

Containment tree:

The parent of interval $v$ is the smallest interval containing $v$
Query:
Starting node $s$ = smallest interval containing the query point

Relevant priorities are on the path from $s$ to the root.

Problem: path may be long…
Hey, dynamic trees know how to do that

We can use a dynamic tree to represent the containment tree.

Query $\text{mincost}()$
Problem: Updates => Many cuts & links
Binarization

Node $v$ $\Rightarrow$ node $v$

Leftmost child of $v$ $\Rightarrow$ Left child of $v$

Any other child of $v$ $\Rightarrow$ right child of its left sibling

Adjust costs:

Left edge $\Rightarrow$ priority of parent

Right edge $\Rightarrow$ $\infty$
Insert (Cont.)

Constant number of links and cuts
Summary

• Containment tree C
  Query = min cost on path from starting point to root

• Represent C by binarized version B

• Represent B by dynamic tree D

• How do you find the point to start the query?
• How do you find the edges to cut?
How do you start the query?

Use a balanced search tree on the endpoints

Min(Mincost(○),pri(○))
query (cont)

Mincost(●)
Let's implement this data type

\begin{verbatim}
maketree(v)

w = findroot(v)

(w,c) = mincost(v)

addcost(v,c)

link(v,w,c)

cut(v)

evert(v)
\end{verbatim}
Simple case -- paths

Assume for a moment that each tree $T$ in the forest is a path. We represent it by a virtual tree which is a simple splay tree.
Findroot(v)

Splay at v, then follow right pointers until you reach the last vertex w on the right path. Return w and splay at w.
Mincost(v)

With every vertex \( x \) we record \( \text{cost}(x) = \) the cost of the edge \((x, p(x))\)

We also record with each vertex \( x \) \( \text{mincost}(x) = \) minimum of \( \text{cost}(y) \) over all descendants \( y \) of \( x \).

\[
\begin{array}{cccccccc}
\text{a} & 1 & \text{b} & 3 & \text{c} & 2 & \text{d} & 7 & \text{e} & 4 & \text{f} \\
\text{b} & 1\text{,}1 & \text{c} & 2\text{,}2 & \text{d} & 3\text{,}1 & \text{e} & 4\text{,}4 & \text{f} & \infty, \infty
\end{array}
\]
Mincost(v)

Splay at v and use mincost values to search for the minimum

Notice: we need to update mincost values as we do rotations.
Addcost(v,c)

Rather than storing cost(x) and mincost(x) we will store

\[ \Delta \text{cost}(x) = \text{cost}(x) - \text{cost}(p(x)) \]

\[ \Delta \text{min}(x) = \text{cost}(x) - \text{mincost}(x) \]

Addcost(v,c) :
Splay at v,
\[ \Delta \text{cost}(v) += c \]
\[ \Delta \text{cost}(\text{left}(v)) -= c \]

similarly update \( \Delta \text{min} \)
Addcost\((v,c)\) (cont)

Notice that now we have to update \(\Delta \text{cost}(x)\) and \(\Delta \text{min}(x)\) through rotations

\[
\Delta \text{cost}'(v) = \Delta \text{cost}(v) + \Delta \text{cost}(w) \\
\Delta \text{cost}'(w) = -\Delta \text{cost}(v) \\
\Delta \text{cost}'(b) = \Delta \text{cost}(v) + \Delta \text{cost}(b)
\]
Addcost(v, c) (cont)

Update $\Delta \text{min}$:

$\Delta \text{min}'(w) = \max \{0, \Delta \text{min}(b) - \Delta \text{cost}'(b), \Delta \text{min}(c) - \Delta \text{cost}(c)\}$

$\Delta \text{min}'(v) = \max \{0, \Delta \text{min}(a) - \Delta \text{cost}(a), \Delta \text{min}'(w) - \Delta \text{cost}'(w)\}$
Link(v, w, c), cut(v)

Translate directly into catenation and split of splay trees if we talk about paths.

Lets do the general case now.
The virtual tree

- We represent each tree $T$ by a virtual tree $V$.

The virtual tree is a binary tree with middle children.

Think of $V$ as partitioned into solid subtrees connected by dashed edges

What is the relation between $V$ and $T$?
Actual tree
Path decomposition

Partition T into disjoint paths
Virtual trees (cont)

Each path in $T$ corresponds to a solid subtree in $V$

The parent of a vertex $x$ in $T$ is the successor of $x$ (in symmetric order) in its solid subtree or the parent of the solid subtree if $x$ is the last in symmetric order in this subtree
Virtual trees (cont)
Virtual trees (representation)

Each vertex points to \( p(x) \) to its left son \( l(x) \) and to its right son \( r(x) \).

A vertex can easily decide if it is a left child a right child or a middle child.

Each solid subtree functions like a splay tree.
The general case

Each solid subtree of a virtual tree is a splay tree.

We represent costs essentially as before.

\[ \Delta \text{cost}(x) = \text{cost}(x) - \text{cost}(p(x)) \text{ or cost}(x) \text{ is x is a root of a solid subtree} \]

\[ \Delta \text{min}(x) = \text{cost}(x) - \text{mincost}(x) \text{ (where mincost is the minimum cost within the subtree)} \]
Splicing

Want to change the path decomposition such that v and the root are on the same path.

Let w be the root of a solid subtree and v a middle child of w

Want to make v the left child of w. It requires:

\[ \Delta \text{cost}'(v) = \Delta \text{cost}(v) - \Delta \text{cost}(w) \]
\[ \Delta \text{cost}'(u) = \Delta \text{cost}(u) + \Delta \text{cost}(w) \]
\[ \Delta \text{min}'(w) = \max \{ 0, \Delta \text{min}(v) - \Delta \text{cost}'(v), \Delta \text{min}(\text{right}(w)) - \Delta \text{cost}(\text{right}(w)) \} \]
Splicing (cont)

What is the effect on the path decomposition of the real tree?

\[
\begin{align*}
\text{u} & \quad \text{v} & \text{right} & \rightarrow & \text{v} & \quad \text{u} & \text{right} \\
\text{w} & \rightarrow & \text{w} & \rightarrow & \text{w} & \rightarrow & \text{w} \\
\text{a} & \quad \text{b} & \text{a} & \quad \text{b} & \text{a} & \quad \text{b} & \text{a} & \quad \text{b} \\
\text{w} & \rightarrow & \text{w} & \rightarrow & \text{w} & \rightarrow & \text{w} & \rightarrow & \text{w} \\
\text{u} & \quad \text{v} & \text{u} & \quad \text{v} & \text{u} & \quad \text{v} & \text{u} & \quad \text{v} \\
\end{align*}
\]
Splaying the virtual tree

Let $x$ be the vertex in which we splay.

We do 3 passes:

1) Walk from $x$ to the root and splay within each solid subtree

   After the first pass the path from $x$ to the root consists entirely of dashed edges

2) Walk from $x$ to the root and splice at each proper ancestor of $x$.

   Now $x$ and the root are in the same solid subtree

3) Splay at $x$

   Now $x$ is the root of the entire virtual tree.
Example
Actual and virtual trees
Splay at m
Splay at m
Splay at m
Splay at m
Splay at m
Splay at m
Splay at m
Dynamic tree operations

\[ w = \text{findroot}(v) : \text{Splay at v, follow right pointers until reaching the last node w, splay at w, and return w.} \]

\((v,c) = \text{mincost}(v) : \text{Splay at v and use } \Delta\text{cost and } \Delta\text{min to follow pointers to the smallest node after v on its path (its in the right subtree of v). Let w be this node, splay at w.} \]

\[ \text{addcost}(v,c) : \text{Splay at v, increase } \Delta\text{cost}(v) \text{ by c and decrease } \Delta\text{cost(left(v))) by c, update } \Delta\text{min(v)} \]

\[ \text{link}(v,w,c(v,w)) : \text{Splay at v, update the cost of v to be c(v,w) (requires updates to } \Delta\text{cost(v), } \Delta\text{min(v), } \Delta\text{cost(left(v))), and } \Delta\text{cost(right(v))), splay at w (so potential does not increase too much when we add v as a child) and make v a middle child of w} \]

\[ \text{cut}(v) : \text{Splay at v, break the link between v and right(v), set } \Delta\text{cost(right(v))) } += \Delta\text{cost(v)} \]
Cut(m)
Splay at m
Cut at m
Dynamic tree (analysis)

It suffices to analyze the amortized time of splay in the virtual tree.

Use the access lemma as follows:

The weight assigned to each node/item $v$ is

$$1 + \sum \text{sizes of subtrees (in the virtual tree) rooted at middle children of } v$$

The size of $v$ is the number of elements in $v$'s subtree in the virtual tree.

Note: Splices do not affect the size of $v$. 
Dynamic tree (analysis)

Analysis of the step (1) of a splay of a node in the virtual tree:

Apply the access lemma to each splay and sum up

\[ 3\log \left( \frac{s(T_k)}{s(x_k)} \right) + \ldots + 3\log \left( \frac{s(T_1)}{s(x_1)} \right) + 3\log \left( \frac{s(T_x)}{s(x)} \right) + k \leq \]

\[ 3\log \left( \frac{s(T_k)}{s(x) + k} \right) \]

k=#solid subtree along the path
Dynamic tree (analysis)

pass 1 takes $3\log n + k$

pass 2 takes $k$

pass 3 takes $3\log n + 1$

How do we get rid of this $k$?
Refining the access lemma

Original version: The amortized time to splay a node $x$ in a tree with root $t$ is at most $3(r(t) - r(x)) + 1 = 3\log(s(t)/s(x)) + 1$

Modified version: For any constant $c \geq 1$, the amortized time to splay a node $x$ in a tree with root $t$ is at most $3c(r(t) - r(x)) + 1 = 3c\log(s(t)/s(x)) + 1 - (c-1)(c-1)$, where $\ell$ is the length of the splay path
Dynamic tree (analysis)

pass 1 takes $3\log n + k$

pass 2 takes $k - 1$

pass 3 takes $3\log n + 1 - (k-2)(c-1)$

$\Rightarrow \mathcal{O}(\log n)$
Proving the modified access lemma

• Same proof, multiply the potential by $c$:

Potential is: $c \cdot \sum r(x) = c \cdot \sum \log_2(s(x))$
Proof of the access lemma (cont)

(1) zig - zig

amortized time(zig-zig) = 2 + ΔΦ =

2 + c(r'(x) + r'(y) + r'(z) - r(x) - r(y) - r(z)) ≤

2 + c(r'(x) + r'(z) - r(x) - r(y)) ≤ 2 + c(r'(x) + r'(z) - r(x) - r(x)) =

2 + c(r(x) - r'(x) + r'(z) - r'(x) + 3(r'(x) - r(x))) ≤

2 + c(\log(s(x)/s'(x)) + \log(s'(z)/s'(x))) + 3c(r'(x) - r(x)) ≤

2 + c(\log([s'(x)/2]/s'(x)) + \log([s'(x)/2]/s'(x)) + 3c(r'(x) - r(x)) =

3c(r'(x) - r(x)) - 2(c-1)
(2) zig - zag

Proof of the access lemma (cont)

Same modification