Maximum Flow in Planar Graphs
Planar Graph and its Dual
Duality is defined for directed planar graphs as well.
Minimum $s$-$t$ cut in undirected planar graphs
An $s-t$ cut (undirected graphs)
An $s-t$ cut
The dual to the cut
Cuts/Cycles

A cut that separates the graph into two connected components one containing $s$ and one containing $t$ (we can assume the min-cut is like this)

A simple cycle with $t$ inside and $s$ outside

→ Look for a shortest such cycle in the dual (lengths in the dual are capacities in the primal)
The face containing $s$
and the face containing t
Let $P$ be the shortest path between them.
Any cycle sep. s and t crosses P
The shortest cycle will cross $P$ once
The shortest cycle will cross it once

We are after the shortest cycle sep. s and t and crosses P once.
Finding such shortest cycle
Finding such shortest cycle

Classify edges incident to the path as left or right
Finding such shortest cycle

Cut the path open
Finding such shortest cycle

Direct the edges incident to the path
Finding such shortest cycle

Find shortest path between every pair $v_1, v_2$
Finding such shortest cycle

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Finding such shortest cycle

Find shortest path between every pair \( v_1, v_2 \)
Finding such shortest cycle

Take the shortest among these shortest paths
Speeding up by divide and conquer
Speeding up by divide and conquer

Shortest cycles do not cross
Speeding up by divide and conquer

Shortest cycles do not cross
Take $v_1$ and $v_2$ to be the middle pair
Take $v_1$ and $v_2$ to be the middle pair.
Split the problem
Split the problem
Add new source/sink
In fact

This state is symmetric to our starting position
Analysis

**Obs1**: Paths are shorter by a factor of 2

→ depth of recursion ≤ log n
Analysis

Obs2: Each red vertex is in one subproblem (+s' and t')

Total size of subproblems at level $k \leq \log(n)$ is $O(n + 2^k) = O(n)$
Summary

Total time $O(n\log^2 n)$ using Dijkstra or $O(n\log(n))$ using the $O(n)$ SSSP algorithm for planar graphs.
Circulations and prices
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Circulations and prices

Decompose the flow into CCW cycles

Start with potentials of 0

For each CCW cycle of value $\beta$, add $\beta$ to the potentials of the faces inside the cycle

The flow along an edge is the difference in the potentials of its incident faces
Circulations and prices

Any face prices define a circulation the same way

The flow is feasible iff \( \beta - \beta' \leq u(e) \)

Iff \( \forall e \ u(e) + \beta' - \beta \geq 0 \) (nonnegative reduced costs)
Circulations and prices

Flow is feasible iff $\forall e \ u(e) + \beta' \geq \beta$

We can get potentials from any shortest path tree in the dual

The reduced costs equal the residual capacities of the corresponding flow.
2 applications for this connection
Feasible circulations

Negative capacity is a lower bound on the flow on the reverse arc

A circulation exists iff there are feasible potentials iff no negative cycles in the dual

Can decide via a shortest path algorithm that can handle negative weights $\Rightarrow O(mn)$
Max s-t flow when s and t are on the same face

Find max flow from s to t when s and t are on the same face
Max $s$-$t$ flow when $s$ and $t$ are on the same face

Add an edge from $t$ to $s$ with $\infty$ capacity
Max $s$-$t$ flow when $s$ and $t$ are on the same face

Infinite face splits

Compute shortest paths from $f_1$ and define a flow according to these potentials
Max $s$-$t$ flow when $s$ and $t$ are on the same face

Infinite face splits

Compute shortest paths from $f_1$ and define a flow according to these potentials
Max $s$-$t$ flow when $s$ and $t$ are on the same face

Delete the new edge and you get a maximum flow from $s$ to $t$

Proof. Its feasible (corres. to sp. in the dual), its maximum because it equals the minimum $s$-$t$ cut (=shortest path from $f_1$ to $f_2$)
Max $s$-$t$ flow when $s$ and $t$ are on the same face

Proof. Its feasible (corres. to sp. in the dual), its maximum because it equals the minimum $st$-cut (=shortest path from $f_1$ to $f_2$)