Exact Bounds for Some Hypergraph Saturation Problems

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Abstract

Let $K_{p_1,\ldots,p_d}$ denote the complete $d$-uniform $d$-partite hypergraph with partition classes of sizes $p_1,\ldots,p_d$. A hypergraph $G \subseteq K_{n,\ldots,n}^d$ is said to be weakly $K_{p_1,\ldots,p_d}$-saturated if one can add the edges of $K_{n,\ldots,n}^d \setminus G$ in some order so that at each step a new copy of $K_{p_1,\ldots,p_d}^d$ is created. Let $W_n(p_1,\ldots,p_d)$ denote the minimum number of edges in such a hypergraph.

The problem of bounding $W_n(p_1,\ldots,p_d)$ was introduced by Balogh, Bollobás, Morris and Riordan who determined it when each $p_i$ is either 1 or some fixed $p$. In this short paper we fully determine $W_n(p_1,\ldots,p_d)$. Our proof applies a reduction to a multi-partite version of the Two Families Theorem obtained by Alon. While the reduction is combinatorial, the main idea behind it is algebraic.

1 Introduction

A graph $G$ is strongly saturated with respect to a graph $H$ (or strongly $H$-saturated) if $G$ does not contain a copy of $H$, yet adding any new edge to $G$ creates a copy of $H$. The problem of strong saturation asks for the minimum number of edges in an $n$-vertex graph that is strongly $H$-saturated, for different graphs $H$ of interest (notice that the “dual” problem, of finding the maximum number of edges in an $n$-vertex $H$-saturated graph, is of course the classical Turán problem). Let $S_n(p)$ be the minimum number of edges in an $n$-vertex graph that is strongly $K_p$-saturated, where $2 \leq p \leq n$. The problem of determining $S_n(p)$ was considered already in the 1940’s by Zykov [14], and later by Erdős, Hajnal and Moon [8] who showed that $S_n(p) = \binom{n}{2} - \binom{n-p+2}{2}$. The upper bound on $S_n(p)$ is easy, as removing the edges of a $K_{n-p+2}$ from $K_n$ clearly gives a strongly $K_p$-saturated graph. Bollobás [3] famously gave a tight lower bound for $S_n(p)$ and for its natural hypergraph generalization using the so-called Two Families Theorem.

Another famous variant of saturation was introduced by Bollobás [5]. A graph $G$ is weakly saturated with respect to a graph $H$ (or weakly $H$-saturated) if all the non-edges of $G$ can be added one at a time, in some order, so that each new edge creates a new copy of $H$. We refer to the corresponding ordering of the non-edges of $G$ as a saturation process of $G$ with respect to $H$. Let $W_n(p)$ be the minimum number of edges in an $n$-vertex graph that is weakly $K_p$-saturated, where $2 \leq p \leq n$. Notice that for any $H$, a strongly $H$-saturated graph is in particular weakly $H$-saturated, so $W_n(p) \leq S_n(p)$. It follows from the famous skew version of the Two Families Theorem [12, 9, 11] that in fact $W_n(p) = S_n(p)$.

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In this paper we focus on saturation problems in the setting of bipartite graphs, and more generally, \(d\)-uniform \(d\)-partite hypergraphs. This variant of the problem was introduced recently by Balogh, Bollobás, Morris and Riordan \([2]\), although, as we mention next, some special cases have already been considered earlier. Unlike the definition of saturation in the previous subsection, here (and henceforth) the only edges that are considered are those containing one vertex from each vertex class. Let \(H\) be a \(d\)-uniform \(d\)-partite hypergraph with vertex classes \(V_1, \ldots, V_d\). We say that \(H\) is weakly \(K_{\overrightarrow{p_1, \ldots, p_d}}\)-saturated\(^1\) if all edges—containing one vertex from each \(V_i\)—that do not belong to \(H\) can be added to \(H\) one after the other so that whenever a new edge is added, a new copy of \(K_{\overrightarrow{p_1, \ldots, p_d}}\) is created.\(^2\) For integers \(1 \leq p_1, \ldots, p_d \leq n\), let \(W_n(p_1, \ldots, p_d)\) be the smallest number of edges in a \(d\)-uniform \(d\)-partite hypergraph, with \(n\) vertices in each vertex class, that is weakly \(K_{\overrightarrow{p_1, \ldots, p_d}}\)-saturated. A similar notion of weak saturation, in which the copies of \(K_{\overrightarrow{p_1, \ldots, p_d}}\) are required to have \(p_i\) vertices in the \(i\)-th vertex class, was considered long before; we refer to this notion as directed weak saturation.

For integers \(1 \leq p_1, \ldots, p_d \leq n\) denote \(\overrightarrow{W}_n(p_1, \ldots, p_d)\) the directed analogue of \(W_n(p_1, \ldots, p_d)\). Alon \([1]\) determined \(\overrightarrow{W}_n(p_1, \ldots, p_d)\) exactly, showing that \(\overrightarrow{W}_n(p_1, \ldots, p_d) = n^d - \prod_{i=1}^d (n - p_i + 1)\). Note that, by definition, \(W_n(p_1, \ldots, p_d) = \overrightarrow{W}_n(p_1, \ldots, p_d)\), and so we can deduce from Alon’s result a partial answer to the question considered in this paper, namely,

\[
W_n(p_1, \ldots, p_d) = n^d - (n - p + 1)d.
\]

A partial answer for a different setting of parameters was given by Balogh et al. \([2]\). Their main result, proved using linear algebraic techniques, determined \(W_n(p_1, \ldots, p_d)\) when the \(p_i\) only take the two values 1 and \(p\), for some positive integer \(p\).

In this paper we determine \(W_n(p_1, \ldots, p_d)\) for all values of \(p_1, \ldots, p_d\) and \(n\). To state our result we need the following definition.

**Definition 1.1.** For integers \(1 \leq p_1 \leq \cdots \leq p_d \leq n\), let \(q_n(p_1, \ldots, p_d)\) be the number of \(d\)-tuples \(x \in [n]^d\) such that \(x(i) \geq p_i\) for every \(1 \leq i \leq d\), where \(x(i)\) is the \(i\)-th smallest element in the sorted \(d\)-tuple of \(x\) (i.e., which includes repetitions).\(^3\)

Our main theorem is as follows.

**Theorem 1.** For all integers \(1 \leq p_1 \leq \cdots \leq p_d \leq n\) we have

\[
W_n(p_1, \ldots, p_d) = n^d - q_n(p_1, \ldots, p_d)\,.
\]

It is not hard to find explicit formulas for \(q_n(p_1, \ldots, p_d)\), and thus for \(W_n(p_1, \ldots, p_d)\). For the sake of brevity, we omit the details which can be found in the arxiv version of the paper. By combining Theorem 1 and the explicit formulas, one can obtain the interesting corollary that if \(p_d = o(n)\) then \(W_n(p_1, \ldots, p_d) = (d(p_1 - 1) + o(1))n^{d-1}\), that is, \(W_n(p_1, \ldots, p_d)\) is asymptotically determined only by \(p_1\). This should be compared with the fact that the directed analogue \(\overrightarrow{W}_n(p_1, \ldots, p_d)\) is asymptotically determined by all \(p_1, \ldots, p_d\); specifically, if \(p_1, \ldots, p_d\) are all of order \(o(n)\) then \(\overrightarrow{W}_n(p_1, \ldots, p_d) = (p_1 + \ldots + p_d - d + o(1))n^{d-1}\).

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\(^1\)\(K_{\overrightarrow{p_1, \ldots, p_d}}\) denotes the complete \(d\)-uniform \(d\)-partite hypergraph with vertex classes of sizes \(p_1, \ldots, p_d\).

\(^2\)Saturation in the setting of \(d\)-partite hypergraphs is referred to in some papers as \(d\)-saturation, or bi-saturation if \(d = 2\). Since we henceforth only consider saturation in this setting, we prefer to keep using the term “saturation”.

\(^3\)For example, if \(x = (5, 2, 5, 1)\) then the sorted 4-tuple of \(x\) is \((1, 2, 5, 5)\).
Organization: We prove Theorem 1 in Section 2. The proof proceeds by a reduction to Alon’s theorem stated in (1). The proof is reminiscent of Blokhuis’ trick [6], and this is not a coincidence since one can also prove Theorem 1 using Blokhuis’ method of resultants of polynomials [7] together with the idea in [6]. In fact, this was the original proof we found before we found the direct reduction. In Section 3 we raise a conjecture regarding a natural variant of the problem studied here, and also mention a generalization of Theorem 1.

2 Proof of Theorem 1

We begin by proving the upper bound in Theorem 1. Let \( G_0 = G_0(p_1, \ldots, p_d) \) be the \( d \)-uniform \( d \)-partite hypergraph whose non-edges are enumerated by \( q_n(p_1, \ldots, p_d) \) (recall Definition 1.1). More formally, let the vertex classes \( V_1, \ldots, V_d \) of \( G_0 \) each contain \( n \) vertices, and let us label the vertices in each set by \( 1, 2, \ldots, n \) (abusing notation slightly). Let us henceforth identify edges with \( d \)-tuples in \([n]^d\).

Then an edge \((x_1, \ldots, x_d) \in V_1 \times \cdots \times V_d\) does not belong to \( G_0 \) if and only if for every \( 1 \leq i \leq d \) the \( i \)-th smallest element, when \( x_1, \ldots, x_d \) are sorted with repetitions, is at least \( p_i \). We now show that \( G_0 \) is weakly \( K_{p_1, \ldots, p_d}^d \)-saturated, proving the upper bound in Theorem 1. Henceforth we use \( \|H\| \) for the number of edges in a hypergraph \( H \).

Lemma 2.1. \( W_n(p_1, \ldots, p_d) \leq \|G_0\| \).

Proof. Call \( x_1 + \cdots + x_d \) the weight of the edge \( e = (x_1, \ldots, x_d) \), and denote \( G_w \) the \( d \)-uniform \( d \)-partite hypergraph obtained from \( G_0 \) by adding every edge of weight at most \( w \). We next prove that adding any new edge of weight \( w \) to \( G_{w-1} \) creates a new copy of \( K_{p_1, \ldots, p_d}^d \). From this it clearly follows by induction on \( w \) that \( G_0 \) is weakly \( K_{p_1, \ldots, p_d}^d \)-saturated, as required.

Let \( e = (x_1, \ldots, x_d) \) be an edge of weight \( w \) and suppose \( e \) is not in \( G_{w-1} \). Fix \( 1 \leq i \leq d \) and suppose that \( x_i \) is the \( j \)-th smallest among \( x_1, \ldots, x_d \) (i.e., when ordered with repetitions). Let \( S_i \subseteq V_i \) be the set of vertices in the \( i \)-th vertex class that are labeled by \( 1, 2, \ldots, p_j - 1 \). Since \( e \) is not in \( G_w \) and hence not in \( G_0 \), it follows from the definition of \( G_0 \) that \( x_i \geq p_j \). Therefore, \( S_i \cup \{x_i\} \) has \( p_j \) distinct elements. Note that every edge spanned by \( \bigcup_{i=1}^d (S_i \cup \{x_i\}) \), except for \( e \), is of weight smaller than that of \( e \), and so is contained in \( G_{w-1} \). This means that adding \( e \) to \( G_0 \) creates a new copy of \( K_{p_1, \ldots, p_d}^d \) spanned by the vertices \( \bigcup_{i=1}^d (S_i \cup \{x_i\}) \), thus completing the proof.

We now turn to prove the lower bound in Theorem 1. To this end we will in fact use the hypergraph \( G_0 \) we constructed earlier to prove that every weakly \( K_{p_1, \ldots, p_d}^d \)-saturated hypergraph must have as many edges as \( G_0 \). For the proof we will need to use the property of \( G_0 \) that its complement \( \overline{G_0} \) contains every possible “orientation” of \( K_{n-p_1+1, \ldots, n-p_d+1}^d \).

Claim 2.2. For every permutation \( \pi : [d] \to [d] \), the hypergraph \( \overline{G_0} \) contains a copy of the hypergraph \( K_{n-p_1+1, \ldots, n-p_d+1}^d \) having \( n - p_{\pi(i)} + 1 \) vertices in the \( i \)-th vertex class.

Proof. We start with a simple observation, claiming that if two tuples of real numbers \( x = (x_1, \ldots, x_d) \) and \( y = (y_1, \ldots, y_d) \) satisfy \( x_i \geq y_i \) for every \( 1 \leq i \leq d \), then they satisfy \( x_{\pi(i)} \geq y_{\pi(i)} \) for every \( 1 \leq i \leq d \) as well (where, as usual, \( x_{\pi(i)} \) is the \( i \)-th smallest element in the sorted tuple of \( x \), and

4I.e., the \( d \)-tuple \((x_1, \ldots, x_d) \in [n]^d\) is identified with the edge containing from each \( V_i \) the vertex labeled \( x_i \).

5By complement we mean relative to \( K_{n, \ldots, n} \), that is, the hypergraph that contains an edge \((x_1, \ldots, x_d) \in V_1 \times \cdots \times V_d \) if and only if \( G_0 \) does not.
similarly for \( y \). To see this, let \( \sigma : [d] \to [d] \) be a permutation sorting \( y \), that is, \( y_{\sigma(1)} \leq \cdots \leq y_{\sigma(d)} \). Now note that for every \( 1 \leq i \leq d \) and \( i \leq j \leq d \) we have \( x_{\sigma(j)} \geq y_{\sigma(j)} \geq y_{\sigma(i)} = y(i) \). This means that \( x \) has at least \( d - i + 1 \) elements that are at least as large as \( y(i) \), which means that we must have \( x(i) \geq y(i) \).

Now, suppose without loss of generality that \( p_1 \leq \cdots \leq p_d \). Let \( \pi : [d] \to [d] \) be an arbitrary permutation and let \( S_i \) be the subset of vertices of the \( i \)-th vertex class containing those vertices labeled by \( p_{\pi(i)} \), \( p_{\pi(i)} + 1 \), \ldots, \( n \). Clearly, for every edge \( e = (x_1, \ldots, x_d) \) spanned by the vertices in \( S := \bigcup_{i = 1}^{d} S_i \) it holds that \( x_i \geq p_{\pi(i)} \). It follows from our observation above that \( x(i) \geq p_i \). From the definition of \( G_0 \) we conclude that \( e \in \overline{G_0} \). Hence, \( S \) spans a copy of \( K_{n-p_1+1,\ldots,n-p_d+1}^d \) in \( \overline{G_0} \) having \( n - p_{\pi(i)} + 1 \) vertices in the \( i \)-th vertex class, as desired.

**Lemma 2.3.** \( W_n(p_1, \ldots, p_d) \geq \|G_0\| \).

**Proof.** Let \( H \) be a \( d \)-uniform \( d \)-partite hypergraph that is weakly \( K_{p_1,\ldots,p_d}^d \)-saturated, where its vertex classes \( (V_1, \ldots, V_d) \) are each of cardinality \( n \). We construct a hypergraph \( H' \) with \( 2n \) vertices in each vertex class by combining it with \( \overline{G_0} \) as follows. Let \( U_1, \ldots, U_d \) be \( d \) sets of new vertices (i.e., disjoint from \( \bigcup_{i = 1}^{d} V_i \) and from each other) with \( |U_i| = n \). We let \( H' \) be the \( d \)-uniform \( d \)-partite hypergraph with vertex classes \( (V_1 \cup U_1, \ldots, V_d \cup U_d) \) whose edges are defined as follows. The edges of \( H' \) that are spanned by the vertices in \( V := \bigcup_{i = 1}^{d} V_i \) are precisely those of \( H \); the edges of \( H' \) that are spanned by the vertices in \( U := \bigcup_{i = 1}^{d} U_i \) are precisely those of \( \overline{G_0} \); finally, all other possible edges (i.e., those containing at least one vertex from \( V \) and at least one vertex from \( U \)) appear in \( H' \) as well. Notice that by counting the non-edges of \( H' \) we get

\[
(2n)^d - \|H'\| = (n^d - \|H\|) + \|G_0\|.
\]

We claim that \( H' \) is weakly \( K_{n+1,\ldots,n+1}^d \)-saturated. Observe that (1) and (2) would then give

\[
(2n)^d - (n^d - \|H\|) - \|G_0\| = \|H'\| \geq W_{2n}(n + 1, \ldots, n + 1) = (2n)^d - n^d,
\]

implying that \( \|H\| \geq \|G_0\| \), thus completing the proof.

To show that \( H' \) is weakly \( K_{n+1,\ldots,n+1}^d \)-saturated, we claim that one obtains a saturation process of \( H' \) with respect to \( K_{n+1,\ldots,n+1}^d \) by first adding the non-edges of \( H \) in the same order they appear in some saturation process of \( H \) (with respect to \( K_{p_1,\ldots,p_d}^d \)), and then adding, in an arbitrary order, all edges of \( G_0 \). To see that this indeed defines a saturation process of \( H' \) with respect to \( K_{n+1,\ldots,n+1}^d \), let \( e \) be a non-edge of \( H \) added at some point. Then adding \( e \) to \( H' \) (after all the edges that precede \( e \) in the saturation process are added) creates a new copy of \( K_{p_1,\ldots,p_d}^d \) in \( H' \), which we denote \( C \). Let \( \pi : [d] \to [d] \) be a permutation such that \( C \) contains \( p_{\pi(i)} \) vertices in the \( i \)-th vertex class for every \( 1 \leq i \leq d \). By Claim 2.2, \( \overline{G_0} \) contains a copy \( C' \) of \( K_{n-p_1+1,\ldots,n-p_d+1}^d \) having \( n - p_{\pi(i)} + 1 \) vertices in the \( i \)-th vertex class. It follows that when adding \( e \) we in fact create a new copy of \( K_{n+1,\ldots,n+1}^d \) in \( H' \), namely, the copy spanned by the union of the vertex sets of \( C \) and \( C' \). To complete the proof of our claim we observe that, after all the edges over \( V \) are added to \( H' \), each edge \( (x_1, \ldots, x_d) \) of \( G_0 \) is the only missing edge in the copy of \( K_{n+1,\ldots,n+1}^d \) spanned by \( \bigcup_{i = 1}^{d} (V_i \cup \{x_i\}) \) (recall that \( |V_i| = n \) for every \( i \)). This completes the proof of the statement.

**Proof of Theorem 1.** Lemmas 2.1 and 2.3 give \( W_n(p_1, \ldots, p_d) = \|G_0\| = n^d - q_n(p_1, \ldots, p_d) \).
3 Concluding Remarks and Open Problems

Undirected strong saturation in bipartite graphs: Theorem 1 shows that in the setting of weak saturation, the undirected version $\overrightarrow{W}_n(p,q)$ requires much fewer edges than its directed analogue $\overrightarrow{W}_n(p,q)$. It is thus natural to ask what happens in the setting of strong saturation. Let $S_n(p,q)$ be the minimum number of edges in an $n \times n$ bipartite graph such that any addition of a new edge between its two classes creates a copy of $K_{p,q}$ (i.e., the graph is strongly $K_{p,q}$-saturated). Let $\overrightarrow{S}_n(p,q)$ denote the directed analogue of $S_n(p,q)$. Answering a conjecture of Erdős-Hajnal-Moon [8], $\overrightarrow{S}_n(p,q)$ was completely determined by Wessel [13] and Bollobás [4] to be $(p+q-2)n - (p-1)(q-1)$. Perhaps surprisingly, there are constructions showing that $S_n(p,q)$ is, in general, strictly smaller than this. To see this, suppose $p \leq q$ and let $G_{k_{p,q}}^k$ be any $n \times n$ bipartite graph having $p-1$ vertices in each class complete to the other class, some $k$ additional vertices in each class spanning a $K_{k,k}$, and where the remaining vertices have degree $q-1$. Note that $G_{p,q}^k$ has the property that any new edge one adds to it has an endpoint of degree at least $q$. The $q$ neighbors, together with the $p$ complete vertices from the other class, then form a $K_{p,q}$, implying that $G_{p,q}^k$ is strongly $K_{p,q}$-saturated. One can check that $G_{p,q}^k$ has in fact $\overrightarrow{S}_n(p,q) - k(q-p-k)$ edges. Optimizing using $k = \lfloor (q-p)/2 \rfloor$ gives $S_n(p,q) \leq \overrightarrow{S}_n(p,q) - \lfloor (q-p)^2/4 \rfloor$, for every $n$ large enough such that $G_{p,q}^k$ is well defined. We raise the following conjecture.

Conjecture 1. For every $p,q$ and $n \geq n_0(p,q)$ we have $S_n(p,q) = \overrightarrow{S}_n(p,q) - \lfloor (q-p)^2/4 \rfloor$.

It is not hard to prove that the above conjecture indeed holds when $p = 1$ and $q$ is arbitrary. Very recently, Gan, Korandi and Sudakov [10] confirmed the conjecture also when $p = 2$ and $q = 3$, and also proved an approximate version of it by showing that $S_n(p,q) \geq (p+q-2)n - (p+q-2)^2$.

A variant of Bollobás’s Two Families Theorem: As mentioned earlier, the classical results on $K_p$-strong/weak saturation follow as applications of Bollobás’s famous Two Families Theorem [3] and its skew analogues. As it turns out, our Theorem 1 can also be deduced from a more general undirected version of Alon’s multi-partite analogue of Bollobás’s theorem. The details appear in the arxiv version of the paper.

References


*I.e., where the copies of $K_{p,q}$ must have their $p$ vertices in the first class and their $q$ vertices in the second class.


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