Computational Aspects of Prediction Markets

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Prediction markets

- Many traders, each holds some partial information.
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- Goal: to aggregate all the traders’ information and learn about the true state of the world.
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Partition model of knowledge

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We assume a common prior distribution $\mathcal{P} \in \Delta(\Omega)$ shared by all agents before receiving any information.
Model of an information market

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- \( n \) agents, each has a single bit of information \( x_i \).
- Agents are rational, truthful, risk neutral and myopic.
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- \( S^r \) denotes the set of commonly known possible states after round \( r \).
- \( S_i^r \) denotes the set of states agent \( i \) consider possible after round \( r \).
- After each round, each agent \( i \) can update their \( S_i^r \) by eliminating all \( x \)'s that would result a different price than what was announced.
Equilibrium price characterization

Note that:

1. $S^0 = \Omega$. 
2. $S^0_i = \{y \in \Omega | y_i = x_i\}$
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- $S^1 = \{ y \in S^0 | \text{price}^1(y) = p^1 \}$, where $\text{price}^1(x)$ is the function that relates each state to a clearing price. Even though $x$ is unknown, any observer can still calculate the clearing price for hypothetical $x$ and eliminate those which are different than what was declared.
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- $S^r_i = \{y \in S^r \text{ | } y_i = x_i\}$
- $S^r_i \subseteq S^{r-1}_i$ and $S^r \subseteq S^{r-1}$
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- $S_i^r = \{y \in S^r | y_i = x_i\}$
- $S_i^r \subseteq S_i^{r-1}$ and $S^r \subseteq S^{r-1}$
- A convergence is reached after a finite number of steps. We denote: $S^\infty$, $S_i^\infty$ and $p^\infty$. 
Example: OR function
the function $C_n$ takes $2n$ bits as input - $(x, y)$ and returns the carry bit for $x + y$. Let us examine the function $C_2$ with the following distribution:
Example: Carry bit function

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- $(x_2, y_2)$ has a distribution conditional on $(x_1, y_1)$. If $(x_1, y_1) = (1, 1)$, then $(x_2, y_2)$ takes the values $(0, 0), (0, 1), (1, 0), (1, 1)$ with probabilities $\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}$ respectively. Otherwise with probabilities $\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{2}$
A *rational expectations equilibrium* is a mapping $P^* : \Omega \rightarrow \mathbb{R}$ s.t. in every state $\omega$, if every trader conditions her demand (or supply) on her private information $\pi_i$ and the price $P^*(\omega)$, the market will clear at the price of exactly $P^*(\omega)$. 
Equilibrium price characterization

**Stochastically monotone function**

A function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is called *Stochastically monotone* if it can be written in the form $g(x) = \sum_i g_i(x_i)$ where each $g_i : \mathbb{R} \rightarrow \mathbb{R}$ is strictly increasing.
**Equilibrium price characterization**

### Stochastically monotone function

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### Stochastically regular function

A function $g : \mathbb{R}^n \to \mathbb{R}$ is called **Stochastically regular** if it can be written in the form $g = h \circ g'$ where $g'$ is stochastically monotone and $h$ is invertible on the range of $g'$.
Suppose that, at equilibrium, the $n$ agents have a common prior, but a possibly different information about the value of $F$. For all $i$, let $p_i^\infty = E(F|x \in S_i^\infty)$. If $g$ is stochastically monotone and $g(p_1^\infty, p_2^\infty, ..., p_n^\infty)$ is common knowledge then:

$$p_1^\infty = p_2^\infty = ... = p_n^\infty = E(F|x \in S_i^\infty) = p^\infty$$
Shared knowledge at equilibrium

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Nielsen et al. 1990

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This means that at equilibrium all agents must have exactly the same expectation of the value of the security and that this must agree with the expectation of an uninformed observer. Is equilibrium enough for the purpose of information aggregation? No. We also want that $p^\infty = f(x)$

Example: XOR with a uniform distribution.
Characterizing computable aggregates

**weighted threshold function**

A function \( f : \{0, 1\}^n \rightarrow \{0, 1\} \) is a *weighted threshold function* iff there are \( w_1, w_2 \ldots w_n \in \mathbb{R} \) s.t

\[
f(x) = 1 \text{ iff } \sum_{i=1}^{n} w_i x_i \geq 1\]
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**Equilibrium of weighted threshold functions**

If $f$ is a weighted threshold function, then for any prior probability distribution, the equilibrium price of $F$ is equal to $f(x)$. 
Proof:

\[ p^\infty = E(F|x \in S_i^\infty) \] thus, if \( p^\infty = 0 \) or \( 1 \) then \( f(x) = p^\infty \)
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Assume towards contradiction that \( 0 < p^\infty < 1 \). from Nielsen et al. we get:

\[ P(f(y) = 1|y \in S^\infty) = p^\infty \]

\[ \forall iP(f(y) = 1|y \in S_i^\infty) = p^\infty \]
Proof:

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\[ P(f(y) = 1|y \in S^\infty) = p^\infty \] (1)
\[ \forall i P(f(y) = 1|y \in S_i^\infty) = p^\infty \] (2)

since \( S_i^\infty = \{y \in S^\infty|y_i = x_i\} \) Equation (2) can be written as:

\[ \forall i P(f(y) = 1|y \in S^\infty, y_i = x_i) = p^\infty \] (3)
Define:

\[ J_i^+ = P(y_i = 1 | y \in S^\infty, f(y) = 1) \]
\[ J_i^- = P(y_i = 1 | y \in S^\infty, f(y) = 0) \]

\[ J^+ = \sum_{i=1}^{n} w_i J_i^+ \]
\[ J^- = \sum_{i=1}^{n} w_i J_i^- \]

Since \( p^\infty \neq 0, 1 \) both \( J_i^+, J_i^- \) are well defined for all \( i \).
Proof cont.

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Claim: Equations (1) and (3) imply that $J_i^+ = J_i^-$ for all $i$
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Proof: Assume $x_i = 1$, we then have:

\[
P(f(y) = 1|y \in S^\infty) \cdot J_i^+ + P(f(y) = 0|y \in S^\infty) \cdot J_i^-
\]

By Bayes’ law:

\[
P(\frac{p^\infty \cdot J_i^+}{p^\infty \cdot J_i^+ + (1 - p^\infty)J_i^-} = p^\infty \text{ by Eqs. (1) and (3)}
\]

\[
\Rightarrow J_i^+ = p^\infty J_i^+ + (1 - p^\infty)J_i^-
\]

\[
\Rightarrow J_i^+ = J_i^-
\]

The proof is symmetrical for $x_i = 0$
Using linearity of expectation we can write:

\[ J^+_i = E \left( \sum_{i=1}^{n} w_i y_i \mid y \in S^\infty, f(y) = 1 \right) \]

\[ J^-_i = E \left( \sum_{i=1}^{n} w_i y_i \mid y \in S^\infty, f(y) = 0 \right) \]
Using linearity of expectation we can write:

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J_i^+ = E \left( \left[ \sum_{i=1}^{n} w_i y_i \right] \mid y \in S^\infty, f(y) = 1 \right)
\]

\[
J_i^- = E \left( \left[ \sum_{i=1}^{n} w_i y_i \right] \mid y \in S^\infty, f(y) = 0 \right)
\]

Since \( f \) is a weighted threshold function, \( f(y) = 1 \) iff \( \sum_{i=1}^{n} w_i y_i \geq 1 \)

and thus \( J_i^+ \geq 1 \), similarly \( J_i^- < 1 \) which implies \( J_i^+ \neq J_i^- \)

And so we have a contradiction. \( \square \)
Upper bound on the number of iterations

In the worst case, at most \( n \) rounds of trading are required to reach an equilibrium.
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- Each set $S^0, S^1 \ldots S^n$ has a strictly lower dimension than the previous one until the market reaches an equilibrium.
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**Dimension**

The *dimension* of a set $S \subseteq \{0, 1\}^n$ is the dimension of the smallest affine subset of $\mathbb{R}^n$ that contains all points in $S$. Denoted $\text{dim}(S)$. 
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**Lemma**

If $S^r \neq S^{r-1}$, then $\text{dim}(S^r) < \text{dim}(S^{r-1})$. 
Proof

Let $k = \text{dim}(S^{r-1})$. Consider the bids in round $r$:

$$b^r_i = E(f(y) = 1 | y \in S^{r-1}, y_i = x_i)$$
Proof

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Depending on the value of \( x_i \), \( b^r_i \) is either \( h^{(0)}_i \) or \( h^{(1)}_i \). These depend only on \( S^{r-1} \) which is common knowledge.
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Depending on the value of $x_i$, $b^r_i$ is either $h_i^{(0)}$ or $h_i^{(1)}$. These depend only on $S^{r-1}$ which is common knowledge.

Denote $d_i = h_i^{(0)} - h_i^{(1)}$ and we get:

$$p^r = \frac{1}{n} \sum_{i=1}^{n} (h_i^{(0)} + d_i x_i) \quad (4)$$
Equation (4) defines a hyperplane the intersection of which with $S^{r-1}$ is the resulting common knowledge $S^r$.
This is because all agents already know all $h_i^{(0)}$ and $d_i$ and so they use the linear equation (4) to rule out any possibility that would not have resulted the price $p^r$. 
Equation (4) defines a hyperplane the intersection of which with $S^{r-1}$ is the resulting common knowledge $S^r$. This is because all agents already know all $h_i^{(0)}$ and $d_i$ and so they use the linear equation (4) to rule out any possibility that would not have resulted the price $p^r$.

If $S^r \neq S^{r-1}$, this intersection defines a linear subspace of dimension $(k-1)$ that contains $S^r$ and so $S^r$ has a dimension of at most $(k - 1)$. 
Upper bound on the number of iterations

Let $f$ be a weighted threshold function, and $\mathcal{P}$ an arbitrary distribution. Then after at most $n$ rounds of trading, the price reaches its equilibrium value $p^\infty = f(x)$.

This follows directly from the Lemma as $\dim(S^0) = n$ and once we have a round $r$ s.t. $S^r = S^{r-1}$ then no trader has gained any new knowledge and thus $p^\infty = f(x)$. 

proof cont.