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Auctions

1.1 Auctions
Auctions are an ancient mechanism for buying and selling goods, and in modern times a huge volume of economic transactions is conducted through auctions: The US government runs auctions to sell treasury bills, spectrum licenses and timber and oil leases, among others. Christie’s and Sotheby’s run auctions to sell art. In the age of the Internet, we can buy and sell goods and services via auction, using the services of companies like eBay. The advertisement auctions that companies like Google, Yahoo! and Microsoft run in order to sell advertisement slots on their web pages bring in a significant fraction of their revenue.

Why might a seller use an auction as opposed to simply fixing a price? Primarily because sellers often don’t know how much buyers value their goods, and don’t want to risk setting prices that are either too low, thereby leaving money on the table, or, so high that nobody will want to buy the item. An auction is a technique for dynamically setting prices. Auctions are particularly important these days because of their prevalence in Internet settings where the participants in the auction are computer programs, or individuals with no direct knowledge of or contact with each other.

1.2 Single Item Auctions
We are all familiar with the famous English or ascending auction for selling a single item: The auctioneer starts by calling out a low price \( p \). As long as there are at least two people willing to pay the price \( p \), he increases \( p \) by a small amount. This continues until there is only one player left willing to pay the current price, at which point that player “wins” the auction, i.e. receives the item at that price.

When multiple rounds of communication are inconvenient, the English
Auctions

Fig. 1.1. The left figure shows an auction for fish in Tokyo. The right figure shows an auction at Sotheby’s for the painting The Scream.

Auction is sometimes replaced by other formats. For example, in a sealed-bid first-price auction, the participants submit sealed bids to the auctioneer. The auctioneer allocates the item to the highest bidder who pays the amount she bid.

We’ll begin by examining auctions from two perspectives: what are equilibrium bidding strategies and what is the resulting revenue of the auctioneer?

To answer these questions, we need to know what value the bidders place on the item and what they know about each other. For example, in an art auction, the value a bidder places on a painting is likely to depend on other people’s values for that painting, whereas in an auction for fish among restaurant owners, each bidder’s value is known to him before the auction and roughly independent of other bidder’s valuations.

**Private Values**

For most of this chapter, we will assume that each player has a private value $v$ for the item being auctioned off. This means that he would not pay more than $v$ for the item, while if he gets the item at a price $p < v$, his utility is $v - p$. Given the rules of the auction, and any knowledge he has about other players’ bids, he will bid so as to maximize his utility.

In the ascending auction, it is a dominant strategy for a bidder to increase his bid as long as the current price is below his value, i.e., doing this maximizes his utility no matter what the other bidders do. But how should a player bid in a sealed-bid first price auction? Clearly, bidding one’s value makes no sense, since even upon winning, this would result in a gain of 0! So a bidder will want to bid lower than their true value. But how much lower? Low bidding has the potential to increase a player’s gain, but at the
same time increases the risk of losing the auction. In fact, the optimal bid in such an auction depends on how the other players are bidding, which in general, a bidder will not know.

**Definition 1.2.1.** A (direct) single-item auction with \( n \) bidders is a mapping that assigns to any vector of bids \((b_1, \ldots, b_n)\) a winner \( i \in \{0 \ldots n\} \) \((i = 0 \text{ means that the item is not allocated})\) and a set of prices \((p_1, \ldots, p_n)\), where \(p_j\) is the price that bidder \( j \) must pay. A bidding strategy for agent \( i \) is a mapping \( \beta_i : [0, \infty) \rightarrow [0, \infty) \) which specifies agent \( i \)'s bid \( \beta_i(v_i) \) for each possible value \( v_i \) she may have.

**Definition 1.2.2.** (Private Values) Suppose that \( n \) bidders are competing in a (direct) single-item auction, and the joint distribution of their values \( V_1, V_2, \ldots, V_n \) is common knowledge. Each bidder \( i \) also knows the realization \( v_i \) of his own value \( V_i \). Fix a bidding strategy \( \beta_i : [0, \infty) \rightarrow [0, \infty) \) for each agent \( i \). Note that we may restrict \( \beta_i \) to the support of \( V_i \).

- The allocation probabilities are:
  \[ a_i[b] := P \text{ [bidder } i \text{ wins bidding } b \text{ when other bids are } \beta_j(V_j), \forall j \neq i]. \]

- The expected payments are:
  \[ p_i[b] := E \text{ [payment of bidder } i \text{ bidding } b \text{ when other bids are } \beta_j(V_j), \forall j \neq i]. \]

- The expected utility of bidder \( i \) with value \( v_i \) bidding \( b \) is:
  \[ u_i[b|v_i] = v_i a_i[b] - p_i[b]. \]

The bidding strategy profile \((\beta_1, \ldots, \beta_n)\) is in Bayes-Nash equilibrium if for all \( i \) and all \( v_i \)

\[ b \rightarrow u_i[b|v_i] \text{ is maximized at } b = \beta_i(v_i). \]

### 1.3 Independent Private Values

Consider a first-price auction, in which each player’s value \( V_i \) is drawn independently from a distribution \( F_i \). If each other bidder \( j \) bids \( \beta_j(V_j) \) and bidder \( i \) bids \( b \), his expected utility is

\[ u_i[b|v_i] = v_i a_i[b] - p_i[b] \]

† The support of a random variable \( V \) with distribution function \( F \) is defined as \( \text{supp}(V) = \text{supp}(F) := \cap_{\epsilon>0} \{ x | F(x + \epsilon) - F(x - \epsilon) > 0 \} \).
\begin{equation}
u_i[b|v_i] = (v_i - b) \cdot a_i[b] = (v_i - b) \cdot \mathbb{P}\left[ b > \max_{j \neq i} \beta_j(V_j) \right]. \tag{1.1}\end{equation}

**Example 1.3.1.** Consider a two-bidder first price auction where the \(V_i\) are independent and uniform on \([0, 1]\). Suppose that \(\beta_1 = \beta_2 = \beta\) is an equilibrium, with \(\beta : [0, 1] \rightarrow [0, \beta(1)]\) differentiable and strictly increasing. Bidder 1 with value \(v_1\), knowing that bidder 2 is bidding \(\beta(V_2)\), compares the utility of alternative bids \(b\) to \(\beta(v_1)\). We may assume that \(b \in [0, \beta(1)]\), since higher bids are dominated by bidding \(\beta(1)\). Thus, \(b = \beta(w)\) for some \(w \neq v_1\). With this bid, the expected utility for bidder 1 is

\[ u_1[b] = \beta(w)|v_1| = (v_1 - b) \cdot \mathbb{P}[b > \beta(V_2)] = (v_1 - b) \cdot w. \]

To eliminate \(b\), we introduce the notation

\[ u_1(w|v_1) := u_1[\beta(w)|v_1] = (v_1 - \beta(w)) \cdot w. \tag{1.2} \]

For \(\beta\) to be an equilibrium, \(w \rightarrow u_1(w|v_1)\) must be maximized when \(w = v_1\), i.e.,

\[ \frac{\partial u_1(w|v_1)}{\partial w} = v_1 - \beta'(w)w - \beta(w) \]

must vanish for \(w = v_1\). Thus, for all \(v_1\),

\[ v_1 = \beta'(v_1)v_1 + \beta(v_1) = (v_1\beta(v_1))'. \]

Integrating both sides, we obtain

\[ \frac{v_1^2}{2} = v_1\beta(v_1) \quad \text{and so} \quad \beta(v_1) = \frac{v_1^2}{2}. \]

We now verify that \(\beta(v) = v/2\) is an equilibrium. Bidder 1’s utility when her value is \(v_1\), she bids \(b\), and bidder 2 bids \(\beta(V_2) = V_2/2\) is

\[ u_1[b|v_1] = \mathbb{P}\left[ \frac{V_2}{2} \leq b \right] (v_1 - b) = 2b(v_1 - b). \]

This function is maximized when \(b = v_1/2\). Thus, since the bidders are symmetric, \(\beta(v) = v/2\) is indeed an equilibrium.

In the example above of an equilibrium for the first price auction, bidders must bid below their values, taking the distribution of competitor’s values into account. This contrasts with the English auction, where no such strategizing is needed. Is strategic bidding (that is, considering competitor’s values
and potential bids) a necessary consequence of the convenience of sealed-bid auctions? No. Nobel-prize winner William Vickrey (1960) discovered that one can combine the low communication cost of sealed-bid auctions with the simplicity of the optimal bidding rule in ascending auctions. We can get a hint on how to construct this combination by determining the revenue of the auctioneer in the ascending auction when all bidders act rationally: The item is sold to the highest bidder when the current price exceeds what other bidders are willing to offer; this threshold price is approximately the value of the item to the second-highest bidder.

**Definition 1.3.2.** In a (sealed bid) **second price auction** (also known as a **Vickrey auction**), the highest bidder wins the auction at a price equal to the second highest bid.

**Theorem 1.3.3.** The second price auction is **truthful**. In other words, for each bidder $i$, and for any fixed set of bids of all other bidders, bidder $i$’s utility is maximized by bidding her true value $v_i$.

**Proof.** Suppose the maximum of the bids submitted by bidders other than $i$ is $m$. If $m > v_i$, bidding truthfully (or bidding any value that is at most $m$) will result in a utility of 0 for bidder $i$. On the other hand, bidding above $m$ would result in a negative utility. Thus, the bidder cannot gain by lying. On the other hand, if $m \leq v$, then as long as the bidder wins the auction, his utility will be $v - m \geq 0$. Thus, the only change in utility that can result due to bidding untruthfully occurs if the bidder bids below $m$, in which case, his utility will be 0 since he then loses the auction.

*Remark.* We emphasize that the theorem statement is not merely saying that truthful bidding is a Nash equilibrium, but rather the much stronger statement that bidding truthfully is a **dominant strategy**, i.e., it maximizes each bidders gain no matter how other bidders play.

In Chapter 2, we show that a variant of this auction applies much more broadly. For example, when an auctioneer has $k$ identical items to sell, and each bidder wants only one, the following auction is also truthful.

**Definition 1.3.4.** In a (sealed bid) **$k$-unit Vickrey auction** the top $k$ bidders win the auction at a price equal to the $(k + 1)^{st}$ highest bid.

**Exercise 1.3.5.** Prove that the $k$-unit Vickrey auction is truthful.
1.3.1 Profit in single-item auctions

From the perspective of the bidders in an auction, a second price auction is appealing. They don’t need to perform any complex strategic calculations. The appeal is less clear, however, from the perspective of the auctioneer. Wouldn’t the auctioneer make more money running a first price auction?

Example 1.3.6. We return to our earlier example of two bidders, each with a value drawn from U[0,1] distribution. From that analysis, we know that if the auctioneer runs a first price auction, then in equilibrium his expected profit will be

$$E \left[ \max \left( \frac{V_1}{2}, \frac{V_2}{2} \right) \right] = \frac{1}{3}.$$ 

On the other hand, suppose that in the exact same setting, the auctioneer runs a second-price auction. Since the bidders will bid truthfully, the auctioneer’s profit will be the expected value of the second highest bid, which is

$$E [\min(V_1, V_2)] = \frac{1}{3},$$

exactly the same as in the 1st price auction!

In fact, in both cases, bidder \( i \) with value \( v_i \) has probability \( v_i \) of winning the auction, and the conditional expectation of his payment given winning is \( v_i/2 \): in the case of the first price auction, this is because he bids \( v_i/2 \) and in the case of the second price auction, this is because the expected bid of the other player is \( v_i/2 \). Thus, overall, in both cases, his expected payment is \( v_i^2/2 \).

Coincidence? No. As we shall see next the amazing revenue equivalence theorem shows that any auction that has the same allocation rule in equilibrium yields the same auctioneer revenue! (This applies even to funky auctions like the all-pay auction; see below.)

1.4 Definitions

To test whether a strategy profile \((\beta_1, \beta_2, \ldots, \beta_n)\) is an equilibrium, it will be important to determine the utility for bidder \( i \) when he bids as if his value is \( w \neq v_i \). We adapt the notation of Definition 1.2.2 as follows:

Definition 1.4.1. Let \((\beta_i)_{i=1}^{n}\) be a strategy profile for \( n \) bidders with values \( V_1, V_2, \ldots, V_n \). Suppose bidder \( i \), knowing his own value \( v_i \), bids \( \beta_i(w) \). Then

† We use square brackets to denote functions of bids and regular parentheses to denote functions of alternative valuations.
1.4 Definitions

- The **allocation probability** to bidder $i$ is $a_i(w) := a_i[\beta_i(w)]$.
- His **expected payment** is $p_i(w) := p_i[\beta_i(w)]$.
- His **expected utility** is $u_i(w|v_i) := u_i[\beta_i(w)|v_i] = v_i a_i(w) - p_i(w)$.

We will assume that $p_i(0) = 0$, as this holds in most auctions.

1.4.1 Payment Equivalence and Characterization of Bayes-Nash equilibrium

Consider the setting of $n$ bidders, with i.i.d. values drawn from $F$. Since the bidders are all symmetric, it’s natural to look for symmetric equilibria, i.e. $\beta_i = \beta$ for all $i$. As above (dropping the subscript in $a_i(\cdot)$ due to symmetry), let

$$a(v) = \mathbb{P} [\text{the item is allocated to a bidder with bid } \beta(v)].$$

For example, if the item goes to the highest bidder, as in a first price auction, then for $\beta(\cdot)$ increasing

$$a(w) = \mathbb{P} \left[ \beta(w) > \max_{j \neq i} \beta(V_j) \right] = \mathbb{P} \left[ w > \max_{j \neq i} V_j \right] = F_{n-1}(w).$$

(The simplicity of this expression is one of the motivations for using the notation $a_i(w)$ rather than $a_i[b_i]$.)

If bidder $i$ bids $\beta(w)$, his expected utility is

$$u(w|v_i) = v_i a(w) - p(w). \quad (1.3)$$
Assume $p(w)$ and $a(w)$ are differentiable. For $\beta$ to be an equilibrium, it must be that for all $v_i$, the derivative $v_i a'(w) - p'(w)$ vanishes at $w = v_i$, so

$$p'(v_i) = v_i a'(v_i) \quad \text{for all } v_i.$$

Hence, if $p(0) = 0$, we get

$$p(v_i) = \int_0^{v_i} v a'(v) dv$$

which, integrating by parts, yields

$$p(v_i) = v_i a(v_i) - \int_0^{v_i} a(w) dw.$$  \hfill (1.4)

In other words, the expected payment in equilibrium is determined by the allocation rule.

This analysis leaves unanswered the question of what properties of the allocation rule guarantee that there is an equilibrium. It turns out that monotonicity is necessary and sufficient.

To see that it is necessary, fix an arbitrary bidder, say $i$. If bidder $i$ has value $v$, then he has higher utility bidding $\beta(v)$ than $\beta(w)$, i.e.,

$$u(v|v) = va(v) - p(v) \geq va(w) - p(w) = u(w|v).$$

Reversing the roles of $v$ and $w$, we have

$$wa(w) - p(w) \geq wa(v) - p(v)$$
1.4 Definitions

Fig. 1.4. This figure shows how a monotone allocation rule and payments determined by (1.4) ensures that no bidder has an incentive to bid as if he had a value other than his true value. Consider a bidder with value $v_1$. On the left side, we see what happens to the bidder’s utility if he bids as if his value is $w' > v_1$. In this case, $u(w'|v_1) = v_1a(w') - p(w')$, which is the yellow area minus the orange area. Note that the yellow area in the left figure is precisely $u(v_1|v_1)$. On the right side we see what happens to the bidder’s expected utility if he bids as if his value is $w < v_1$. In this case again $u(w'|v_1) = v_1a(w') - p(w)$ is less than the expected utility he would have obtained by bidding $\beta(v)$.

Adding these two inequalities, we obtain that for all $v$ and $w$

$$(v - w)(a(v) - a(w)) \geq 0.$$  

Thus, if $v \geq w$, then $a(v) \geq a(w)$, and we conclude that $a(\cdot)$ is monotone nondecreasing.

The fact that monotonicity, combined with the payment rule given by (1.4), is sufficient for $\beta(\cdot)$ to be an equilibrium is shown pictorially in Figure 1.4.

The next theorem shows that the characterization of Bayes-Nash equilibrium just discussed holds under more general conditions. The proof of this more general version can be found in Chapter 3.

**Theorem 1.4.2 (Characterization of Bayes-Nash Equilibrium).** Let $A$ be an auction for selling $k$ identical items, where bidder $i$’s value $V_i$ is drawn independently from $F_i$. We assume that $F_i$ is strictly increasing and continuous on $[0, h_i]$, with $F(0) = 0$ and $F(h_i) = 1$. ($h_i$ can be $\infty$.)

(a) If $(\beta_1, \ldots, \beta_n)$ is a Bayes-Nash equilibrium, then for each agent $i$:

(i) The probability of allocation $a_i(v_i)$ is monotone increasing in $v_i$. 

(ii) The utility $u_i(v_i)$ is a convex function of $v_i$, with

$$u_i(v_i) = \int_0^{v_i} a_i(z)dz.$$ 

(iii) The expected payment is determined by the allocation probabilities:

$$p_i(v_i) = v a_i(v) - \int_0^{v_i} a_i(z)dz = \int_0^{v_i} z a_i'(z)dz.$$ 

(b) Conversely, if $(\beta_1, \ldots, \beta_n)$ is a set of bidder strategies for which (i) and (iii) hold, then for all bidders $i$, and values $v$ and $w$

$$u_i(v|v) \geq u_i(w|v).$$  \hfill (1.5)

Remarks:

• The converse in the theorem statement implies that if (i) and (iii) hold for a set of bidder strategies, then these bidding strategies are an equilibrium relative to alternatives in the image of the bidding strategies $\beta_i$. In fact, showing that (i) and (iii) hold can be a shorter route to proving that a set of bidding strategies is a Bayes-Nash equilibrium, since strategies that are outside the range of $\beta$ can often be ruled out easily by other means. We will see this in the examples below.

The famous revenue equivalence theorem is a corollary of this characterization.

Corollary 1.4.3 (Revenue Equivalence). If $A$ and $\tilde{A}$ are two $k$-unit auctions with the same allocation rule in equilibrium, i.e., $a_i^A(v_i) = a_i^{\tilde{A}}(v_i)$, then for all bidders $i$ and values $v_i$: $p_i^A(v_i) = p_i^{\tilde{A}}(v_i)$, whence the expected auctioneer revenue is the same.

Corollary 1.4.4. Suppose that each agents’ value $V_i$ is drawn independently from the same strictly increasing distribution $F \in [0, h]$. Consider any $n$-bidder single-item auction in which the item is allocated to the highest bidder. If $\beta_i = \beta$ is a symmetric Bayes-Nash equilibrium and $\beta$ is strictly increasing in $[0, h]$, then

$$a(v) = F(v)^{n-1} \quad \text{and} \quad p(v) = \int_0^{v} [a(v) - a(w)]dw.$$  \hfill (1.6)
Moreover,

\[ p(v) = F(v)^{n-1}E \left[ \max_{i \leq n-1} V_i \mid \max_{i \leq n-1} V_i \leq v \right] \quad (1.7) \]

**Proof.** Since \( \beta \) is a symmetric equilibrium, and is strictly increasing, an agent wins precisely if her value \( v \) is higher than the values of all the other agents, which happens with probability \( F(v)^{n-1} \). The equalities in (1.6) follow from the preceding theorem. Any such auction has the same allocation rule as the truthful second price auction, for which each agent has the expected payment given in (1.7).

We now use this corollary to derive equilibrium strategies in a number of auctions:

**First-price auction:**

(a) Suppose that \( \beta \) is a strictly increasing in \([0, h]\) and defines a symmetric equilibrium. Then \( a(v) \) and \( p(v) \) are given by (1.6). Since the expected payment \( p(v) \) in a first price auction is \( F(v)^{n-1} \beta(v) \), it follows that

\[ \beta(v) = E \left[ \max_{i \leq n-1} V_i \mid \max_{i \leq n-1} V_i \leq v \right] = \int_0^v 1 - \frac{F(w)F(v)}{F(v)}^{n-1} dw. \]

(b) Suppose that \( \beta \) is defined by the preceding equation. We verify that this formula actually defines an equilibrium. Since \( F \) is strictly increasing, by the preceding equation, \( \beta \) is also strictly increasing. Therefore \( a(v) = F(v)^{n-1} \) and conditions (i) and (iii) of the theorem hold. Hence (3.8) holds. Finally, bidding more than \( \beta(h) \) is dominated by bidding \( \beta(h) \). Hence this bidding strategy is in fact an equilibrium.

**Examples:**

With \( n \) bidders each with a value that is \( U[0, 1] \), we obtain that \( \beta(v) = \frac{n-1}{n} v \).

With 2 bidders, each with a value that is exponential with parameter 1 (that is \( F(v) = 1 - e^{-v} \)), we obtain

\[ \beta(v) = \int_0^v 1 - \left( \frac{1 - e^{-w}}{1 - e^{-v}} \right) dw = 1 - \frac{ve^{-v}}{1 - e^{-v}}. \]

This function is always below 1. Thus, even a bidder with a value of 100 would bid below 1!
**Auctions**

**All-pay auction:**
This auction allocates to the player that bids the highest, but charges every player their bid. For example, architects competing for a construction project submit design proposals. While only one architect wins the contest, all competitors expend the effort to prepare their proposals. Thus, participants need to make the strategic decision as to how much effort to put in.

Using Corollary 1.4.4 and arguments similar to those just used for the first price auction, it follows that the only symmetric increasing equilibrium is given by

$$\beta(v) = F(v)^{n-1} \mathbb{E} \left[ \max_{i \leq n-1} V_i \mid \max_{i \leq n-1} V_i \leq v \right].$$

For example, if $F$ is uniform on $[0,1]$, then $\beta(v) = \frac{n-1}{n} v^n$.

**War-of-attrition auction:**

This auction allocates to the player that bids the highest, charges the highest bidder the second-highest bid, and charges all other players their bid. For example, animals fighting over territory expend energy. A winner emerges when the fighting ends, and each animal has expended energy up to the point at which he dropped out or, in the case of the winner, until he was the last one left.

Again, let $\beta$ be a symmetric strictly increasing equilibrium strategy. The
expected payment \( p(v) \) of an agent in a war-of-attrition auction in which all bidders use \( \beta \) is

\[
p(v) = F(v)^{n-1}E \left[ \max_{i \leq n-1} \beta(V_i) \mid \max_{i \leq n-1} V_i \leq v \right] + (1 - F(v)^{n-1})\beta(v).
\]

Equating this with \( p(v) \) from (1.6), we have

\[
\int_0^v (F(v)^{n-1} - F(w)^{n-1})dw = \int_0^v \beta(w)(n-1)F(w)^{n-2}f(w)dw + (1-F(v)^{n-1})\beta(v).
\]

Differentiating both sides with respect to \( v \), cancelling common terms and simplifying yields

\[
\beta'(v) = \frac{(n-1)vF(v)^{n-2}f(v)}{1-F(v)^{n-1}},
\]

and hence

\[
\beta(v) = \int_0^v \frac{(n-1)wF(w)^{n-2}f(w)}{1-F(w)^{n-1}}dw.
\]

For two players with \( F \) uniform on \([0,1]\) this yields

\[
\beta(v) = \int_0^v \frac{w}{1-w}dw = -v - \log(1-v).
\]

**Remark.** There is an important subtlety related to the equilibrium just derived for the war of attrition. It is only valid for more than two players in a setting in which bids are committed to up-front, rather than in the more natural setting where bids (the decision as to how long to stay in) can be adjusted over the course of the auction. See Chapter 3 for a discussion.

**IPL cricket auction:** In the India Premier League, cricket franchises can acquire a player by participating in the annual auction. The rules of the auction are as follows. An English auction is run until either only one bidder remains or the price reaches \( \$m \) (for example \( \$m \) could be \( \$750,000 \)). In the latter case, a sealed bid first-price auction is run on the remaining bidders.

**Exercise 1.4.5.** Use the revenue equivalence theorem to determine equilibrium bidding strategies in the IPL cricket auction.
1.4.2 Designing auctions to maximize profit

We have seen that in equilibrium, with players whose values for the item being sold are drawn independently from the same distribution, the expected seller profit is the same for any auction that always allocates to the highest bidder. How should the seller choose which auction to run? As we have discussed, an appealing feature of the second-price auction is that it induces truthful bidding. On the other hand, the auctioneer’s revenue might be lower than his own value for the item. A notorious example was the 1990 New Zealand sale of spectrum licenses in which a 2nd price auction was used, the winning bidder bid $100,000, but paid only $6! A natural remedy for situations like this is for the auctioneer to impose a reserve price.

Definition 1.4.6. The Vickrey auction with a reserve price $r$ is a sealed-bid auction in which the item is not allocated if all bids are below $r$. Otherwise, the item is allocated to the highest bidder, who pays the maximum of the second highest bid and $r$.

A virtually identical argument to that of Theorem 1.3.3 shows that the Vickrey auction with a reserve price is truthful. Alternatively, the truthfulness follows by imagining that there is an extra bidder whose value/bid is the reserve price.

Perhaps surprisingly, an auctioneer may want to impose a reserve price even if his own value for the item is zero. For example, we have seen that for
two bidders with values independent and U[0,1], all auctions that allocate
the highest bidder have an expected auctioneer revenue of 1/3.

Now consider the expected revenue if, instead, the auctioneer uses the
Vickrey auction with a reserve of r. Relative to the case of no reserve price,
the auctioneer loses an expected profit of r/3 if both bidders have values
below r, for a total expected loss of r^3/3. On the other hand, he gains if one
bidder is above r and one below. This occurs with probability 2r(1−r), and
the gain is r minus the expected value of the bidder below r, i.e. r − r/2.
Altogether, the expected revenue is
\[
\frac{1}{3} - \frac{r^3}{3} + 2r(1-r)\frac{r}{2} = \frac{1}{3} + r^2 - \frac{4}{3}r^3.
\]
Differentiating shows that this is maximized at r = 1/2 yielding an expected
auctioneer revenue of 5/12. (This is not a violation of the revenue equiva-
rence theorem, because imposition of a reserve price changes the allocation
rule.)

Remarkably, this simple auction optimizes the auctioneer’s expected rev-
enue over all possible auctions. It is a special case of Myerson’s optimal
auction, a broadly applicable technique for maximizing auctioneer revenue
when agents values are drawn from known prior distributions. We will de-
velop the theory of optimal auctions in Chapter 3. For now, we will just
explore some examples.

**Example 1.4.7. (Single bidder, one item:)** Consider a seller with a
single item to sell to a single buyer whose private value is publicly known
to be drawn from distribution F. Suppose the seller plans to make a take-
it-or-leave it offer to the buyer. What price should the seller set in order
to maximize her profit? If she sets a price of w, the buyer will accept the
offer if his value for the item is at least w, i.e. with probability 1 − F(w).
Thus, the seller should choose w to maximize her expected revenue \( R(w) = w(1 − F(w)). \)

**Exercise 1.4.8.** Use Theorem 1.4.2 to show that the selling procedure de-
scribed in Example 1.4.9 is the optimal single bidder deterministic auction.

**Example 1.4.9. (Single bidder, two items:)** Now suppose the seller
has two items, and there is a buyer whose private values \((v_1, v_2)\) for the two
items are known to be independent samples from distribution F. Suppose
further that the buyers value for getting both items is \( v_1 + v_2 \). Since the
values for the two items are independent, one might think that the seller
should sell each of them separately in the optimal way, resulting in twice
the expected revenue from a single item. This is not necessarily the case. Consider the following examples:

(i) Suppose that each $v_i$ is equally likely to be 1 or 2. Then the optimal revenue the seller can get separately from each item is 1: If he sells at price 1, the buyer will always buy. If he sells at price 2, the buyer will buy with probability $1/2$. Thus, selling separately yields a total expected seller profit of 2. However, if the seller offers the buyer the bundle of both items at a price of 3, the probability the buyer has $v_1 + v_2 \geq 3$ is $3/4$, and so the expected revenue is $3 \cdot 3/4 = 2.25$, more than the optimal profit from selling the items separately.

(ii) If $v_i$ is equally likely to be 0 or 1, then selling the bundle can yield expected revenue of at most $3/4$, which is less than twice the expected revenue from each item separately (which is $1/2$).

(iii) When $v_i$ is equally likely to be 0, 1 or 2, the unique optimal auction offers the buyer the choice between any single item at price 2, or the bundle of both items at 3. This yields expected revenue $13/9$, which is larger than the revenue of $4/3$ from selling separately, or from selling as a bundle.

(iv) The optimal mechanism may not even be deterministic! For example, suppose that $v_i$ is equally likely to be 1 or 2, and $v_2$ is equally likely to be 1 or 3. In this case, the optimal mechanism offers the buyer two options. He can either take the bundle of both items at price 4, or he can pay 2.5 to participate in a lottery that with probability $1/2$ gives him both items and with probability $1/2$ gives just the first item.

For details on why the above statements are true, see exercise 3.10.

**Example 1.4.10. Profit when values are correlated:** Earlier, we considered the optimal auction when there are two bidders whose values are uniform on [0,1]. In this case, the expected value of the highest bidder is $2/3$, and yet the auction which maximizes the sellers revenue obtains an expected profit of only $5/12$. This loss is the “price” the auctioneer has to pay because the values of the bidders are private. However, when bidders values are correlated, the auctioneer can take advantage of the correlation to obtain essentially the expected maximum value.
1.4 Definitions

For example, suppose that there are two agents, and they each either have value 10 or 100, with the following joint distribution.

<table>
<thead>
<tr>
<th></th>
<th>$v_2 = 10$</th>
<th>$v_2 = 100$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1 = 10$</td>
<td>1/3</td>
<td>1/6</td>
</tr>
<tr>
<td>$v_1 = 100$</td>
<td>1/6</td>
<td>1/3</td>
</tr>
</tbody>
</table>

If the auctioneer runs a second price auction, the expected utility of an agent is 0 when her value is 10, and $\frac{1}{3}(90) = 30$ when her value is 100.

If the auctioneer wants to extract the full surplus, he can take advantage of the correlation to add some additional charges that will reduce each agent’s expected utility to 0. Specifically, suppose he charges an agent $p(10)$ when the other agent’s value is 10 and $p(100)$ when the other agent’s value is 100. This can be achieved by choosing $p(10)$ and $p(100)$ as the solution to the following system of equations.

\[
\begin{pmatrix}
\mathbb{P}[v_2 = 10|v_1 = 10] & \mathbb{P}[v_2 = 100|v_1 = 10] \\
\mathbb{P}[v_2 = 10|v_1 = 100] & \mathbb{P}[v_2 = 100|v_1 = 100]
\end{pmatrix} \cdot \begin{pmatrix}
p(10) \\
p(100)
\end{pmatrix} = \begin{pmatrix}
0 \\
30
\end{pmatrix}
\]

Since this matrix of conditional probabilities is full rank, this system has a solution: $p(10) = -30$ and $p(100) = 60$.

Now suppose the auctioneer runs a second price auction with the following additional rules:

- Pay a bidder $30 if the other bidder’s value is 10.
- Charge a bidder $60 if the other bidder’s value is 100.

The discussion above implies that each agent’s expected utility is 0, and hence the auctioneer’s expected profit is $\mathbb{E}[\max(v_1, v_2)]$. In addition, truth-telling is a Bayes-Nash equilibrium, with zero expected utility to each player. (The auctioneer can in fact slightly reduce the payment when the other bidder’s value is 100 to make their expected utility just barely positive.) However, a bidder may be very unhappy with this auction after the fact. For example, if both agents have value 100, they both end up with negative utility ($-60$)!

Example 1.4.11. Purple pricing: The industry which sells tickets to concerts and athletic events does not necessarily operate efficiently or, in some cases, profitably due to complex and unknown demand. This can lead to tickets being sold on secondary markets at exorbitant prices and/or unfilled seats. An auction format, known as purple pricing, has been proposed in
order to mitigate these problems. The auction works as follows: prices start out high and drop until the tickets sell out; however, as shown in Figure 1.7, all buyers pay the price at which the final ticket is sold.

**Exercise 1.4.12.** Use revenue equivalence (Corollary 1.4.3) and the equilibrium of the $k$-unit Vickrey auction to determine the expected price at which the tickets sell for a venue that holds $k$ people, assuming that each person’s value is drawn independently from prior distribution $F$. (You may assume that the population of possible ticket-purchasers has size $n > k$.)

**Example 1.4.13. Google IPO auction:** When Google went public in 2004, they implemented their initial public offering using an auction. When the auction began, each bidder was able to submit a bid indicating the number of shares they want and the price they were willing to pay for them. The final IPO price was determined after the auction closes. It was determined by gathering all the bids and calculating the price at which all the shares available can be sold.

---

Fig. 1.7. This is a picture of the website where one can buy tickets to Northwestern basketball games using purple pricing.
1.5 Notes

There are several excellent texts on auction theory with fairly exhaustive bibliographies, including the books by Klemperer [Kle99], Krishna [Kri09], Menezes and Monteiro [MM05] and Milgrom [Mil04]. These books cover most of the material in this chapter.

Example 1.4.9 is taken from Hart and Nisan [HN12] and Daskalakis et al [DDT12]. Purple pricing (example 1.4.11) was proposed by economists Baliga and Ely. Example 1.4.10 is inspired by [Mye81] and [CM88].

Exercises

1.1
Allocation and pricing mechanisms

We turn next to more general allocation and pricing mechanisms. In these settings, the mechanism designer might be an entity such as the government whose responsibility is to make sure that society as a whole is happy with the outcome of its decisions. In other situations, the goal might be to maximize profit. Note though that even companies often make decisions so as to maximize the happiness of their customers on the theory that long-term customer loyalty may be more important than short-term profit.

2.1 Examples

![Example 2.1](http://www.popularmechanics.com/technology/gadgets/news/4246037)

**Example 2.1.1. Spectrum Auctions:** In a spectrum auction, the government is selling licenses for the use of some band of electromagnetic spectrum in a certain geographic area. The participants in the auction are cell phone companies such who need such licenses to operate. Each company has a value for each combination of licenses. The government wishes to design a procedure for allocating and pricing the licenses that maximizes the cumu-
2.1 Examples

What procedure should be used?

Fig. 2.2. A typical page of search results: some organic, and some sponsored. If you click on a sponsored search result, the associated entity pays the search engine some amount of money. Some of the most expensive keywords relate to mesothelioma, asbestos, annuity and auto donation. For example, in recent years, the cost per click for "mesothelioma settlement" has been over $100. Clicks on sponsored search slots for other keywords can go for as low as a penny.

Example 2.1.2. Sponsored Search Auctions: An individual performing a search, say for the term "mesothelioma", in a search engine receives a page of results containing the links the search engine has deemed relevant to the search, together with sponsored links, i.e. “advertisements”. For this particular term, these links might lead to the web pages of law firms or medical clinics. To have their ads shown in these slots, these companies participate in an auction. How should the search engine design the mechanism for allocating and pricing these ad slots?

Example 2.1.3. The federal government is trying to determine which roads to build to connect a new city C to cities A and B (which already have a road between them). The options are to build a road from A to C or a road from B to C, both roads, or neither. Each road will cost the government 10 million dollars to build. Each city obtains a certain economic/social benefit for each outcome. For example, city A might obtain a 5 million dollar benefit from the creation of a road to city C, but no real benefit from the creation of a road between B and C. City C, on the other hand, currently disconnected
Allocation and pricing mechanisms

Fig. 2.3. Behind the scenes when you do a query for a keyword, like "hotel", in a search engine: At that moment, some of the advertisers who have bid on the keyword participate in an instantaneous auction that determines which sponsored search slot, if any, they are allocated to, and how much they will have to pay the search engine in the event of a user click.

from the others, obtains a significant benefit (9 million) from the creation of each road, but the marginal benefit of adding a second connection is not as great as the benefit creating a first connection. The following table summarizes these values (in millions), and the cost to the government for each option.

<table>
<thead>
<tr>
<th></th>
<th>road A-C</th>
<th>road B-C</th>
<th>both</th>
<th>none</th>
</tr>
</thead>
<tbody>
<tr>
<td>City A</td>
<td>5</td>
<td>0</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>City B</td>
<td>0</td>
<td>5</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>City C</td>
<td>9</td>
<td>9</td>
<td>15</td>
<td>0</td>
</tr>
<tr>
<td>Government</td>
<td>-10</td>
<td>-10</td>
<td>-20</td>
<td>0</td>
</tr>
</tbody>
</table>

The government’s goal might be to choose the option that yields the highest total benefit to society which, for these numbers, is the creation of both roads. However, these numbers are reported to the government by the cities themselves, who may have an incentive to exaggerate their values, so that their preferred option will be selected. Thus, the government would like to employ a mechanism for learning the values and making the decision that provides the correct incentive structure.
2.2 Social Welfare Maximization

Suppose the goal of the designer is to choose the outcome that maximizes social welfare, the cumulative value of the outcome to all participants.

Formally, each player $i$ has a valuation function $v_i : A → \mathbb{R}$, that maps the possible outcomes $A$ to real numbers. The quantity $v_i(a)$ represents the "value" that $i$ assigns to outcome $a ∈ A$, measured in a common currency, such as dollars.

The goal is to design a mechanism $M$ that takes reported valuation functions from the agents (which they may or may not report truthfully) and based on those reports, selects an outcome $a^*$ that maximizes social welfare, $\sum_i v_i(a^*)$, and a set of payments. The payments should be chosen so that it is a dominant strategy for each agent to report his valuation function truthfully. (Truthfulness is a necessity when the goal is to make sure that the correct outcome is selected.)

**Definition 2.2.1.** We say that a mechanism $M$ is **truthful** if, for each player $i$, each valuation function $v_i(\cdot)$ and each possible report $b_{−i}$ of the other players, it is a dominant strategy for player $i$ to report their valuation truthfully. Formally, for all $b_{−i}$, all $i$, $v_i(\cdot)$, and $b_i(\cdot)$

$$u_i[v_i, b_{–i}|v_i] ≥ u_i[b_i, b_{–i}|v_i],$$

where $u_i[b_i, b_{–i}|v_i] = v_i(a(b)) – p_i(b)$.

We use the following mechanism to solve this problem:

![Fig. 2.4. A depiction of the VCG Mechanism. The outcome selected is $a^*$, and the payment of agent $i$ is $p_i(b)$. Note that Theorem 2.2.6 holds for any choice of functions $b_i(b_{–i})$, as long as it is independent of $b_i(\cdot)$. In Definition 2.2.2, we take $h_i(b_{–i}) = \max_a \sum_{j \neq i} b_j(a)$.](image)

**Definition 2.2.2.** The Vickrey-Clarke-Groves (VCG) mechanism, illustrated in Figure 2.4, works as follows: The agents are asked to report their valuation functions. Say they report $b = (b_1(\cdot), \ldots, b_n(\cdot))$ (where $b_i(\cdot)$ may
or may not equal their true valuation function \( v_i(\cdot) \). The outcome selected is

\[
a^* = a(b) = \arg\max_a \sum_j b_j(a),
\]

which maximizes social welfare with respect to the reported valuations. The payment \( p_i(b) \) player \( i \) makes is the loss his presence causes others (with respect to the reported bids), formally:

\[
p_i(b) = \left( \max_a \sum_{j \neq i} b_j(a) \right) - \sum_{j \neq i} b_j(a^*).
\]

The first term is the total reported value the other players would obtain if \( i \) was absent, and the term being subtracted is the total reported value the others obtain when \( i \) is present.

**Exercise 2.2.3.** Check that the Vickrey second price auction and the Vickrey \( k \)-unit auction are both special cases of the VCG mechanism.

**Example 2.2.4.** Consider a search engine selling advertising slots on one of its pages. There are two advertising slots with publicly-known clickthrough rates (probability that an individual viewing the web page will click on the ad) of 1 and 0.5 respectively, and three advertisers whose values per click are 7, 6 and 1 respectively. We assume that the expected value for an advertiser to have his ad shown in a particular slot is his value times the clickthrough rate. Suppose the search engine runs a VCG auction in order to decide which advertiser gets which slot.

Figure 2.5 shows the resulting allocation, payments and advertiser utility.

**Remark.** Often clickthrough rates depend on the advertiser quality and not only on the slot the advertiser’s ad is shown in. See exercise 2.3.

**Example 2.2.5.** Consider the outcome and payments for the VCG mechanism on example 2.1.3, assuming that the cities report truthfully. As the social welfare of each outcome is the sum of the values to each of the participants for that outcome (the final row in the following table), the social welfare maximizing outcome would be to build both roads.
2.2 Social Welfare Maximization

Fig. 2.5. VCG on sponsored search example: An advertiser’s value for a slot is her value times the click through rate of the slot. For example, the blue advertiser’s value for slot 1 is 6, and her value for slot 2 is $6 \cdot 0.5 = 3$. Her expected payment is the value other players would obtain if she wasn’t there ($7 \cdot 1 + 1 \cdot 0.5$), since the green player would get the second slot) minus the value other players get when she is present ($7 \cdot 1$). Her expected payment is the price-per-click (PPC) times the click through rate.

<table>
<thead>
<tr>
<th></th>
<th>road A-C</th>
<th>road B-C</th>
<th>both</th>
<th>none</th>
</tr>
</thead>
<tbody>
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<td>5</td>
<td>5</td>
<td>0</td>
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<td>9</td>
<td>15</td>
<td>0</td>
</tr>
<tr>
<td>Government</td>
<td>-10</td>
<td>-10</td>
<td>-20</td>
<td>0</td>
</tr>
<tr>
<td>Social welfare</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>0</td>
</tr>
</tbody>
</table>

What about the payments using VCG? For city A, the total value attained by others in A’s absence is 4 (road B-C only would be built), whereas with city A, the total value attained by others is 0, and therefore player A’s payment, the harm his presence causes others is 4. By symmetry, B’s payment is the same. For city C, the total value attained by other’s in C’s absence is 0, whereas the total value attained by others in C’s presence is -10, and therefore the difference, and C’s payment is 10. Notice that the total payment is 18, whereas the government spends 20.

**Theorem 2.2.6.** VCG is a truthful mechanism for maximizing social welfare.

**Proof.** Fix the reports $b_{-i}$ of all agents except agent $i$ (that may or may not
be truthful. Suppose that agent $i$’s true valuation function is $v_i(\cdot)$ and he reports $b_i(\cdot)$. This results in outcome

$$a' = \arg\max_a \sum_j b_j(a)$$

and payment

$$p_i(b) = \max_a \sum_{j \neq i} b_j(a) - \sum_{j \neq i} b_j(a') = - \sum_{j \neq i} b_j(a') + C.$$ 

where $C := \max_a \sum_{j \neq i} b_j(a)$ is a constant that agent $i$’s report has no influence on. Thus, agent $i$’s utility is

$$u_i(b|v_i) = v_i(a') - p_i(b) = v_i(a') + \sum_{j \neq i} b_j(a') - C.$$ 

The only effect his bid has on his utility is in the choice of $a'$. By reporting $b_i(\cdot) = v_i(\cdot)$, he ensures that $a'$ is chosen to maximize his utility, i.e.,

$$u_i(v_i, b_{-i}|v_i) = \max_a \left( v_i(a) + \sum_{j \neq i} b_j(a) \right) - C.$$ 

Hence, for every $b_{-i}$ and $v_i$,

$$u_i(b|v_i) \leq u_i(v_i, b_{-i}|v_i).$$

Remark. Social welfare maximization is one of the few objectives for which it is known how to design truthful mechanisms that work in such extremely general settings, and don’t require the designer to have any prior information about the agents.

The following examples illustrate a few of the deficiencies of the VCG mechanism.

**Example 2.2.7. Spectrum Auctions:** In a spectrum auction, the government is selling licenses for the use of some band of electromagnetic spectrum in a certain geographic area. The participants in the auction are cell phone companies who need such licenses to operate. Company A has recently entered the market and needs two licenses in order to operate efficiently enough to compete with the established companies. Thus, A has no value for a single license, but values a pair of licenses at 1 billion dollars. Companies B
and C are already well established and only seek to expand capacity. Thus, each one needs just one license and values that license at 1 billion.

Suppose the government runs a VCG auction to sell 2 licenses. If only companies A and B compete in the auction, the government revenue is 1 billion dollars (either A or B can win). However, if A, B and C all compete, then companies B and C will each receive a license, but pay nothing. Thus, VCG revenue is not necessarily monotonic in participation or bidder values.

A variant on this same setting illustrates another problem with the VCG mechanism and that is susceptibility to collusion. Suppose that company A’s preferences are as above, and companies B and C still only need one license each, but now they only value a license at 25 million. In this case, if companies B and C bid honestly, they lose the auction. However, if they collude and each bid 1 billion, they both win at a price of 0.

**Example 2.2.8. Concert tickets:** Consider a singer planning a concert in a 10,000 seat arena to raise money for her favorite charity. In this scenario, the singer is the auctioneer and has a set of 10,000 identical items (tickets) she wishes to sell. Assume that there are 100,000 potential buyers, each interested in exactly one ticket, and each with his or her own private value for the ticket. If the singer runs the VCG mechanism, she will maximize social welfare. In this case, it is easy to check that this will result in selling the tickets to the top 10,000 bidders at the price bid by the 10,001st bidder. But this price could be tiny, resulting in a small donation to charity, for example, if the prices bidders are willing to pay drop off precipitously.

If she has good information about the distribution of bidder values, then she can run the optimal (profit-maximizing) auction or the prior-free auction that we develop in Chapter 3. An alternative is for her to set a target $T$, say a million dollars, and try to design an auction that raises $T$ dollars. One approach she can take is run an ascending auction in which the price $p$ starts at 0, and continuously rises until it first reaches a price $p$ at which there are $T/p$ buyers willing to pay $p$. The auction then ends with a ticket sale to each of those buyers at price $p$. This auction is truthful, and that it will successfully raise $T$ dollars if there is a set of buyers that are willing to equally share the burden of paying the $T$ dollars.

### 2.3 Generalized Second Price Mechanism

As discussed in example 2.1.2, search engines run millions of auctions every second. It is unclear exactly what goal the search engines should be or, indeed, are optimizing; there are many possibilities, including social wel-
Allocation and pricing mechanisms

Fig. 2.6. The left side of the figure shows an execution of GSP when advertisers bid truthfully. Bidding truthfully is not an equilibrium though. For example, if bidder 1 reduces his bid to 5, as shown on the right side of the figure, then his utility is higher.

fare, profit, simplicity and usability from the perspective of advertisers, user happiness, etc.

The most common auction in use is called the Generalized Second Price (GSP) mechanism, named so because it generalizes the Vickrey second price auction. It is used each time a user searches for a particular keyword, say $K$, and is used to allocate advertisers that have bid on that keyword to advertisement slots.

The GSP auction works (roughly) as follows:

- Each advertiser interested in bidding on keyword $K$ submits a bid $b_i$, indicating the price they are willing to pay to have their ad shown alongside organic search results on a search for keyword $K$.
- Associated with each advertiser/keyword pair is a publicly known click through rate, say $c_i$ for advertiser $i$.
- Bids are ranked according to their bid $b_i$, and allocated to slots in this order. (The higher the slot is on the page of search results the more desirable it is considered, since users are more likely to click on them.)
- Each winning advertiser pays the minimum bid needed to win the allocated slot. For example, if the advertisers are indexed according to the slot they are assigned to, with advertiser 1 assigned to the highest slot (slot 1), then advertiser $i$’s payment $p_i$ is

$$p_i = b_{i+1}.$$  

(Wlog, there are more advertisers than slots. If not, add dummy advertisers with bids of value 0.)

An example execution of GSP is shown on the left side of Figure 2.6.
Fig. 2.7. The left side of the figure illustrates the allocation and payments using VCG. The right side of the figure illustrates the allocation and payments in GSP when bidders bid so as to obtain the same allocation and payments. Bidding this way is a Nash equilibrium for GSP.

In the special case where there is only one slot (and all advertisers have the same click through rates), GSP is the same as a second-price auction, and hence the name. However, when there is more than one slot, GSP is no longer a truthful auction as Figure 2.6 shows.

However, we observe the following:

**Lemma 2.3.1.** In a full information version of GSP, that is, when all competing advertisers in an auction have publicly known values $v_i$, assumed to be sorted so that $v_1 \geq v_2 \geq \ldots \geq v_n$, it is a Nash equilibrium if bidder 1 bids above $p_1$, and the remaining bidders $i \geq 2$ bid $b_i = p_{i-1}$, where $p_i$ is the VCG payment of bidder $i$ given in 2.2.

*Proof.* This follows from the fact that the GSP outcome (allocation and payments) when bidders bid this way is precisely that of VCG and from the result of exercise 2.6. \qed
Notes

The VCG mechanism is named for Vickrey [Vic61], Clarke [Cla71] and Groves [Gro79]. An excellent introduction to mechanism design with significant emphasis on allocation and pricing mechanisms and especially VCG can be found in Chapter 9 of [Nis07]. For analysis of the GSP auction, see [Var07] and [EOS05]. Chapter 28 in [Nis07] gives an overview of sponsored search auctions.

Exercises

2.1 Consider a search engine selling advertising slots on one of its pages. There are three advertising slots with publicly-known clickthrough rates (probability that an individual viewing the web page will click on the ad) of 0.08, 0.03 and 0.01 respectively, and four advertisers whose values per click are 10, 8, 2 and 1 respectively. Assume that the expected value for an advertiser to have his ad shown in a particular slot is his value times the clickthrough rate. What is the allocation and payments if the search engine runs VCG? How about GSP?

2.2 Show that for an ad-auction with \( k \) slots, and agents with values \( v_i \) and click through rates \( c_i \) (ordered in decreasing order of \( v_i \cdot c_i \)), the payment of the \( i \)-th advertiser, for \( 1 \leq i \leq k \) is: ??

2.3 VCG/GSP when bidders have quality scores as well.

2.4 bridge example VCG not budget balanced

2.5 VCG with weights

2.6 VCG outcome envy-free.

2.7 Other equilibria of GSP.
Auctions: Some more advanced topics

3.1 Characterization of Equilibrium

We begin with the characterization of Bayes-Nash equilibrium discussed in Chapter 1.

Theorem 3.1.1 (Characterization of Equilibria). Let $A$ be an auction for selling a single item, where bidder $i$’s value $V_i$ is drawn independently from $F_i$. We assume that $F_i$ is strictly increasing and continuous on $[0,h_i]$, with $F(0) = 0$ and $F(h_i) = 1$. ($h_i$ can be $\infty$.)

(a) If $(\beta_1, \ldots, \beta_n)$ is a Bayes-Nash equilibrium, then for each agent $i$:

(i) The probability of allocation $a_i(v_i)$ is monotone increasing in $v_i$.

(ii) The utility $u_i(v_i)$ is a convex function of $v_i$, with

$$u_i(v_i) = \int_0^{v_i} a_i(z)dz.$$ 

(iii) The expected payment is determined by the allocation probabilities:

$$p_i(v_i) = v a_i(v) - \int_0^{v} a_i(z)dz = \int_0^{v} za_{i}'(z)dz.$$ 

(b) Conversely, if $(\beta_1, \ldots, \beta_n)$ is a set of bidder strategies for which (i) and (iii) hold (or alternatively (i) and (ii)), then for all bidders $i$, and values $v$ and $w$

$$u_i(v|v) \geq u_i(w|v).$$  

Proof. (a): Suppose that $(\beta_1, \ldots, \beta_n)$ is a Bayes-Nash equilibrium. In what follows, all quantities refer to bidder $i$, so for notational simplicity we usually
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drop the subscript $i$. If bidder $i$ has value $v$, then he has higher utility bidding
$\beta_i(v)$ than $\beta_i(w)$, i.e.,
$$u(v|v) = va(v) - p(v) \geq va(w) - p(w) = u(w|v).$$

Reversing the roles of $v$ and $w$, we have
$$wa(w) - p(w) \geq wa(v) - p(v)$$

Adding these two inequalities, we obtain that for all $v$ and $w$
$$(v - w)(a(v) - a(w)) \geq 0.$$ Thus, if $v \geq w$, then $a(v) \geq a(w)$, and we conclude that $a_i(v_i)$ is monotone
nondecreasing.

Also, since for agent $i$,
$$u(v) := u(v|v) = \sup_w u(w|v) = \sup_w \{va(w) - p(w)\},$$
by Appendix 1: (i) and (ii) it follows that $u(v)$ is a convex function of $v$.
In addition, we observe that for every $v$ and $w$
$$u(v) \geq u(w|v) = va(w) - p(w) = u(w) + (v - w)a(w).$$

Thus, letting $v \downarrow w$ gives right derivative $u'_+(w) \geq a(w)$ and letting $v \uparrow w$
gives left derivative $u'_-(w) \leq a(w)$.

We conclude that where $u(v)$ is differentiable,
$$u'(v) = a(v).$$

Finally, since a convex function is the integral of its derivative (see Appendix 1: (vi), (vii) and (xi)), we have
$$u(v) - u(0) = \int_0^v a(z)dz.$$ The assumption $p(0) = 0$ gives (ii) Finally, since $u(v) = va(v) - p(v)$, (iii) follows.

(b): For the converse, from condition (iii) it follows that
$$u(v) = \int_0^v a(z)dz$$
whereas
$$u(w|v) = va(w) - p(w) = (v - w)a(w) + \int_0^w a(z)dz,$$
whence, by condition (i)

\[ u(v) \geq u(w|v). \]

\[ \square \]

### 3.1.1 When is truthfulness dominant?

In the setting of i.i.d. bidders, there is a dominant strategy auction (the Vickrey auction) that delivers the same expected revenue to the auctioneer as the Bayes-Nash equilibria in other auctions that allocate to the highest bidder. A dominant strategy equilibrium is more robust since it does not rely on knowledge of value distributions of other players, and bidders will not regret their bids even when all other bids are revealed. The next theorem characterizes auctions where bidding truthfully is a dominant strategy.

**Theorem 3.1.2.** Let \( A \) be an auction for selling a single item. It is a dominant strategy in \( A \) for bidder \( i \) to bid truthfully if and only if, for any bids \( b_{-i} \) of the other bidders:

1. The probability of allocation \( \alpha_i(v_i, b_{-i}) \) is (weakly) increasing in \( v_i \).
   (This probability is over the randomness in the auction.)
2. The expected payment of bidder \( i \) is determined by the allocation probabilities:
   \[
   p_i(v_i, b_{-i}) = v_i \cdot \alpha_i(v_i, b_{-i}) - \int_0^{v_i} \alpha_i(z, b_{-i}) \, dz
   \]

**Exercise 3.1.3.** Prove Theorem 3.1.2 by adapting the proof of Theorem 3.1.1.

Observe that if bidding truthfully is a dominant strategy for an auction \( A \), then \( A \) is BIC.

**Corollary 3.1.4.** Let \( A \) be a deterministic auction. Then it is a dominant strategy for bidder \( i \) to bid truthfully if and only if for each \( b_{-i} \),

1. There is a threshold \( \theta_i(b_{-i}) \) such that the item is allocated to bidder \( i \) if \( v_i > \theta_i(b_{-i}) \) but not if \( v_i < \theta_i(b_{-i}) \).
2. If \( i \) receives the item, then his payment is \( \theta_i(b_{-i}) \), and otherwise is 0.
3.2 The revelation principle

An extremely useful insight that simplifies the design and analysis of auctions is the revelation principle. It says that for every auction with a Bayes-Nash equilibrium, there is another “equivalent” auction in which bidding truthfully is a Bayes-Nash equilibrium.

**Definition 3.2.1.** If bidding truthfully (i.e., $\beta_i(v) = v$ for all $i$) is a Bayes-Nash equilibrium for auction $\mathcal{A}$, then $\mathcal{A}$ is said to be Bayes-Nash incentive compatible (BIC).

**Definition 3.2.2.** Let $\mathcal{A}$ be a single-item auction. The allocation rule of $\mathcal{A}$ is denoted by $a^{\mathcal{A}}[b] = (a_1[b], \ldots, a_n[b])$ where $a_i[b]$ is the probability of allocation to bidder $i$, when the bid vector is $b = (b_1, \ldots, b_n)$. The payment rule of $\mathcal{A}$ is denoted by $p^{\mathcal{A}}[b] = (p_1[b], \ldots, p_n[b])$ where $p_i[b]$ is the expected payment of bidder $i$ when the bid vector is $b$. (The probability is taken over the randomness in the auction itself.)

Consider a first price auction $\mathcal{A}$ in which bidders values are drawn from known prior distribution $F$. A bidder that is not adept at computing his equilibrium bid might hire an agent to do this for him, and submit bids on his behalf. The revelation principle changes this perspective, and considers the bidding agents and auction together as a new more complex auction $\tilde{\mathcal{A}}$, for which bidding truthfully is an equilibrium. The key advantage of this transformation, which works for any auction, not just first-price, is that it enables us to reduce the problem of designing an auction with "good" properties in Bayes-Nash equilibrium to the problem of designing an auction with good properties that is BIC.
3.2 The revelation principle

**Theorem 3.2.3** (The Revelation Principle). Let \( \mathcal{A} \) be an auction with Bayes-Nash equilibrium strategies \( \{\beta_i\}_{i=1}^n \). Then there is another auction \( \tilde{\mathcal{A}} \) which is BIC, and which has the same winner and payments as \( \mathcal{A} \) in equilibrium, i.e. for all \( v \), if \( b = \beta(v) \), then

\[
\alpha^\mathcal{A}[b] = \alpha^{\tilde{\mathcal{A}}}[v] \quad \text{and} \quad \mathbf{p}^\mathcal{A}[b] = \mathbf{p}^{\tilde{\mathcal{A}}}[v].
\]

**Proof.** The auction \( \tilde{\mathcal{A}} \) operates as follows: On each input \( v \), \( \tilde{\mathcal{A}} \) computes \( \beta(v) = (\beta_1(v_1), \ldots, \beta_n(v_n)) \), and then runs \( \mathcal{A} \) on \( \beta(v) \) to compute the output and payments. (See Figure 3.1.) It is straightforward to check that if \( \beta \) is in Bayes-Nash equilibrium for \( \mathcal{A} \), then bidding truthfully is a Bayes-Nash equilibrium for \( \tilde{\mathcal{A}} \), i.e. \( \tilde{\mathcal{A}} \) is BIC.

\[
\square
\]

**Remarks:**

- In real-life auctions, the actions of the bidder often go beyond submitting a single bid e.g., in an English auction, a bidder’s strategy may involve submitting a sequence a bids. The revelation principle can be extended to these more general settings, via essentially the same proof.

- The revelation principle applies to other equilibrium concepts, other than Bayes-Nash equilibrium. For example, the revelation principle for dominant strategies says that if \( \mathcal{A} \) is an auction with dominant strategies \( \{\beta_i\}_{i=1}^n \), then there is another auction \( \tilde{\mathcal{A}} \) for which truth-telling is a dominant strategy which has the same winner and payments as \( \mathcal{A} \). The proof is essentially the same.

### 3.2.1 Myerson’s Optimal Auction

We now consider the design of optimal auctions with \( n \) bidders, where bidder \( i \)'s value is drawn from strictly increasing distribution \( F_i \) on \([0,h]\) with density \( f_i \). By the revelation principle (Theorem 3.2.3), we need consider optimizing only over BIC auctions. Moreover, by Theorem 3.1.1, we only need to select the allocation rule, since it determines the payment rule (and we will fix \( p_i(0) = 0 \) for all \( i \)).

Consider an auction \( \mathcal{A} \) where truthful bidding \( (\beta_i(v) = v \text{ for all } i) \) is a Bayes-Nash equilibrium, and suppose that its allocation rule is \( \alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n \). (Recall that \( \alpha[v] = (\alpha_1[v], \ldots, \alpha_n[v]) \), with \( \alpha_i[v] \) the probability that the item is allocated to bidder \( i \) on bid vector \( v = (v_1, \ldots, v_n) \), and \( a_i(v_i) = \mathbb{E} [\alpha_i(v_i, V_{-i})] \).)
The goal of the auctioneer is to choose $\alpha[\cdot]$ to maximize

$$\mathbb{E} \left[ \sum_i p_i(V_i) \right].$$

Fix an allocation rule $\alpha[\cdot]$ and a specific bidder with value $V$ that was drawn from the density $f(\cdot)$. As usual, let $a(v)$, $u(v)$ and $p(v)$ denote his allocation probability, expected utility and expected payment, respectively, given that $V = v$. Using condition (iii) from Theorem 3.1.1 we have

$$\mathbb{E} \left[ u(V) \right] = \int_0^\infty \int_v^\infty a(w) f(v) dv dw.$$

Reversing the order of integration, we get

$$\mathbb{E} \left[ u(V) \right] = \int_0^\infty a(w) \int_w^\infty f(v) dv dw = \int_0^\infty a(w)(1 - F(w)) dw.$$

Thus, since $u(v) = va(v) - p(v)$, we obtain

$$\mathbb{E} \left[ p(V) \right] = \int_0^\infty va(v) f(v) dv - \int_0^\infty a(w)(1 - F(w)) dw$$

$$= \int_0^\infty a(v) \left[ v - \frac{1 - F(v)}{f(v)} \right] f(v) dv.$$

**Definition 3.2.4.** For agent $i$ with value $v_i$ drawn from distribution $F_i$, the virtual value of agent $i$ is

$$\phi_i(v_i) := v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}.$$

We have proved the following proposition:

**Lemma 3.2.5.** The expected payment of agent $i$ in an auction with allocation rule $\alpha(\cdot)$ is

$$\mathbb{E} \left[ p_i(V_i) \right] = \mathbb{E} \left[ a_i(V_i) \phi_i(V_i) \right].$$

Summing over all bidders, this means that the expected auctioneer profit is the expected virtual value of the winning bidder. Note, however, that the auctioneer directly controls $\alpha(v)$ rather than $a_i(v_i) = \mathbb{E} \left[ \alpha(v_i, V_{-i}) \right]$. 
Expressing the expected profit in terms of $\alpha(\cdot)$, we obtain:

\[
\mathbb{E} \left[ \sum_i p_i(V_i) \right] = \mathbb{E} \left[ \sum_i a_i(V_i) \phi_i(V_i) \right] \\
= \int_0^\infty \cdots \int_0^\infty \left[ \sum_i \alpha_i(v) \phi_i(v_i) \right] f_1(v_1) \cdots f_n(v_n) \, dv_1 \cdots dv_n.
\]

(3.2)

(3.3)

The auctioneer’s goal is to choose $\alpha(\cdot)$ to maximize this expression. Since we are designing a single-item auction, the key constraint on $\alpha(\cdot)$ is that

\[
\sum_i \alpha_i(v) \leq 1.
\]

Thus, if on bid vector $v$ the item is allocated, the contribution to (3.3) will be maximized by allocating to a bidder $i^*$ with maximum $\phi_i(v_i)$. However, we only want to do this if $\phi_i(v_i^*) \geq 0$. Summarizing, to maximize (3.3), on each bid vector $v$, allocate to a bidder with the highest\[†\] virtual value $\phi_i(v_i)$, if this virtual value is positive. Otherwise, do not allocate the item.

One crucial issue remains: Are the resulting allocation probabilities $a_i(v_i)$ increasing? Unfortunately, not always, and hence the proposed auction is not always BIC. Nevertheless, in many cases, the required monotonicity does hold: whenever the virtual valuations $\phi_i(v_i)$ are increasing in $v_i$ for all $i$. In this case, for each $i$ and every $b_{-i}$, the allocation function $\alpha_i(v_i, b_{-i})$ is increasing in $v_i$, and hence, by choosing payments according to Theorem 3.1.2(ii), truthfulness is a dominant strategy in the resulting auction.

Exercise 3.2.6. (i) Show that the uniform, Gaussian, exponential and the equal-revenue distribution ($F(x) = 1 - 1/x$ for $x \geq 1$) have weakly increasing virtual valuations.

(ii) Show that the following bimodal distribution does not have increasing virtual valuations: Draw a random variable that is $U[1, 1.75]$ with probability $3/4$ and a random variable that is $U[4, 5]$ with probability $1/4$.

Definition 3.2.7. The Myerson auction for distributions with strictly\[‡\] increasing virtual valuations is defined by the following steps:

(i) Solicit a bid vector $b$ from the agents.

(ii) Allocate the item to the bidder with the largest virtual value $\phi_i(b_i)$, if positive, and otherwise, do not allocate.

† Break ties according to value.

‡ We discuss the weakly increasing case below.
(iii) Charge the winning bidder $i$, if any, her threshold bid, the minimum value she could bid and still win, i.e.,

$$\phi_i^{-1}\left(\max(0, \{\phi_j(b_j)\}_{j \neq i})\right).$$

Specializing to the i.i.d. case, we obtain:

**Observation 3.2.8.** The Myerson auction for i.i.d. bidders with increasing virtual valuations is the Vickrey auction with a reserve price of $\phi^{-1}(0)$.

The discussion above proves the following:

**Theorem 3.2.9.** The Myerson auction is optimal, i.e., it maximizes the expected auctioneer revenue in Bayes-Nash equilibrium when bidders values are drawn from independent distributions with increasing virtual valuations.

**Example 3.2.10.** Consider $n$ bidders, each with value known to be drawn from an exponential distribution with parameter $\lambda$. For this distribution

$$\phi(v) = v - \frac{1 - F(v)}{f(v)} = v - \frac{e^{-\lambda v}}{\lambda e^{-\lambda v}} = v - \frac{1}{\lambda},$$

The resulting optimal auction is Vickrey with a reserve price of $\lambda^{-1}$.

**Example 3.2.11.** Consider a 2-bidder auction, where bidder 1’s value is drawn from an exponential distribution with parameter 1, and bidder 2’s value is drawn independently from a uniform distribution $U[0, 1]$. Then

$$\phi_1(v_1) = v_1 - 1 \quad \text{and} \quad \phi_2(v_2) = v_2 - \frac{1 - v_2}{1} = 2v_2 - 1.$$

Thus, bidder 1 wins when $\phi_1(v_1) \geq \max(0, \phi_2(v_2))$, i.e., when $v_1 \geq \max(1, 2v_2)$, whereas bidder 2 wins when $\phi_2(v_2) > \max(0, \phi_1(v_1))$ i.e., when $v_2 \geq \max(1/2, v_1/2)$.

For example, on input $(v_1, v_2) = (1.5, 0.8)$, we have $(\phi_1(v_1), \phi_2(v_2)) = (0.5, 0.6)$. Thus, bidder 2 wins and pays $\phi_2^{-1}(\phi_1(1.5)) = 0.75$. This example shows that in the optimal auction with non-i.i.d. bidders, the highest bidder may not win!

**Exercise 3.2.12.** Show that if the auctioneer has a value of $C$ for the item, i.e., his profit in a single item auction is the payment he receives minus $C$, then with $n$ i.i.d. bidders (with strictly increasing virtual valuation functions), the auction which maximizes his expected profit is Vickrey with a reserve price of $\phi^{-1}(C)$.

**Remark.** In the case where virtual valuations are weakly increasing, there may be a tie in step (ii) of the Myerson auction [3.2.7].

For a BIC auction, it is crucial to use a tie-breaking rule that retains the
monotonicity of the allocation probabilities $a_i(\cdot)$. Three natural tie-breaking rules are

- break ties by value;
- break ties according to a fixed ranking over the bidders, and
- break ties uniformly at random (equivalently, assign a random ranking to the bidders).

In all cases, the payment the winner pays is still the threshold bid, the minimum value for the winner to obtain the item.

**Exercise 3.2.13.** Determine the explicit payment rule for the three tie-breaking rules just discussed.

**Solution:** Suppose that

$$\varphi = \max_{i \geq 2} \phi_i(b_i)$$

is attained $k$ times by bidders $i \geq 2$. Let

$$[v_-(\varphi), v_+(\varphi)] = \{b : \phi_1(b) = \varphi\}, \quad \text{and} \quad b_* = \max\{b_i : \phi_i(b_i) = \varphi, i \geq 2\}.$$

- Tie-breaking by bid:
  - If $\phi_1(b_1) = \varphi$ and $b_1$ is largest among those with virtual valuation $\varphi$, then bidder 1 wins and pays $\max\{b_*, v_-(\varphi)\}$.
  - If $\phi_1(b_1) > \varphi$, then he wins and pays $\max\{\min\{b_*v_+(\varphi)\}, v_-(\varphi)\}$.

- Tie-breaking according to a fixed ranking of bidders: If $\phi_1(b_1) = \varphi$ and bidder 1 wins (has the highest rank), then his payment is $v_-(\varphi)$. If $\phi_1(b_1) > \varphi$, then his payment is $v_-(\varphi)$ if he has the highest rank, and $v_+(\varphi)$ otherwise.

- Random tie-breaking:
  - If $\phi_1(b_1) = \varphi$, then bidder 1 wins with probability $\frac{1}{k+1}$, and if bidder 1 wins, he is charged $v_-(\varphi)$.
  - If $\phi_1(b_1) > \varphi$, then bidder 1 wins, and he is charged

$$\frac{1}{k+1}v_-(\varphi) + \frac{k}{k+1}v_+(\varphi),$$

because in $\frac{1}{k+1}$ of the permutations he will be ranked above the other $k$ bidders with virtual value $\varphi$. 
3.2.2 Optimal Mechanism

We now derive the general version of Myerson’s optimal mechanism, that does not require that virtual valuations be increasing. We begin with the formula for payment that we derived earlier (Theorem 3.1.1) and make a change of variable to quantile space (i.e., \( q = F(v) \)). To this end, define \( v(q) = F^{-1}(q) \), the payment function \( p(q) = p(v(q)) \) and allocation function \( a(q) = a(v(q)) \) in quantile space. Given any \( v_0 \) and \( q_0 = F(v_0) \), we have

\[
\hat{p}(q_0) = p(v_0) = \int_0^{v_0} a'(v)dv = \int_0^{v_0} \hat{a}'(F(v))vf(v)\,dv = \int_{q_0}^{1} \hat{a}'(q)v(q)\,dq,
\]

since \( q = F(v) \) implies that \( dq = f(v)dv \).

From this formula, we derive the expected revenue from this bidder. Let \( Q \) be the random variable representing this bidder’s draw from the distribution in quantile space, i.e., \( Q = F(V) \). Then

\[
E[\hat{p}(Q)] = \int_0^1 \int_{q_0}^1 \hat{a}'(q)v(q)\,dq\,dq_0.
\]

Reversing the order of integration, we get

\[
E[\hat{p}(Q)] = \int_0^1 \hat{a}'(q)\int_q^1 v(q)\,dq_0\,dq = \int_0^1 \hat{a}'(q)(1-q)v(q)\,dq = \int_0^1 \hat{a}'(q)R(q)\,dq,
\]

where \( R(q) = (1-q)v(q) \) is called the revenue curve. It represents the expected revenue to a seller from offering a price of \( v(q) \) to a buyer whose value \( V \) is drawn from \( F \). Integrating by parts, we obtain

\[
E[p(Q)] = -\int_0^1 \hat{a}(q)R'(q)dq = E\left[-\hat{a}(Q)R'(Q)\right].
\]

Summarizing:

**Lemma 3.2.14.** Consider a bidder with value \( V \) drawn from distribution \( F \), with \( Q = F(V) \). Then his expected payment in a BIC auction is

\[
E[p(Q)] = E[\hat{a}'(Q)R(Q)] = E[-\hat{a}(Q)R'(Q)]
\]

where \( R(q) = v(q)(1-q) \) is the revenue curve.
3.2 The revelation principle

Next we show that this is a rewriting of Lemma 3.2.5.

**Lemma 3.2.15.** Let $q = F(v)$. Then

$$\phi(v) = v - \frac{1 - F(v)}{f(v)} = -R'(q).$$

*Proof.*

$$R'(q) = \frac{d}{dq} (v(q)(1 - q)) = -v + (1 - q) \frac{dv(q)}{dq} = -v + \frac{1 - F(v)}{f(v)}.$$

\[ \square \]

As we discussed in section 3.2.1, allocating to the bidder with the highest virtual value (or equivalently, the largest $-R'(q)$) yields the optimal auction, provided that virtual valuations are increasing.

**Observation 3.2.16.** Let $R(q) = (1 - q)v(q)$ be the revenue curve with $q = F(v)$. Then $\phi(v) = -R'(q)$ is (weakly) increasing if and only if $R(q)$ is concave.

To derive an optimal mechanism for the case where $R(q)$ is not concave consider the concave envelope $\overline{R}(q)$ of $R(q)$, that is, the infimum over concave functions $g(q)$ such that $g(q) \geq R(q)$ for all $q \in [0, 1]$. Passing from $R(\cdot)$ to $\overline{R}(\cdot)$ is called ironing. As we will see below $\overline{R}(\cdot)$ can also be interpreted as a revenue curve when randomization is allowed.

**Definition 3.2.17.** The ironed virtual value of bidder $i$ with value $v(q_i)$ is

$$\bar{\phi}_i(v) = -\overline{R}_i(q_i).$$

We will replace virtual values with ironed virtual values to obtain an optimal auction even when virtual valuations are not increasing.

**Definition 3.2.18.** The Myerson auction with ironing:

(i) Solicit a bid vector $b$ from the bidders.

(ii) Allocate the item to the bidder with the largest value of $\bar{\phi}_i(b_i)$, if positive, and otherwise, do not allocate. (In the event of ties, allocate according to a fixed ranking of the bidders, or uniformly at random among those with largest ironed virtual values).

(iii) Charge the winning bidder $i$, if any, her threshold bid, the minimum value she could bid and still win.†

† In the case of random tie-breaking, payments are determined as in Exercise 3.2.16 with $\bar{\phi}(\cdot)$ replacing $\phi(\cdot)$. 
Theorem 3.2.19. The Myerson Auction described above is optimal.

Proof. The expected profit from a BIC auction is

\[ E \left[ \sum_i \hat{\mu}(Q_i) \right] = E \left[ \sum_i \hat{\alpha}_i(Q_i)(-R'_i(Q)) \right] = E \left[ - \sum_i \hat{\alpha}'_i(Q_i)R_i(Q_i) \right] \]

where the second equality above is from Lemma 3.2.14

\[ = E \left[ \sum_i \hat{\alpha}_i(Q_i)(-\overline{R}'_i(Q_i)) \right] + E \left[ \sum_i \hat{\alpha}'_i(Q_i) \left[ R_i(Q_i) - \overline{R}_i(Q_i) \right] \right] \]

(add and subtract \(-E \left[ \sum_i \hat{\alpha}_i(Q_i)\overline{R}'_i(Q_i) \right] = E \left[ \sum_i \hat{\alpha}'_i(Q_i)\overline{R}_i(Q_i) \right] \)).

Consider choosing a BIC allocation rule \(\alpha(\cdot)\) to maximize the first term:

\[ E \left[ \sum_i \hat{\alpha}_i(Q_1, \ldots, Q_n)(-\overline{R}_i)'(Q_i) \right] \]

This is optimized pointwise by allocating to the bidder with the largest \(-\overline{R}'_i(q_i)\), if positive. Moreover, because \(\overline{R}(\cdot)\) is concave, this is an increasing allocation rule and hence, by Theorem 3.1.2 yields a dominant strategy auction. Notice also that \(-\overline{R}'_i(\cdot)\) is constant in each interval of non-concavity of \(R(\cdot)\), and hence in each such interval \(\hat{\alpha}_i(q)\) is constant and thus \(\hat{\alpha}'_i(q) = 0\).

Consider now the second term: In any BIC auction, \(\hat{\alpha}_i(\cdot)\) must be increasing and hence \(\hat{\alpha}'_i(q) \geq 0\) for all \(q\). But \(R(q) \leq \overline{R}(q)\) for all \(q\) and hence the second term is non-positive. Since the allocation rule that optimizes the first term has \(\hat{\alpha}'(q) = 0\) whenever \(R(q) < \overline{R}(q)\), it ensures that the second term is zero, which is best possible.

\[ \square \]

Remark. Tie-breaking by value would still yield a BIC auction, but it would no longer be optimal, because it wouldn’t have \(\hat{\alpha}'(q) = 0\) for all \(R(q) < \overline{R}(q)\).

3.2.3 The advantages of just one more bidder...

One of the downsides of implementing the optimal auction is that it requires that the auctioneer know the distributions from which agents values are drawn. The following result shows that in lieu of knowing the distribution from which \(n\) i.i.d. bidders are drawn, it suffices to recruit just one more bidder into the auction.
3.3 War of Attrition

**Theorem 3.2.20.** Let $F$ be a distribution for which virtual valuations are increasing. The expected revenue in the optimal auction with $n$ i.i.d. bidders with values drawn from $F$ is upper bounded by the expected revenue in a Vickrey auction with $n+1$ i.i.d. bidders with values drawn from $F$.

**Proof.** First, the optimal (profit-maximizing) auction that is *required* to sell the item is the Vickrey auction. This follows from Lemma 3.2.5 which says that for any auction, the expected profit is equal to the expected virtual value of the winner.

Second, observe that one possible $n+1$-bidder auction that always sells the item consists of, first, running the optimal auction with $n$ bidders, and then, if the item is unsold, giving the item to the $n+1$-st bidder for free. 

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### 3.3 War of Attrition

In Section 1.4.1, we considered the war of attrition in which bidders decided at the beginning of the game when to drop out. A more natural model is for bidders to dynamically decide when to drop out. The last player to drop out is then the winner of the item. With two players this is equivalent to the model discussed in Section 1.4.1 and the equilibrium strategy $\beta(v)$, how long a player with value $v$ waits before dropping out, satisfies:

$$\beta(v) = \int_0^v wh(w)dw,$$

where $h(w) = f(w)/(1 - F(w)) = f(w)/F(1)$ is the hazard rate of the distribution $F$.

We rederive this here without the use of revenue equivalence. To this end, assume that the opponent plays $\beta(\cdot)$. The agent’s utility from playing $\beta(w)$ when his value is $v$ is

$$u(w|v) = vF(w) - F(w)\beta(w) - \int_0^w \beta(z)f(z)dz.$$

Differentiating with respect to $w$, we get

$$\frac{\partial u(w|v)}{\partial w} = vf(w) + f(w)\beta(w) - F(w)\beta'(w) - \beta(w)f(w) = vf(w) - F(w)\beta'(w).$$

For this to be maximized at $w = v$, we must have

$$vf(v) = F(w)\beta'(v),$$

implying (3.4).

With three players, a strategy has two components, $\gamma(v)$ and $\beta(y, v)$. For a player with value $v$, he will drop out at time $\gamma(v)$ if no one else dropped
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out earlier. Otherwise, if another player dropped out at time $\gamma(y) < \gamma(v)$, our player will continue until time $\gamma(y) + \beta(y,v)$.

The case of two players applied at time $y$ implies that in equilibrium

$$\beta(y,v) = \int_y^v zh(z)dz,$$

since the updated density $\frac{f(z)}{F(y)}1_{z \geq y}$ has the same hazard rate as $f$.

Unfortunately, and somewhat surprisingly, there is no equilibrium once there are three players. To see this, suppose that players II and III are playing $\gamma(\cdot)$ and $\beta(\cdot, \cdot)$, and player I with value $v$ plays $\gamma(v - \epsilon)$ instead of $\gamma(v)$. Then

$$0 \leq u(v|v) - u(v - \epsilon|v) \leq Ce^2 - \overline{F}(v)^2(\gamma(v) - \gamma(v - \epsilon))$$

since with probability $\overline{F}(v)^2$, the other two players outlast player I (and then he pays an additional $\gamma(v) - \gamma(v - \epsilon)$ for naught), and with probability $(\int_{v-\epsilon}^v f(z))^2dz \leq Ce^2$ for some constant $C$, both of the other players drop out.

Thus, it must be that $\gamma(v) - \gamma(v - \epsilon) \leq Ce^2 \overline{F}(v)^{-2}$, and for any $k$, we have

$$\gamma(v - ke) - \gamma(v - ke - \epsilon) \leq \frac{C}{\overline{F}(v)^2}e^2$$

(since $\overline{F}(v)$ is a non-increasing function). Summing from $k = 0$ to $v/\epsilon$, we obtain $\gamma(v) \leq C\overline{F}(v)^{-2}v$, and hence $\gamma(v) = 0$.

Finally, we observe that $\gamma(v) = 0$ is not an equilibrium since a player, knowing that the other players are going to drop out immediately, will prefer to stay in.

**Generalized Bulow-Klemperer War of Attrition**

An alternative model considered by Bulow and Klemperer is motivated by battles between companies to control and set the standards for a new technology. In these settings, as long as a company is still competing to set the standard, it incurs significant expenses, e.g. due to advertising and promotions. However, once a company has given up, it still incurs a (lower) delay cost until a clear winner emerges and a standard is determined.

This scenario is modeled as follows: Fix a constant $c$ between 0 and 1. There are $n$ agents competing for an item, with values drawn independently from distribution $F$. Their strategic decision is how long to stay in the game. The game is over when only one agent remains, say at time $T$. The
3.3 War of Attrition

remaining agent is the winner, and his payment is $T$. If a losing agent drops out at time $t$, his payment is $t + c(T - t)$.

We derive equilibrium strategies for this model when there are three players. Again, let $\gamma(v)$ be the time a player with value $v$ waits before dropping out, assuming that no other player has dropped out before that. If, however, another player drops out before time $\gamma(v)$, say at time $\gamma(y)$, the player stays in for an additional $\beta(y, v)$ time units. Here $\beta(y, v)$ is the equilibrium strategy of a player with value $v$ whose opponent has value at least $y$ in a two-player game. The same derivation as above shows that

$$\beta(y, v) = \int_y^v xh(x)dx$$

(3.5)

Next, we derive the initial part $\gamma(\cdot)$ of the equilibrium strategy. This strategy is only relevant for the player with the lowest value, since in equilibrium the higher valued players will not want to drop out before the lowest valued player.

We therefore focus on the lowest valued player, say with value $v_3$. Suppose he is even told the value $v_2$ of the second highest player. In this case, he knows that if he plays as if his value is $w < v_2$, then after he drops out the game will last for another $\beta(w, v_2)$ period of time. Thus, for this player, we have

$$u(w|v_3) = \gamma(w) + c\beta(w, v_2).$$

Differentiating with respect to $w$ and setting this derivative equal to 0 at $w = v_3$, we obtain

$$\gamma'(v_3) + c\frac{\partial\beta(v_3, v_2)}{\partial v_3} = 0$$

or

$$\gamma'(v_3) = cv_3h(v_3),$$

yielding

$$\gamma(v) = c\int_0^v zh(z)dz.$$  

(3.6)

Finally, we verify that $\gamma(\cdot)$ given by (3.6) is indeed an equilibrium. Consider first the player with the lowest value $v_3$. For $0 \leq w < v_2$, his payment is

$$\gamma(w) + c\beta(w, v_2) = c\int_0^{v_2} xh(x)dx.$$  

For $w > v_2$, his payment is in fact the same, since the moment the player with value $v_2$ drops out (at time $\gamma(v_2)$), he will want to drop out as well.
As for either of the top two players, say with value $v > v_3$, we know that
at time $\gamma(v_3)$, they will want to stay in for an additional $\beta(v_3, v)$ time, rather
than drop out immediately. But dropping out at time $\gamma(w)$ for $w < v_3$ is equivalent to dropping out at time $\gamma(v_3)$ since $\gamma(w) + c\beta(w, v_3) = \gamma(v_3)$. Hence $\gamma$ is an equilibrium (even in the setting where all players are told the value of the lowest valued player).

### 3.4 On Asymmetric Bidding

So far, all equilibrium bidding strategies we have derived have been symmetric equilibria for i.i.d. bidders. It is natural to ask whether this is always the case. As we’ll see next, the answer depends on the auction format.

**Example 3.4.1.** Consider a Sotheby’s style English auction, where at any
time there is a current price (initialized to zero) that any bidder can choose
to raise. The auction terminates when no bidder wishes to raise the price
further; the winner is the bidder that set the final price.

Suppose that there are two bidders and it is public knowledge that their
values $V_1$ and $V_2$ are drawn independently from the same continuous, in-
creasing distribution $F$. The following is a Bayes-Nash equilibrium strategy
pair:

- Player I bids $\beta(v_1) = \mathbb{E}[V_2 | V_2 \leq v_1]$, the equilibrium bid in a first price auction.
- Player I infers $v_1$ from $\beta(v_1)$ (which is possible since $\beta(\cdot)$ is strictly
increasing). If $v_2 > v_1$, player II bids $v_1$. Otherwise, II drops out
and I wins at the price $\beta(v_1)$.

The previous example shows that even symmetric settings can sometimes
admit an asymmetric equilibrium! However, for first price auctions with
symmetric bidders, there are no asymmetric Bayes-Nash equilibria:

**Theorem 3.4.2.** Consider $n$ bidders with values drawn i.i.d. from a con-
tinuous, increasing distribution $F$ with bounded support $[0, h]$. There is no
asymmetric Bayes-Nash equilibrium $\beta(\cdot)$ in a first-price auction that is con-
tinuous and strictly increasing in $[0, h]$.

**Proof.** Suppose there is an asymmetric Bayes-Nash equilibrium. It suffices
to show that for two arbitrary bidders say 1 and 2, that $\beta_1(\cdot) = \beta_2(\cdot)$, since
such an argument applied to each pair of bidders implies that $\beta_i(\cdot) = \beta(\cdot)$
for all $i$. 
3.4 On Asymmetric Bidding

Focusing in then on bidders 1 and 2, we see that from their perspective, the maximum bid of the other bidders can be viewed as a random “reserve” price \( R \), a price below which they cannot win.

**Step 1:** If \( v \) satisfies \( \beta_1(v) > \beta_2(v) \), then \( a_1(v) > a_2(v) \). To see this, observe that player 1 wins if the independent events \( \{ \beta_1(v) > R \} \) and \( \{ \beta_1(v) > \beta_2(V_2) \} \) both occur. But

\[
P[\beta_1(v) \geq \beta_2(V_2)] > P[\beta_2(v) \geq \beta_1(V_1)],
\]

since \( \beta_2^{-1}(\beta_1(v)) > v > \beta_1^{-1}(\beta_2(v)) \), the \( \beta_i(\cdot) \) are continuous, and \( F \) is strictly increasing. Additionally,

\[
P[\beta_1(v) \geq R] \geq P[\beta_2(v) \geq R] > 0.
\]

Combining these two facts, we get that \( a_1(v) > a_2(v) \).

**Step 2:** A similar argument shows that if \( \beta_1(v) = \beta_2(v) \), then \( a_1(v) = a_2(v) \) and thus

\[
u_1(v) = (v - \beta_1(v))a_1(v) = (v - \beta_2(v))a_2(v) = u_2(v).
\]

**Step 3:** Now for sake of contradiction, suppose that \( \beta_1(\cdot) \neq \beta_2(\cdot) \). Let \( \hat{v} \) be such that, say, \( \beta_1(\hat{v}) > \beta_2(\hat{v}) \), and let \( v \) be the maximum over \( v \leq \hat{v} \) for which \( \beta_1(v) = \beta_2(v) \). This maximum exists since the \( \beta_i(\cdot) \) are continuous and \( \beta_1(0) = \beta_2(0) \). (If, on the other hand, \( \beta_1(0) > 0 \) and wlog \( \beta_1(0) = \max_i \beta_i(0) \), then take \( 0 < \epsilon < \beta_1(0) \) and observe that \( a_1(\epsilon) = P[\max_{i \geq 2} \beta_i(V_i) < \beta_1(\epsilon)] > 0 \) so \( u_1(\epsilon) < 0 \).)

- **Case 3a:** \( \beta_1(v) > \beta_2(v) \) in \( [\hat{v}, h] \). This implies that \( a_1(v) > a_2(v) \) for \( v \in [\hat{v}, h] \), and hence

\[
u_1(h) - u_1(v) = \int_{\hat{v}}^{h} a_1(v) > \int_{\hat{v}}^{h} a_2(v) = u_2(h) - u_2(v),
\]

so \( u_1(h) > u_2(h) \) by Step 2. But if player 2 plays \( b = \beta_1(h) \) instead of \( \beta_2(h) \) when her value is \( h \), then she will outbid player 1 with probability 1, and therefore she will also win with probability \( a_1(h) \). This increases her utility from \( u_2(h) \) to \( u_1(h) \) and contradicts the assumption that \( \beta_2(\cdot) \) was an equilibrium.

- **Case 3b:** There is \( v > \hat{v} \) such that \( \beta_1(v) = \beta_2(v) \). Let \( \overline{v} \) be the
minimum such \( v \). Then by Step 1,

\[
u_1(\bar{v}) - u_1(\bar{v}) = \int_{\bar{v}}^{v} a_1(v) \, dv > \int_{\bar{v}}^{v} a_2(v) \, dv = u_2(\bar{v}) - u_2(\bar{v}). \tag{3.7}
\]

But this contradicts Step 2, which implies that \( u_1(\bar{v}) = u_2(\bar{v}) \) and \( u_1(\bar{v}) = u_2(\bar{v}) \).

\[
\square
\]

3.5 More general distributions

We next consider auctions which allocate to the highest bidder with i.i.d. bidders, however, in the setting where bidders’ distributions \( F \) are neither strictly increasing or atomless. The following example shows that in such settings, there is no longer guaranteed to be a pure equilibrium.

**Example 3.5.1.** Consider two bidders participating in a first price auction, each with a value equally likely to be 0 or 1. We assume random tie-breaking. Wlog each player bids 0 when their value is 0. Suppose that player II bids \( 0 < b \leq 1 \) when his value is 1. Now consider player I’s best response when her value is 1. Comparing the case where she bids above \( b \) to the case where she bids exactly \( b \), we see that

\[
u_1[b + \epsilon|v_1 = 1] = 1 - b - \epsilon > u_1[b|1] = \frac{3}{4}(1 - b),
\]

and thus if outbidding II is a best response, there is no pure equilibrium. Similarly, when bidding below \( b \), say \( \epsilon \), we have

\[
u_1[\epsilon|v_1 = 1] = \frac{1}{2}(1 - \epsilon) > u_1[0|1] = \frac{1}{4},
\]

and again there is no pure equilibrium. (The case where \( b = 0 \) can be handled similarly.)

**Definition 3.5.2.** Consider an auction in which each player’s value is drawn independently from distribution \( F \). A **mixed strategy** is implemented by providing each player with a realization \( z_i \) of a random variable \( Z_i \sim U[0,1] \). Suppose that the bid of agent \( i \) is \( \beta(v_i, z_i) \). We assume without loss of generality that \( \beta(v_i, z_i) \) is monotone increasing in \( z_i \).

The strategy profile in which all players play strategy \( \beta(\cdot, \cdot) \) is in **Bayes-Nash (mixed) equilibrium** if for all \( i \), all \( v_i \) and all \( z_i \)

\[
b \to u_i[b|v_i, z_i] \text{ is (weakly) maximized at } b = \beta(v_i, z_i).
\]
Remark. The previous definition is non-standard. See the notes for a discussion.

**Theorem 3.5.3.** Let $A$ be an auction for selling a single item in which the item is allocated to the highest bidder. Suppose that each bidder’s value is drawn independently from the same distribution $F$.

(a) If $\beta(\cdot, \cdot)$ is a symmetric mixed Bayes-Nash equilibrium, then for each agent $i$:

(i) The probability of allocation $a(v, z)$ and the associated bid $\beta(v, z)$ are monotone increasing in $v$ (and $a(v, z) \leq a(v, z')$ for $z \leq z'$).

(ii) The utility $u(v, z)$ does not depend on $z$, so can be denoted $u(v)$. It is a convex function of $v$, with

$$u(v) = \int_0^v a(x, 0)dx.$$  

(iii) The payment rule is determined by the allocation rule:

$$p(v, z) = va(v, z) - \int_0^v a(x, 0)dx.$$  

(b) Conversely, if all bidders use strategy $\beta(\cdot, \cdot)$ and (i) and (ii) hold (or alternatively (i) and (iii)), then for all bidders $i$, values $v$, $w$, and $z$

$$u(v|v) \geq u(w, z|v).$$  

(3.8)

Remark. The proof of Theorem 3.5.3 is similar to the proof of Theorem 3.1.1 and can be found in the notes.

**Corollary 3.5.4 (Revenue Equivalence).** Let $A$ and $A'$ be two standard single-item auctions resulting in the same allocation rule $a(v, z)$. Then the expected payment of each bidder is the same whence the expected auctioneer revenue is the same.

Proof. The expected payment of an agent with value $v$ is

$$p(v) = \int_0^1 p(v, z)dz.$$  

By Theorem 3.5.3 $p(v, z)$ is determined by the allocation rule.  

As we saw before, revenue equivalence is helpful for finding Bayes-Nash equilibrium strategies. We now show how to use the equilibrium in the second price auction to derive an equilibrium in the first-price auction for general distributions $F$. The relevant calculations are simplified if we use the quantile transformation.

The quantile transformation

For a distribution function $F$ that is strictly increasing and continuous, a standard way to generate a random variable $V$ with distribution $F$ is to take $q \sim U[0,1]$ and define $V = F^{-1}(q)$. Equivalently, the random variable $F(V)$ is uniform on $[0,1]$. For general distribution functions, $V(q) = \sup \{ v : F(v) \leq q \}$

(3.9)

has distribution $F$. If $V$ is defined as in (3.9) and $F$ has atoms, then the function $V(q)$ is not invertible. (See Figure ??.) To extend the quantile transformation to this case, we define

$$q = q(V,Z) = F_-(V) + Z(F_+(V) - F_-(V)),$$

with $Z \sim U[0,1]$ independent of $V$. Then, $q = q(V,Z)$ is also $U[0,1]$.

Second price auction:

Consider an implementation of the second price auction in a setting with atoms. In this case, tie-breaking can be done by the agents via their use of the randomness in $Z \sim U[0,1]$. Specifically, we can imagine that each agent submits both $v$ and $z$, or equivalently, submits $q = q(v,z)$, and the player with the largest value of $q$ wins. In this case,

$$a(q) := a(v,z) = q^{n-1},$$

(3.10)

and

$$p(q) := p(v,z) = F_-(v)^{n-1} \mathbb{E} \left[ \max_{i \leq n-1} V_i \middle| \max_{i \leq n-1} V_i < v \right] + (a(v,z) - F_-(v)^{n-1})v.$$

This latter expression is simpler when written in terms of the quantile $q$. For a given value $v$, let $[q(v), q(v)]$ be the interval of quantiles $q$ for which $F(v) = q$. Then the bid $(v,z)$ corresponds to quantile $(1-z)\tilde{q}(v) + z\tilde{q}(v)$,
3.6 Risk Averse bidders

and we have payment equal to the second highest value, i.e.,

\[ p(q) = \int_0^q v(r)(n-1)r^{n-2}dr. \tag{3.11} \]

We next apply revenue equivalence to compute equilibrium strategies in the first price auction.

First price auction:

(a) Suppose that \( \beta(q) \) defines a symmetric mixed equilibrium in the first price auction. Then by revenue equivalence, \( a(q) \) and \( p(q) \) are given by (3.10) and (3.11). Since the expected payment satisfies

\[ p(q) = a(q)\beta(q), \]

it follows that

\[ \beta(q) = \int_0^q v(r)(n-1)\frac{r^{n-2}}{q^{n-1}}dr. \]

(v) Suppose that \( \beta \) is defined by the preceding equation. We verify that this formula actually defines an equilibrium. Since \( q \) is strictly increasing, by the preceding equation, \( \beta \) is also strictly increasing. Therefore \( a(q) = q^{n-1} \) and conditions (i) and (iii) of the theorem hold. Hence (3.8) holds. Moreover, \( \beta(q) \) is a continuous function of \( q \). Finally, we observe that bidding above \( \beta(1) \) is dominated by bidding \( \beta(1) \).

In particular, for two agents where \( F \) is 0 or 1 with probability 1/2 each, the equilibrium mixed strategy is to bid 0 at value 0, and to bid \( 1 - 1/2q \) for \( q \) uniform in [1/2, 1] otherwise.

3.6 Risk Averse bidders

Until now the utility of a bidder from an auction was simply his expected gain, namely, the expected value of the item received \( a[b] \cdot v \) minus the expected payment \( p[b] \). In particular, such a bidder is indifferent between participating in the auction and receiving \( a[b] \cdot v - p[b] \) dollars outright.

In reality though, many bidders would prefer the second option, and would even be willing to pay for the risk reduction it entails. Formally, we assume a bidder attaches utility \( U(x) \) to receiving \( x \) dollars, and his goal is to maximize his expected utility. A bidder is called (strictly) risk-averse if he strictly prefers receiving \( \alpha x + (1 - \alpha)y \) dollars to a lottery in which he receives \( x \) with probability \( \alpha \) and \( y \) with probability \( 1 - \alpha \), i.e., for all \( x, y \) and \( 1 \leq \alpha \leq 1 \),

\[ \alpha \cdot U(x) + (1 - \alpha) \cdot U(y) < U(\alpha x + (1 - \alpha)y). \]
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This is precisely the definition of $U(\cdot)$ being a strictly concave function. The following proposition is proven in Appendix 1.

**Proposition 3.6.1.** The following are equivalent for a function $U(\cdot)$ defined on interval $I$.

(i) $U(\cdot)$ is a concave function.

(ii) The mapping

$$x \mapsto U(x + z) - U(x)$$

is non-increasing in $x$ for $x, x + z \in I$.

(iii)

$$U'_+(x + z) \leq \frac{U(x + z) - U(x)}{z}.$$  

(iv) (Jensen’s Inequality)

$$U(E[X]) \geq E[U(X)]$$

for every non-constant random variable $X$ taking values in $I$.

For a strictly concave function, all of the above inequalities become strict inequalities.

The revenue equivalence theorem no longer holds when bidders are risk-averse. We show next that for i.i.d. risk-averse bidders with the same concave utility function, the expected auctioneer profit from a first-price auction is greater than that from a second-price auction.

Intuitively, this follows from two observations:

- In the second-price auction, risk-averse bidders have the same dominant strategy equilibrium: report their value truthfully.

- In the first-price auction, bidding $\beta(v + \epsilon)$ instead of $\beta(v)$ increases the allocation probability, but decreases the profit upon winning. In the risk-neutral case, at the equilibrium bid, these effects cancel. The risk-averse bidder, on the other hand, will prefer to reduce his profit upon winning a little bit in order to reduce the risk of losing the valuable item.

Formally, if $\beta(\cdot)$ is the equilibrium bidding strategy for risk-neutral bidders in a first-price auction, then the expected utility of a risk-averse bidder bidding $\beta(w)$ when his value is $v$ is

$$u(w|v) = a(v)U(v - \beta(v))$$
3.6 Risk Averse bidders

and thus

\[
\frac{\partial}{\partial w} \log(u(w|v)) = \frac{a'(w)}{a(w)} \cdot \frac{U'(v - \beta(w))[\beta'(w)]}{U(v - \beta(w))} > \frac{a'(w)}{a(w)} - \frac{\beta'(w)}{v - \beta(w)},
\]

where the inequality comes from Proposition 3.6.1(iii).

For a linear utility function, with equilibrium strategies \(\beta(\cdot)\), the expected utility \(u(w|v)\) is maximized at \(w = v\) for all \(v\), and thus

\[
\left. \frac{\partial}{\partial w} \log(u(w|v)) \right|_{w=v} > \frac{a'(v)}{a(v)} - \frac{\beta'(v)}{v - \beta(v)} = 0.
\]

Hence, a risk-averse bidder with value \(v\) would want to bid above \(\beta(v)\) when other bidders use \(\beta(\cdot)\).

**Theorem 3.6.2.** Let \(V_1, \ldots, V_n\) be the private values of \(n\) bidders, each i.i.d. with increasing distribution function \(F\). Suppose also that all bidders have the same risk-averse utility function \(U(\cdot)\), with \(U\) increasing and strictly concave, and \(U(0) = 0\). Then

\[
\mathbb{E}[\text{profit in first-price auction}] > \mathbb{E}[\text{profit in 2nd-price auction}].
\]

**Proof.** Let \(b(v)\) be a symmetric, monotone equilibrium strategy for each bidder in the first-price auction with risk-averse bidders. Then the winner will be the bidder with the highest value and the probability that a bidder with value \(v\) wins the auction is \(F(v)^{n-1}\).

As usual, we have that bidder \(i\) with value \(v\), knowing that the other bidders are bidding according to \(b(\cdot)\), bids

\[
b(z) = \arg\max_z F(z)^{n-1}U(v - b(z)).
\]

To find the maximum of this function, we set the derivative to 0, and using our standard notation \(a(z) = F(z)^{n-1}\), we obtain that the maximizing \(z\) satisfies

\[
a'(z) \cdot U(v - b(z)) - a(z) \cdot U'(v - b(z)) \cdot b'(z) = 0
\]

or

\[
b'(z) = \frac{U(v - b(z))}{U'(v - b(z))} \cdot \frac{a'(z)}{a(z)}.
\]

For \(b(\cdot)\) to be a Bayes-Nash equilibrium, this equation must hold everywhere with \(z = v\), i.e., for all \(v\),

\[
b'(v) = \frac{U(v - b(v))}{U'(v - b(v))} \cdot \frac{a'(v)}{a(v)}.
\]
By Proposition 3.6.1(iii), since \( u(0) = 0 \), we have \( u(x) > xu'(x) \), and thus we obtain that
\[
b'(v) = \frac{U(v - b(v))}{U'(v - b(v))} \cdot \frac{a'(v)}{a(v)} > (v - b(v)) \frac{a'(v)}{a(v)}.
\]
If \( b(\tilde{v}) \leq \beta(\tilde{v}) \) for some \( \tilde{v} > 0 \) (where \( \beta(v) \) is the Bayes-Nash equilibrium strategy in the first-price auction with risk-neutral bidders), then
\[
b'(\tilde{v}) > (\tilde{v} - b(\tilde{v})) \frac{a'(v)}{a(v)} \geq (\tilde{v} - \beta(\tilde{v})) \frac{a'(\tilde{v})}{a(\tilde{v})} = \beta'(\tilde{v}).
\]
Considering \( v_0 \in [0, \tilde{v}] \) where \( b - \beta \) is minimized yields a contradiction. Thus \( b(v) > \beta(v) \) for all \( v > 0 \). Revenue equivalence completes the argument.

For the rest of the chapter, we revert to the standard assumption of risk-neutral bidders.

3.7 Common or Interdependent Values

In this section, we turn to settings in which the bidders’ values for the item being sold are correlated or common. Scenarios where this might be the case are auctions for an oil field, where the value of the field relates to the amount of oil and cost of production in that field, or a painting by a famous painter, where the value of the painting relates to its resale value and the prestige of owning it. We model these scenarios by assuming that each bidder \( i \) has a signal \( X_i \) about the value of the item being auctioned and that \( (X_1, \ldots, X_n) \) is drawn from joint distribution \( F \). The value of the item to each agent is then some function \( V(X_1, \ldots, X_n) \) of these signals.

A phenomenon known as winner’s curse has been observed empirically in auctions for items with common values. This arises when the winner of the auction pays more than the value of the item. Intuitively, winning means that other bidders believe the value of the item is lower than the winner does, and the expected value of the item conditioned on these beliefs could be lower than what the winner might have expected based solely on his signal.

A new equilibrium notion will be relevant here:

**Definition 3.7.1.** Consider \( n \) agents participating in an auction with private signals \( X_1, \ldots, X_n \). A set of bidding strategies \( (\beta_1(\cdot), \beta_2(\cdot), \ldots, \beta_n(\cdot)) \) is an **ex-post equilibrium** if for each possible signal vector \( (x_1, \ldots, x_n) \), and each bidder \( i \), bidding \( \beta_i(x_i) \) is a best response to \( \beta_{-i}(x_{-i}) \).
3.7 Common or Interdependent Values

3.7.1 Second-price Auctions with Common Value

Example 3.7.2. Consider two bidders, with signals \( X_1 \) and \( X_2 \) drawn from joint distribution \( F \). Suppose that the common value of the item being auctioned is the same for both agents, equal to \( V = X_1 + X_2 \). We claim that in a second price auction for the item, it is an ex-post equilibrium for each agent to bid \( \beta(z) = 2z \).

To see this, suppose that bidder 1’s value is \( x \) and bidder 2’s value is \( y \). Then the value of the item to both bidders is \( x + y \). Assuming bidder 2 bids \( \beta_2(y) = 2y \), player 1 wants to win if \( x > y \) (obtaining a utility of \( x + y - 2y = x - y > 0 \), and wants to lose if \( x < y \) (or will obtain negative utility). Bidding \( \beta(x) = 2x \) accomplishes this.

Some comments are in order:

- The equilibrium just discussed is not a dominant-strategy equilibrium. For example, if bidder 1’s value \( x \) is greater than bidder 2’s value \( y \), and bidder 2 bids between \( x + y \) and \( 2x \), then it is not a best response for bidder 1 to bid \( 2x \).
- The symmetric ex-post equilibrium just derived is not the only equilibrium in this auction. For example, one can check that \( \beta_1(z) = 2z - c \) and \( \beta_2(z) = 2z + c \) is also an ex-post equilibrium.

Exercise 3.7.3. Determine all continuous ex-post equilibria in the setting of Example 3.7.2. Assume that \( X \) and \( Y \) are in \( [0,h] \), that \( \beta_i(z) = z + \gamma_i(z) \) is strictly increasing and continuous. (Note that without loss of generality \( \beta_i(x) \geq x \), since any lower bid is dominated by bidding \( x \).)

Solution: Observe that

\[
x + y > \beta_2(y) \implies \beta_1(x) > \beta_2(y) \implies \beta_1(x) \geq x + y.
\]

Therefore

\[
x > \gamma_2(y) \implies \gamma_1(x) \geq y.
\]

Similarly

\[
x < \gamma_2(y) \implies \gamma_1(x) \leq y.
\]

Continuity of \( \gamma_1 \) and \( \gamma_2 \) then imply that

\[
\gamma_1(\gamma_2(y)) = y \quad \text{and} \quad \gamma_2(\gamma_2(x)) = x.
\]

Symmetric Bidders
Auctions: Some more advanced topics

We next consider a setting where the joint density of the signals $X_1, \ldots, X_n$ is a symmetric function of its arguments. We assume that the value of the item to each of the agents is $V(x_1, \ldots, x_n)$, also a symmetric function of its arguments. The following quantity will be important:

$$v(x, y) := \mathbb{E} \left[ V(X_1, \ldots, X_n) | X_i = x, \max_{j \neq i} X_j = y \right].$$

Assumption 3.7.4. We assume that $v(x, y)$ is strictly increasing in $x$ and increasing in $y$.

Theorem 3.7.5. In the symmetric setting, under Assumption 3.7.4, $\beta(x) = v(x, x)$ is a Bayes-Nash equilibrium bidding strategy in the second price auction.

Proof. Suppose that all bidders other than bidder $i$ bid $\beta(\cdot)$ and bidder $i$’s signal is $x$. If the maximum of the other signals is $y$, then she wins if her bid $b > \beta(y) = v(y, y)$. Thus, her expected utility if she bids $b$ when her signal is $x$ is

$$u[b|x] = \int_{\beta(y) \leq b} \left( v(x, y) - v(y, y) \right) g(y|x) \, dy$$

where $g(y|x)$ is the conditional density of $\max_{j \neq i} X_j$ at $y$, given that $X_i = x$. By Assumption 3.7.4

$$\{ y \mid v(x, y) - v(y, y) > 0 \} = \{ y \mid y < x \} = \{ y \mid \beta(y) < \beta(x) \}.$$

The integral is maximized by integrating over the set of $y$ where the integrand is positive and this is achieved by bidding $b = \beta(x) = v(x, x)$. \qed

We note that this Bayes-Nash equilibrium is no longer an ex-post equilibrium for three or more bidders. For example, suppose that $X_1, X_2, X_3$ are all uniform on $[0, 1]$ and $V = X_1 + X_2 + X_3$. Then $\beta(w) = 5w/2$, so if the three signals are $x > y > z$, then the bidder with signal $x$ will win at price $5y/2$, which could be more than $x + y + z$ and would result in negative utility.

Exercise 3.7.6. Show that the Bayes-Nash equilibrium of Theorem 3.7.5 is an ex-post equilibrium for two bidders.

Example 3.7.7. Another setting for which Assumption 3.7.4 holds is the following: Consider a two bidder auction where the value of the item to both players is a random variable $V$, where $V \sim N(c, \sigma^2)$, and the signals
the bidders receive are noisy versions of $V$, specifically $X = V + Z_1$ and $Y = V + Z_2$ where $Z_1 \sim N(0, \sigma_1^2)$ and $Z_2 \sim N(0, \sigma_2^2)$. Here

$$v(x, y) = \mathbb{E}[V|X = x, Y = y] = c + \frac{\sigma_1^{-2}(x + y)}{2\sigma_1^{-2} + \sigma^{-2}},$$

which is obviously increasing in both $X$ and $Y$. See Exercise 3.7.

We next generalize the previous example.

**Theorem 3.7.8.** Suppose the common value of the item for sale is $V$, and the signals of the bidders are $X_i = V + Z_i$, where $Z_1, \ldots, Z_n$ are i.i.d. with a log-concave, differentiable density. Then for any bidder $i$, say, $i = 1$,

$$v(x, y) = \mathbb{E} \left[ V | X_1 = x \text{ and } \max_{j \geq 2} X_j = y \right]$$

satisfies Assumption 3.7.4.

**Proof.** We have

$$v(x, y) = \frac{\int v f_V(v)f(x - v)f(y - v)F^{n-2}(y - v) dv}{\int f_V(v)f(x - v)f(y - v)F^{n-2}(y - v) dv},$$

where $f_V(\cdot)$ is the density of $V$ and $f(\cdot)$ is the density of $Z_i$. We need to show that

$$\frac{\partial v(x, y)}{\partial x} > 0 \text{ and } \frac{\partial v(x, y)}{\partial y} \geq 0.$$

Letting

$$\Psi(x, y, v) = f_V(v)f(x - v)f(y - v)F^{n-2}(y - v),$$

we have

$$v(x, y) = \frac{\int v \Psi(x, y, v) dv}{\int \Psi(x, y, v) dv}.$$

Since the density $f$ is log-concave, it can be written $f(\cdot) = e^{h(\cdot)}$ where $h(\cdot)$ is a concave function. It follows that

$$\frac{\partial \Psi(x, y, v)}{\partial x} = h'(x - v)\Psi(x, y, v).$$

Thus,

$$\frac{\partial v(x, y)}{\partial x} = \frac{\int \Psi dv \int vh'(x - v)\Psi dv - \int h'(x - v)\Psi dv \int v \Psi dv}{(\int \Psi dv)^2}.$$
Auctions: Some more advanced topics

Let \( \mu := \mu_{xy} \) be a probability density on \( \mathbb{R} \) with density

\[
\frac{\Psi(x, y, \cdot)}{\int \Psi(x, y, v) dv}.
\]

Then

\[
\frac{\partial v(x, y)}{\partial x} = \int vh'(x - v) d\mu(v) - \int h'(x - v) d\mu(v) \int v d\mu(v).
\]

That this is positive follows from Lemma 3.7.10 below, observing that \( v \) and \( h'(x - v) \) are both increasing functions of \( v \). \( h(z) \) is concave and hence \( h'(z) \) is decreasing.)

The same argument shows that \( \frac{\partial v(x, y)}{\partial y} \geq 0 \) for \( n = 2 \) even if \( Z_1 \) and \( Z_2 \) are not identically distributed, but independent and log-concave.

To prove that \( \frac{\partial v(x, y)}{\partial y} \geq 0 \) for more than two i.i.d. bidders, a calculation similar to the above shows that it suffices to verify that

\[
g(v) = f(v)F^{n-2}(v)
\]

is log-concave. This follows from the log-concavity of \( f \) and Lemma 3.7.9.

**Lemma 3.7.9.** If \( f \) is a log-concave, positive, differentiable function, then

\[
F(x) = \int_{x_0}^{x} f(t) dt
\]

is also log-concave.

**Proof.** Write \( f = e^h \) with \( h \) concave. To show this, we verify that

\[
\frac{d}{dv} \log(F(v)) = \frac{f(v)}{F(v)}
\]

is decreasing. Taking a derivative, we obtain

\[
\left( \frac{f}{F} \right)' = \frac{F f' - f^2}{F^2} = \frac{F h' - f^2}{F^2}.
\]

Since \( h'(t) \) is decreasing,

\[
h'(x)F(x) \leq \int_{x_0}^{x} h'(t) f(t) dt = \int_{x_0}^{x} f'(t) dt \leq f(x).
\]

Therefore, by (3.12), \( f/F \) is decreasing.

**Lemma 3.7.10.** (Chebyshev’s Inequality) Let \( \mu(\cdot) \) be a density function, and suppose that \( f(v) \) and \( g(v) \) are both increasing functions of \( v \). Then

\[
\int f(v)g(v) d\mu \geq \int f(v) d\mu \int g(v) d\mu.
\]

If either \( f(v) \) or \( g(v) \) are strictly increasing, then this inequality becomes strict.
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Proof. Observe that
\[ \int_w \int_v \left[ f(v) - f(w) \right] \left[ g(v) - g(w) \right] d\mu(v)d\mu(w) \geq 0, \]
since \( f(v) - f(w) \geq 0 \) if and only if \( g(v) - g(w) \geq 0 \). But the left hand side of this inequality is the same as
\[ 2 \int f(v)g(v)d\mu - 2 \int v \int w f(v)g(w)d\mu(v)d\mu(w) = \]
\[ 2 \int f(v)g(v)d\mu - 2 \int v \int w f(v)d\mu(v) \int w g(v)d\mu(w), \]
completing the proof of the lemma.

3.7.2 English Auctions

We turn next to equilibria of the English auction. As before, we assume that \( V = V(X_1, \ldots, X_n) \) is a symmetric function of the signals \( X_i \).

For \( y_1 \geq y_2 \geq \ldots \geq y_n \), we need the following definition:
\[ v(y_1, y_2, \ldots, y_n) := \mathbb{E}[V | \text{the order statistics of } X_1, \ldots, X_n \text{ are } y_1, y_2, \ldots, y_n]. \]

Assumption 3.7.11. We assume that \( v(y_1, y_2, \ldots, y_n) \) is strictly increasing in all coordinates.

As we shall see equilibrium bidding in the English auction will have the property that the price at which a bidder drops out reveals his signal, and that bidders drop out in order of increasing signal. With these assumptions, it is clear that when two bidders remain, the English auction is equivalent to the 2nd price auction. The equilibrium we found there suggests that the bidder with signal \( y_2 \) should drop out when the price reaches \( v(y_2, y_2, y_3, \ldots, y_n) \). A natural generalization yields the general strategy \( \beta(x) \) of a bidder with signal \( x \). Each time some bidder drops out, each remaining bidder updates the maximum price he is willing to pay as follows: If \( k \) bidders remain in the auction, and it is known that the signals of the bidders that dropped out are \( y_{k+1}, \ldots, y_n \), the bidder with signal \( x \) updates the maximum price she is willing to pay to \( v(x, x, \ldots, x, y_{k+1}, \ldots, y_n) \).

With this bidding strategy, what will happen is the following:

- The bidder with the lowest signal \( y_n \) drops out at price \( v(y_n, \ldots, y_n) \).
- The bidder with second lowest signal \( y_{n-1} \) drops out at price \( v(y_{n-1}, \ldots, y_{n-1}, y_n) \).
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- More generally, for each \( k \geq 2 \), the bidder with the \( k \)-th highest signal \( y_k \) drops out at price \( v(y_k, \ldots, y_k, y_{k+1}, \ldots, y_n) \).
- The auction ends when the bidder with second highest signal \( y_2 \) drops out at price \( v(y_2, y_2, y_3, \ldots, y_n) \), and that is what the bidder with the highest signal pays.

Notice that, by assumption 3.7.11, the maximum price a bidder is willing to pay decreases each time some other bidder drops out.

**Theorem 3.7.12.** The equilibrium strategy just defined is an ex-post equilibrium if assumption 3.7.11 holds.

**Proof.** Since the highest bidder pays \( v(y_2, y_2, y_3, \ldots, y_n) \) and has value \( v(y_1, y_2, y_3, \ldots, y_n) \), he has positive gain and therefore, knowing that other bidders are playing \( \beta(\cdot) \), wants to win the auction.

Now consider a losing bidder with signal \( y_k \), where \( k \geq 2 \). If he deviates from strategy \( \beta(\cdot) \) and stays in long enough to win, the other bidders will adjust their bids according to the strategy just described, so he will end up winning at price \( v(y_1, y_2, \ldots, y_{k-1}, y_k, y_{k+1}, \ldots, y_n) \) which exceeds his expected value \( v(y_1, y_2, \ldots, y_{k-1}, y_k, y_{k+1}, \ldots, y_n) \) given the observed signals.

### 3.7.3 An approximately optimal algorithm

A seller wishes to sell a house by auction. Suppose it is known that the buyers valuations \( V_1, V_2, \ldots, V_n \) are drawn from joint distribution \( F \) (not necessarily a product distribution). Consider the following auction:

- Ask the bidders to report their values. Wlog, suppose that agent 1 has the highest report \( v_1 \) and agent 2 has the second highest report \( v_2 \).
- Now run the optimal auction on agent 1, conditioned on \( v_2, \ldots, v_n \) and the fact that \( V_1 \geq \max(v_2, \ldots, v_n) \). (Letting \( \tilde{F} \) be this conditional distribution, this auction simply offers agent 1 the price \( p \) that maximizes \( p(1 - \tilde{F}(p)) \)).

**Theorem 3.7.13.** The auction just described is truthful, ex-post individually rational and has an expected auctioneer profit at least half that of the optimal BIC, ex-post individually rational auction.

**Proof.** That the auction is ex-post individually rational is immediate from the construction. That the auction is truthful follows from the observation that the auction is bid-independent (Theorem ??): the price offered to a bidder is a function only of the other bidders bids.
Now consider the optimal BIC, ex-post IR auction. The expected revenue of this auction is the sum of its expected profit from bidders that are not the highest plus its expected profit from the highest bidder. It is immediate that the expected profit of the LA auction is at least the latter, since we run the optimal auction for this bidder conditioned on being highest and conditioned on the other bids. As for the other bidders, the optimal profit achievable from these bidders is upper bounded by the maximum value of bidders in this set, that is $v_2$. But the expected profit from the highest bidder is at least $v_2$, since one of the possible auctions is to just offer him a price of $v_2$. Therefore the expected profit of the LA auction is also at least the optimal profit achievable from the other bidders.

Corollary 3.7.14. Consider a single-item auction where agents values are independent but not identically distributed. The second price auction with individual monopoly reserves is a 2-approximation to the optimal auction.

Proof. This auction obtains at least the revenue of the lookahead auction.

3.7.4 Profit maximization without priors

So far, we have assumed access to the prior distributions from which agents values are drawn. An attractive alternative is to try to design an auctions whose performance does not depend on knowledge of priors. Indeed, determining priors may not be convenient, reasonable or even possible. This is particularly true in small markets or when mechanisms are executed repeatedly, where the process of collecting information may negatively impact both the incentives of the agents and the performance of the mechanism. Priors may also change over time.

In this section, we present an auction that is guaranteed to achieve high profit without any prior information about the bidders. We do this in the context of digital goods auction. These are auctions to sell digital goods such as mp3’s, digital video, pay-per view TV, etc. The unique aspect of digital goods is that the cost of reproducing the items is negligible and therefore the auctioneer effectively has an unlimited supply of the items. This means that there is no constraint on how many of the items can be sold, or to whom.

For digital goods auctions, the VCG mechanism allocates to all of the bidders, and charges them all nothing! Thus, while VCG perfectly maximizes social welfare, it can be disastrous when the goal is to maximize profit.

In this section, we present a truthful auction that does much better.
Specifically, we present an auction that always gets within a factor of four of the profit obtained by the auction that sells the items at a fixed price.

**Definition 3.7.15.** The **optimal fixed price profit** that can be obtained from bidders with bid vector \( b = (b_1, b_2, \ldots, b_n) \) is

\[
\text{OFP}(b) = \max_p \{ p \cdot (\text{the number of bids in } b \text{ at or above } p) \},
\]

and the **optimal fixed price** is

\[
p^*(b) = \argmax_p \{ p \cdot (\text{the number of bids in } b \text{ at or above } p) \}.
\]

If we knew the true values \( v \) of the agents, a profit of \( \text{OFP}(v) \) would be trivial to obtain. We would just offer the price \( p^*(v) \) to all the bidders, and sell at that price to all bidders whose values are above \( p^* \). But we can’t do this truthfully.

**Exercise 3.7.16.** Show that no truthful auction can obtain a profit of \( \text{OFP}(v) \) for every bid vector \( v \).

What we can do truthfully is offer each agent a price which does not depend on their own bid. The following auction is perhaps the first thing one might try along these lines:

**The Deterministic Optimal Price Auction (DOP):** For each bidder \( i \), compute \( t_i = p^*(b_{-i}) \), the optimal fixed price for the remaining bidders, and use that as the threshold bid for bidder \( i \).

Unfortunately, this auction does not work well, as the following example shows.

**Example 3.7.17.** Consider a group of bidders of which 11 bidders have value 100, and 1001 bidders have value 1. The best fixed price is 100 – at that price 11 items can be sold for a total profit of 1100. (The only plausible alternative is to sell to all 1001 bidders at price $1, which would result in a lower profit.)

Unfortunately, if we run the DOT auction on this bid vector, then for each bidder of value 100, the threshold price that will be used is 1, whereas for each bidder of value 1, the threshold price is of value 100, for a total profit of only 11!

In fact, the DOT auction can obtain arbitrarily poor profit compared to the optimal fixed price profit. Moreover, it is possible to prove that **any** deterministic truthful auction that treats the bidders symmetrically will fail to consistently obtain a constant fraction of the optimal fixed price profit.
The key to overcoming this problem is to use randomization. First though, we show how to solve a somewhat easier problem.

### 3.7.5 Profit Extraction

A key ingredient in the auction we will develop is the notion of a profit extractor (discussed briefly in example ??). Suppose that we lower our sights and rather than shooting for the best fixed price profit possible for each input, we set a specific target, say $1000, and ask if we can design an auction that guarantees us a profit of $1000, *when the bidders can “afford it”*. Formally:

**Definition 3.7.18.** A digital goods profit extractor with parameter $T$, denoted by $pe_T (\cdot)$, is a truthful auction that, given a set of sealed bids $b$ and a target profit $T$, is guaranteed to obtain a profit of $T$ as long as the optimal fixed price profit $OFP(b)$ is at least $T$. If the optimal fixed price profit $OFP(b)$ is less than $T$, there is no guarantee, and the profit extractor could, in the worst case, obtain no profit.

It turns out that such an auction is easy to design:

**Definition 3.7.19 (A Profit Extractor:).** The digital goods profit extractor $pe_T (b)$ with target profit $T$ sells to the largest group of $k$ bidders that can equally share the cost $T$ and charges each $T/k$.

Using Theorem ??, it is straightforward to verify that:

**Lemma 3.7.20.** The digital goods profit extractor $pe_T$ is truthful, and guarantees a profit of $T$ on any $b$ such that $OFP(b) \geq T$.

### 3.7.6 A profit-making digital goods auction

The following auction is near optimal:

**Definition 3.7.21 (RSPE).** The Random Sampling Profit Extraction auction (RSPE) works as follows:

- Randomly partition the bids $b$ into two by flipping a fair coin for each bidder and assigning her to $b'$ or $b''$.
- Compute the optimal fixed price profit for each part: $T' = OFP(b')$ and $T'' = OFP(b'')$.
- Run the profit extractors: $pe_{T'}$ on $b''$ and $pe_{T''}$ on $b'$.

Our main theorem is the following:
Theorem 3.7.22. The Random Sampling Profit Extraction (RSPE) auction is truthful, and for all bid vectors $v$ for which there are at least two values at or above $p^*(v)$, RSPE obtains at least $1/4$ of the optimal fixed profit $OFP(v)$.

Proof. The fact that the RSPE auction is truthful is straightforward since it is simply randomizing over truthful auctions, one for each possible partition of the bids. (Note that any target profit used in step 3 of the auction is independent of the bids to which it is applied.) So we have only to lower bound the profit obtained by RSPE on each input $v$. The crucial observation is that for any particular partition of the bids, the profit of RSPE is at least $\min(T', T'')$. This follows from the fact that if, say $T' \leq T''$, then $OFP(b'') = T''$ is large enough to ensure the success of $pe_{T'}(b')$, namely the extraction of a profit of $T'$.

Thus, we just need to analyze $E(\min(T', T''))$.

Assume that $OFP(b) = kp^*$ has with $k \geq 2$ winners at price $p^*$. Of the $k$ winners in $OFP$, let $k'$ be the number of them that are in $b'$ and $k''$ the
3.7 Common or Interdependent Values

number that are in $b''$. Thus, $T' \geq k'p^*$ and $T'' \geq k''p^*$. Therefore

$$\frac{E(\text{RSPE}(b))}{\text{OFP}(b)} = \frac{E(\min(T', T''))}{kp^*} \geq \frac{E(\min(k'p^*, k''p^*))}{kp^*} = \frac{E(\min(k', k''))}{k} \geq \frac{k/4}{k} = 1/4.$$

The last inequality follows from the fact that, for $k \geq 2$,

$$E(\min(k', k'')) = \sum_{0 \leq i \leq k} \min(i, k - i) \binom{k}{i} 2^{-k} = k \left( \frac{1}{2} - \left( \frac{k - 1}{k} \right)^2 \right) \geq \frac{k}{4}.$$

**Remark.** Notice that if the bidders actually had values i.i.d. from a distribution $F$, then the optimal auction would be to offer each bidder the price $p$ that maximizes $p(1 - F(p))$. Thus, the optimal auction would in fact be a fixed price auction.
3.8 Notes

Much of the material in this chapter is discussed in the auction theory books [Kle99, Kri09, and MM05]. The generalized war of attrition is from [BK99]. Theorem 3.4.2 is from [CH13]. The material on the revelation principle and Myerson’s optimal auction is from Myerson’s classic paper [Mye81]. Another excellent source for the material in this chapter (as well as the previous two) is [Har12]. The material in Section ?? is from Ronen, and the material in Section ?? is from ??.

- Explain non-standard definition for Bayes-Nash mixed equilibrium.
- When don’t have atoms, no need to randomize. d
- Utility is sometimes called surplus.
- F not strictly increasing.
- In characterization of BNE, a(w) is no longer a function only of w when values are not independent. In independent case, winning probability depends on my bid and not on my value.
- Characterization in more general single-parameter settings.

3.8.1 Proof of Theorem 3.5.3

(a) Suppose that it is a Bayes-Nash equilibrium for all bidders to bid $\beta(\cdot,\cdot)$. If bidder $i$ has values $(v,z)$, then he has higher utility bidding $\beta(v,z)$ than $\beta(w,\tilde{z})$, i.e.,

$$va(v,z) - p(v,z) \geq va(w,\tilde{z}) - p(w,\tilde{z})$$ \hfill (3.13)

and similarly

$$wa(w,\tilde{z}) - p(w,\tilde{z}) \geq wa(v,z) - p(v,z)$$ \hfill (3.14)

Adding these two inequalities, we obtain that for all $v,w$ and $z,\tilde{z}$

$$(v-w)(a(v,z) - a(w,\tilde{z})) \geq 0.$$  

Thus, if $v > w$, then for all $z$ and $\tilde{z}$, we have $a(v,z) \geq a(w,\tilde{z})$. Setting $v = w$ in (3.13) and (3.14) shows that $u(v,z) = u(v,\tilde{z})$ for all $z,\tilde{z}$, and hence we can denote it by $u(v)$. The monotonicity of $a(v,z)$ in $z$ follows from the assumption that $\beta(v,z)$ is increasing in $z$ and the fact that the item goes to the highest bidder.

We also have that

$$u(v) = va(v,z) - p(v,z) = \sup_{w,\tilde{z}}\{va(w,\tilde{z}) - p(w,\tilde{z})\}. \hfill (3.15)$$

Since the right hand side of (3.15) is defined for all $v$, we use it to extend $u(v)$ to be a function of all $v$ (still assuming that $w$ is in the support of $V_i$ and $\tilde{z} \in [0,1]$).

By Appendix I (i) and (ii) it follows that $u(v)$ is a convex function of $v$.

We also note that if $F(w_1) = F_-(w_2)$, then $a(w_1,1) = a(w_2,0)$. This follows from the fact that $a(w_1,1) \leq a(w_2,0)$, and hence $\beta(w_1,1) \leq \beta(w_2,0)$. Since the bidders are symmetric, there can be no bids between $\beta(w_1,1)$ and $\beta(w_2,0)$ and hence the allocation probability cannot change. It follows that $u(v)$ is linear with slope $a(w_1,1) = a(w_2,0)$ in the interval $(w_1,w_2)$. Finally, as in the proof of Theorem 3.1.1 we conclude that for all $w$, we have $u'_+(w) \geq a(w,1)$ and $u'_-(w) \leq a(w,0)$. 

Therefore, where $u(v)$ is differentiable (i.e., for all but countably many $v$), we have $a(v, z) = a(v, 0)$. Hence,

$$u'(v) = a(v, 0) = a(v, z)$$

for all $z \in [0, 1]$.

Since a convex function is the integral of its derivative (see Appendix 1: (vi), (vii) and (xi)), we conclude that

$$u(v) - u(0) = \int_0^v a(x, 0)dx.$$

The assumption $p(0, z) = 0$ gives (iii). Finally, since $u(v) = va(v, z) - p(v, z)$ for all $z$, (iii) follows.

(b): For the converse, from condition (iii) it follows that

$$u(v) = \int_0^v a(x, 0)dx$$

whereas

$$u(w, z|v) = va(w, z) - p(w, z) = (v - w)a(w, z) + \int_0^w a(x, 0)dx,$$

whence, by condition (i)

$$u(v) \geq u(w, z|v).$$

\begin{exercises}

3.1 Generalize the equilibrium for the Bulow-Klemperer war of attrition considered in the notes to 4 or more players. Hint: In the equilibrium for 4 players, the player with lowest valuation $v_4$ will drop out at time $c^2 \beta(0, v_4)$.

3.2 Take the case $c = 1$. What is the expected auctioneer revenue with $k$ players uniform 0,1. If my value is $v$, should I enter the auction or not? What if you don’t know ahead of time.

3.3 Show that there is no BIC auction which allocates to the player with the second highest bid (value) in a symmetric setting.

3.4 Consider an auction, with reserve chosen from distribution $G$, and allocation to a random bidder above the reserve. Show that this auction is truthful and that it could have either higher or lower revenue than the Vickrey auction depending on the distribution of reserve price and the number of bidders. How does this reconcile with revenue equivalence?

\end{exercises}
3. Che and Kim, exercise in Krishna

Consider the following two-bidder auction, with two agents whose values are uniform on \([0, 1]\). The low bidder pays the auctioneer \(1/3\) and the high bidder pays the low bidder his bid.

- Find the symmetric Bayes-Nash equilibrium in this auction.
- Show that this auction is not vulnerable to collusion.

3.5 Use revenue equivalence to compute BNE bidding strategies in the first price auction, and in purple price auction for \(k\) items.

3.6 Apply revenue equivalence to compute equilibrium bidding in cricket auction: English auction up to price \(p\) and whoever survives is invited to submit sealed bid for first price auction.

3.7 Consider a two bidder auction where the value of the item to both players is a random variable \(V\), where \(V \sim N(c, \sigma)\), and the signals the bidders receive are noisy versions of \(V\), specifically \(X = V + Z_1\), and \(Y = V + Z_2\) where \(Z_1 \sim N(0, \sigma_1)\) and \(Z_2 \sim N(0, \sigma_2)\). Show that \(\beta(x) = v(x, x)\) is an equilibrium in this auction, where

\[
v(x, y) = \mathbb{E}[V | X = x, Y = y] = c + \frac{\sigma_1^{-2}x + \sigma_2^{-2}y}{\sigma_1^{-2} + \sigma_2^{-2} + \sigma^{-2}}.
\]

Solution:

This fact can be proved by direct calculation:

\[
\mathbb{E}[V | X = x, Y = y] = c + \frac{\int_{-\infty}^{\infty} v e^{-\frac{v^2}{2\sigma_1^2}} e^{-\frac{(x-v)^2}{2\sigma_1^2}} dv}{\int_{-\infty}^{\infty} e^{-\frac{v^2}{2\sigma_1^2}} e^{-\frac{(x-v)^2}{2\sigma_1^2}} dv} = c + \frac{\sigma_1^{-2}x + \sigma_2^{-2}y}{\sigma_1^{-2} + \sigma_2^{-2} + \sigma^{-2}}
\]

An alternative approach is the following (where we take \(c = 0\) for simplicity). Let

\[
W = \mathbb{E}[V | X, Y]
\]

and guess that

\[
W = aX + bY.
\]

We will solve for \(a\) and \(b\) so that

\[
V - W \perp X \quad \text{and} \quad V - W \perp Y.
\] (E3.1) (Recall that \(Z_1 \perp Z_2\) means that \(\mathbb{E}[Z_1 \cdot Z_2] = 0\). Thus, we need

\[
\mathbb{E}[(V - aX - bY)X] = 0 \quad \text{and} \quad \mathbb{E}[(V - aX - bY)Y] = 0.
\]
Indeed with
\[ a = \frac{\sigma_1^{-2}}{\sigma_1^{-2} + \sigma_2^{-2} + \sigma^{-2}} \quad \text{and} \quad b = \frac{\sigma_2^{-2}}{\sigma_1^{-2} + \sigma_2^{-2} + \sigma^{-2}}, \]

(E3.1) holds since
\[
\left(\sigma_1^{-2} + \sigma_2^{-2} + \sigma^{-2}\right)(V - W) = \left(\sigma_1^{-2} + \sigma_2^{-2} + \sigma^{-2}\right)V
- \sigma_1^{-2}(V + Z_1) - \sigma_2^{-2}(V + Z_2)
= \sigma^{-2}V - \sigma_1^{-2}Z_1 - \sigma_2^{-2}Z_2
\]
which is readily checked to be perpendicular to \(V + Z_1 = X\) and \(V + Z_2 = Y\). Perpendicular normal random variables are independent, therefore \(V - W\) is independent of \(X\) and \(V - W\) is independent of \(Y\). Basic properties of Gaussian variables imply that \(\mathbb{E}[V - W|X, Y] = 0\) implying that
\[
\mathbb{E}[V|X, Y] = \mathbb{E}[W|X, Y] + \mathbb{E}[V - W|X, Y] = W.
\]

3.8 Suppose bidders’ valuations are drawn independently from regular distributions. Show that second price auction with eager monopoly reserves generates at least as much revenue as the second price auction with lazy monopoly reserves.

Solution:

Fix a bidder \(i\) and compute the expected price he pays conditioned on the values of the other bidders.

Let
\[
h^L_{-i} := \max_j \{v_j | j \neq i\} \quad \text{and} \quad h^E_{-i} := \max_j \{v_j | j \neq i, v_j \geq r_j\}
\]
Obviously
\[
h^L_{-i} \geq h^E_{-i}.
\]
Consider the following cases:
- \(h^L_{-i} \leq r_i\). Then price offered to bidder \(i\) in both cases is \(r_i\).
- \(h^L_{-i} > r_i\). Expected revenue from bidder \(i\) in VCG-L is
\[
h^L_{-i} (1 - F_i(h^L_{-i}))
\]
Expected revenue from bidder \(i\) in VCG-E is
\[
p^E_i (1 - F_i(p^E_i))
\]
where
\[
p^E_i = \max(r_i, h^E_{-i}).
\]
Since \(r_i \leq p^E_i \leq p^L_i = h^L_{-i}\), the expected profit from \(i\) is higher in VCG-E.
3.9 Let $\mathcal{M}$ be a mechanism for an arbitrary allocation problem. $\mathcal{M}$ takes as input preferences for the agents $v_i(\cdot)$, which specifies the value to agent $i$ for each possible allocation and produces as output a feasible allocation $a^* = f(v_i, v_{-i})$ and payments $p_i$ for each agent $i$. Prove that the mechanism is truthful if and only if it satisfies the following conditions for every $i$:

- The payment does not depend on $v_i(\cdot)$, but only on the alternative $a^*$ selected.
- The mechanism optimizes for each player, that is, for every $v_i(\cdot)$, it holds that $f(v_i, v_i)$ is one of the outcomes $a \in A_{v_{-i}}$, that maximizes $v_i(a) - p_a$. Here $A_{v_{-i}}$ is the set of possible outcomes selected when the other agents submit $v_{-i}$.

3.10 Use the previous exercise to prove the statements made in parts (ii) and (iii) of Example 1.4.9. For part (iv), show that the proposed randomized mechanism is better than any deterministic mechanism.
Appendix 1
Convex functions

We review basic facts about convex functions:

(i) A function \( f : [a, b] \to \mathbb{R} \) is convex if for all \( x, z \in [a, b] \) and \( \alpha \in (0, 1) \) we have
\[
f(\alpha x + (1 - \alpha)z) \leq \alpha f(x) + (1 - \alpha)f(z).
\]

(ii) The definition implies that the supremum of any family of convex functions is convex.

(iii) For \( x < y \) in \([a,b]\) denote by \( S(x, y) = \frac{f(y) - f(x)}{y - x} \) the slope of \( f \) on \([x, y]\). Convexity of \( f \) is equivalent to the inequality
\[
S(x, y) \leq S(y, z)
\]
holding for all \( x < y < z \) in \([a, b]\).

(iv) The inequality in (iii) is also equivalent to \( S(x, y) \leq S(x, z) \) and to \( S(y, z) \leq S(y, z) \). Thus for \( f \) convex in \([a, b]\), the slope \( S(x, y) \) is (weakly) monotone increasing in \( x \) and in \( y \) as long as \( x, y \) are in \([a, b]\). This implies continuity of \( f \) in \((a, b)\).

(v) It follows from (iii) and the mean value theorem that if \( f \) is continuous in \([a, b]\) and has a (weakly) increasing derivative in \((a, b)\) then \( f \) is convex in \([a, b]\). E.g. this applies to \( e^x \).

(vi) The monotonicity in (iv) implies that a convex function \( f \) in \([a, b]\) has an increasing right derivative \( f_+ \) in \([a, b]\) and an increasing left derivative \( f_- \) in \((a, b)\). Since \( f_+(x) \leq f_-(y) \) for any \( x < y \), we infer that \( f \) is differentiable at every point of continuity in \((a, b)\) of \( f_+ \).

(vii) Since increasing functions can have only countably many discontinuities, a convex function is differentiable with at most countably many exceptions. The convex function \( f(x) = \sum_{n \geq 1} |x - 1/n|/n^2 \) indeed has countably many points of nondifferentiability.
(viii) (Supporting lines) If \( f \) is convex in \([a, b]\) then for every \( t \) in \([a, b]\) the straight line \( \ell_t(x) = f(t) + f_+(t)(x - t) \) lies below \( f \) in \([a, b]\). This follows from (iv). Also, \( \ell \) satisfies \( \ell(t) = f(t) \). Thus \( f \) is the supremum of a family of straight lines; recall from (ii) that conversely, any such supremum is convex.

(ix) **Jensens inequality:** If \( f : [a, b] \to \mathbb{R} \) is convex and \( X \) is a random variable taking values in \([a, b]\) then \( f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)] \). (Note that for \( X \) taking just two values, this is the definition of convexity).

**Proof:** Let \( t = \mathbb{E}[X] \) and \( \ell = \ell_t \). Then by linearity of expectation,

\[
\mathbb{E}[f(X)] = \ell(\mathbb{E}[X]) = \mathbb{E}[\ell(X)] \leq \mathbb{E}[f(X)].
\]

(x) Claim: If \( f : [a, b] \to \mathbb{R} \) is convex then it is the integral of its (right) derivative, i.e., for \( t \in (a, b) \) we have

\[
f(t) = f(a) + \int_a^t f_+(x)\,dx.
\]

**Proof:** By translation, we may assume that \( a = 0 \). Fix \( t \in (0, b) \) and consider, for each \( n \), the step functions \( g_n = \sum_{k=1}^n f_+(\frac{k-1}{n})1_{[(k-1)t/n, kt/n]} \) and \( h_n = \sum_{k=1}^n f_+(\frac{k}{n})1_{[(k-1)t/n, kt/n]} \).

Then \( g_n \leq f_+ \leq h_n \) in \((0, t]\) so

\[
\int_0^t g_n \,dx \leq \int_0^t f_+ \,dx \leq \int_0^t h_n \,dx. \tag{1.1}
\]

Monotonicity of slopes yields that

\[
(z - y)f_+(y) \leq f(z) - f(y) \leq (z - y)f_+(z)
\]

for \( y < z \), whence

\[
\int_0^t g_n \,dx \leq f(t) - f(0) \leq \int_0^t h_n \,dx \tag{1.2}
\]

Direct calculation gives that

\[
\int_0^t h_n \,dx - \int_0^t h_n \,dx = [f_+(t) - f_+(0)]t/n
\]

so by (1.1) and (1.2), we deduce that

\[
|f(t) - f(0) - \int_0^t f_+ \,dx| \leq |f_+(t) - f_+(0)|t/n.
\]

Taking \( n \to \infty \) completes the proof.
(xi) For the left derivative, the claim in also holds; consider $f(-x)$ or use the fact that $f_+$ and $f_-$ coincide at all but countably many points.


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