On the Inefficiency of Equilibria in Congestion Games

Jose R. Correa, UAI, Chile

Andreas S. Schulz, MIT

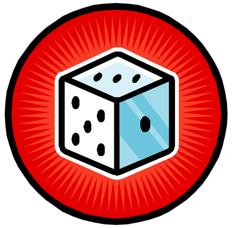
Nicolas E. Stier-Moses, Columbia

Presented by Ophir Setter

Outline

- Nonatomic Congestion Games - Definition
- Separable, Affine Cost Functions
- General Cost Functions
- Pseudo-Approximation (Bicriteria)
- Improved Models
  - Limited Congestion
  - No Fixed Costs
- Nonseparable Costs
- Conclusion

Nonatomic congestion games - definition



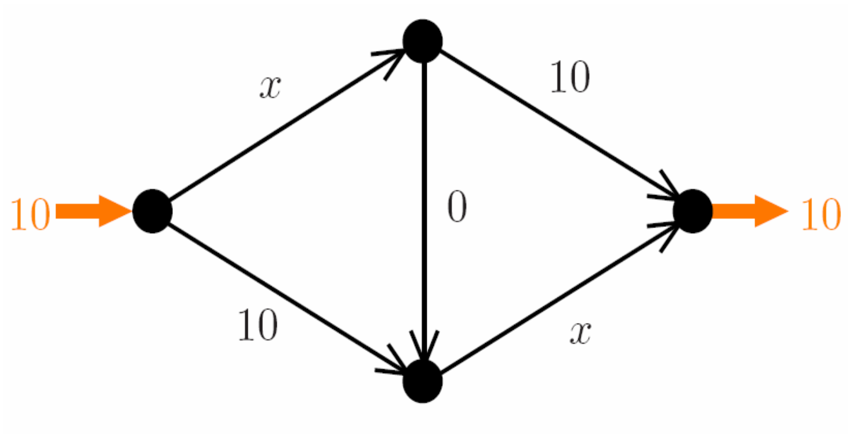
- A finite set of resources A
- Different types of players
- Each player type i has a set of strategies $\mathcal{S}_i$ to choose from
- Each strategy consists of a subset of resources

Nonatomic congestion games - Costs

- The cost of a resource is given by a nondecreasing and continuous function $c_a : \mathbb{R}_+^A \rightarrow \mathbb{R}_+$
- The social cost is defined to be the total cost of players

$$C(x) := \sum_{a \in A} c_a(x) x_a$$

- **Example**: Breass' Instance



Nonatomic congestion games – Nash Equilibria

- Social Optimum x^{OPT} is a strategy distribution of minimum social cost $C(x^{\mathrm{OPT}}) \leq C(x)$
- Nash Equilibrium is a strategy distribution where no player has in incentive to change his strategy

$$c_S(x^{\mathrm{NE}}) \leq c_{S'}(x^{\mathrm{NE}}) \qquad S, S' \in \mathcal{S}_i$$

- Characterized by a Variational Inequality (Smith '79)

$$\boxed{\sum_{a \in A} c_a(x^{NE}) x_a^{NE} \leq \sum_{a \in A} c_a(x^{NE}) x_a}$$

Price of Anarchy

You all know what it is…

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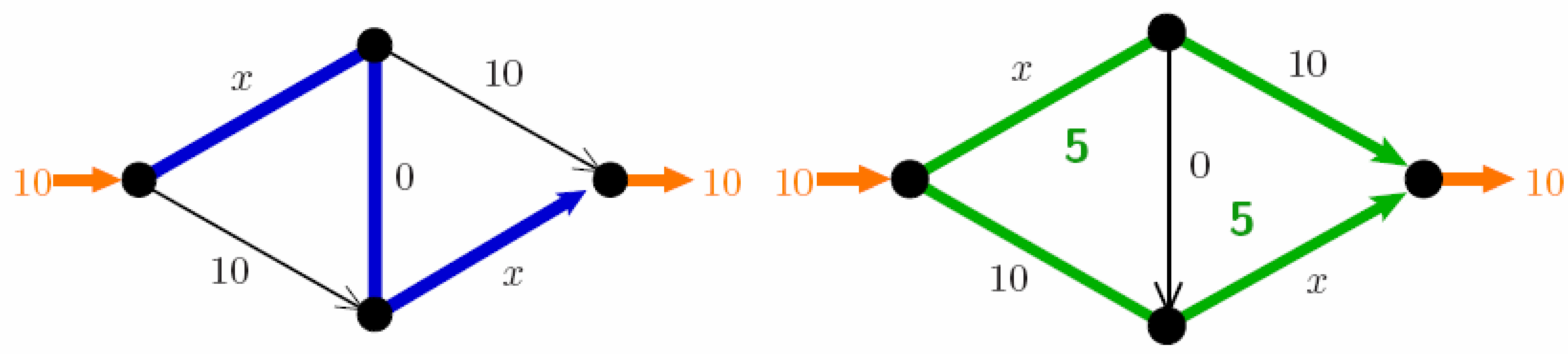
Separable Affine Cost Functions

Theorem (Roughgarden & Tardos 2004): In nonatomic congestion games with separable and affine cost functions

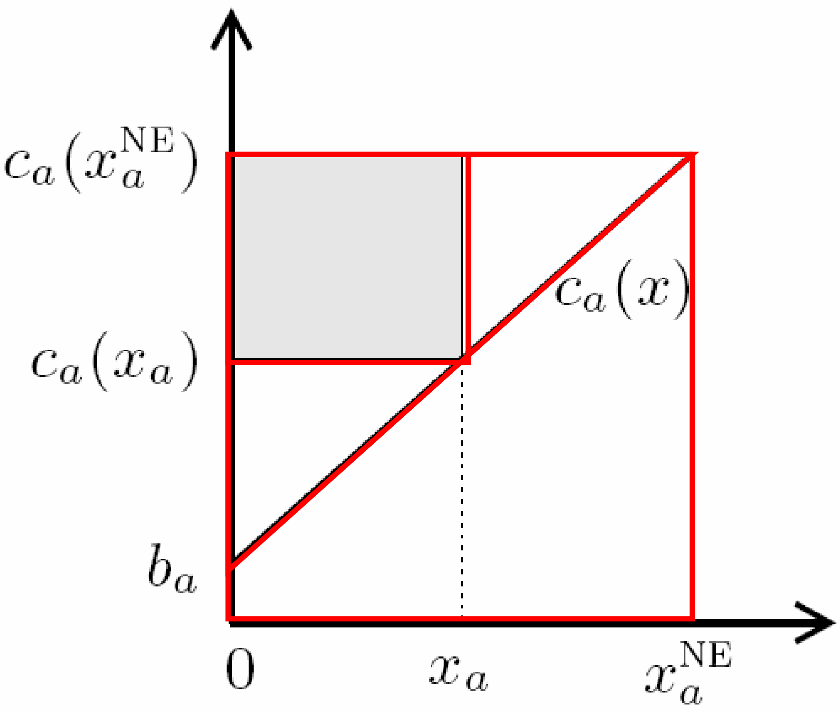
$$C(x^{\mathrm{NE}}) \leq 4/3\, C(x^{\mathrm{OPT}})$$

Corollary: Breass' is a tight bound example



Proof of 4/3

$$C(x^{\mathrm{NE}}) \le \sum_{a\in A} c_a(x_a^{\mathrm{NE}})x_a = \sum_{a\in A} c_a(x_a)x_a + \sum_{a\in A} (c_a(x_a^{\mathrm{NE}}) - c_a(x_a))x_a \le$$



$$C(x) + \frac{1}{4}C(x^{\mathrm{NE}})$$

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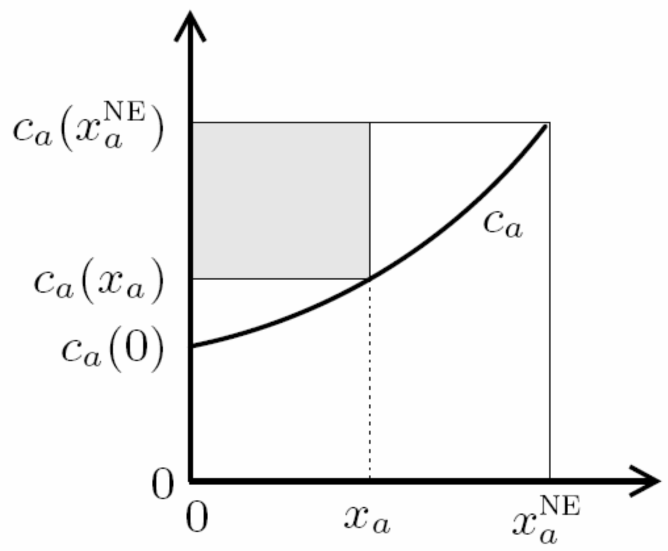
General Separable Functions

- We general previous proofs by introducing for a given class $\mathcal{C}$ of cost functions (polynomials of a certain degree)

$$\beta(\mathcal{C}) := \max \left\{ \frac{\text{shaded area}}{\text{big rectangle}} \right\}$$



Theorem: $C(x^{\text{NE}}) \leq (1-\beta(\mathcal{C}))^{-1} \, C(x^{\text{OPT}})$

Computing β for Polynomials

Assume polynomials with positive coefficients:

For $m \in [0,1]$ $c(mx) \geq m^n c(x)$

$$\beta = \max_{0 \leq x \leq v} \frac{x(c(v) - c(x))}{vc(v)} = \max_{0 \leq x \leq v} \frac{x}{v}\left(1 - \frac{c(x)}{c(v)}\right) \leq \qquad (x = v\frac{x}{v})$$

$$\leq \sup_{0 \leq x \leq v} \frac{x}{v}\left(1 - \left(\frac{x}{v}\right)^n\right) = \sup_{0 \leq x \leq 1} x(1 - x^n) = \frac{n}{(n+1)^{1+1/n}}$$

Computing POA

$$\mathrm{POA} = \left(1 - \frac{n}{(n+1)^{1+1/n}}\right)^{-1}$$

Degree	2	3	4	…	N
POA	1.626	1.896	2.151		Ω(N/ln(N))

- These POA are tight

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Pseudo-Approximation (Bicriteria)

- **Theorem**: If x^{OPT} is the social optimum of the same game with $1 + \beta(\mathcal{C})$ times as many players of each type, then

$$C(x^{\mathrm{NE}}) \leq C(x^{\mathrm{OPT}})$$

- Simple proof for the results of Roughgarden & Tardos '02 '04

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Improved Models

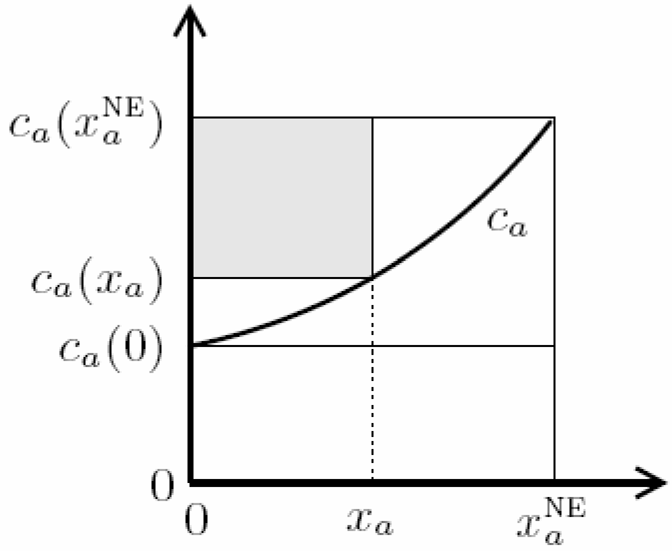
- In traffic networks the ratio between the equilibrium and the social optimum was observed to be significantly smaller in practice

- Divide into two cases:
  - Limited Congestion (Low traffic)
  - No Fixed Costs (Heavy traffic)

Limited Congestion

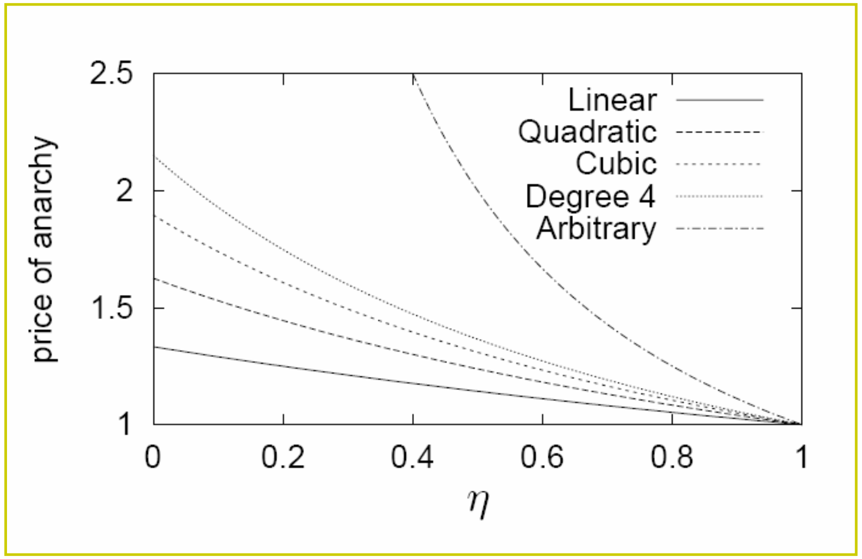
- Assume that:

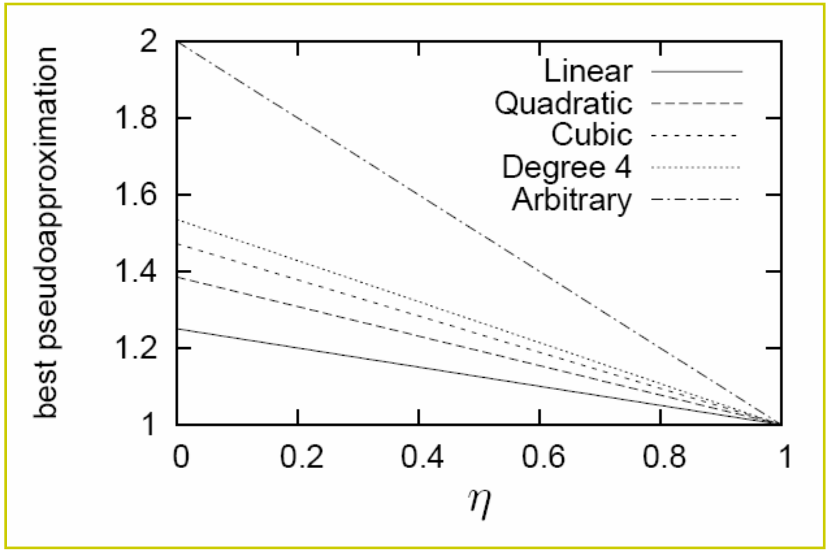
$$c_a(0) \geq \eta\, c_a(x_a^{\mathrm{NE}})$$



- With this assumption we get:
  - $C(x^{\mathrm{NE}}) \leq (1-(1-\eta)\beta(\mathcal{C}))^{-1}\, C(x^{\mathrm{OPT}})$
  - Pseudo-approximation factor: $1+(1-\eta)\beta(\mathcal{C})$

POA & Bicriteria - Limited Congestion





No Fixed Costs

- Assume that: $c_a(0) = 0$

- With that assumption we get:

Set $\mathcal{C}$ of allowable cost functions	Example	Price of Anarchy $\alpha(\mathcal{C})$		
		$c_a(0) = 0$		$c_a(0)$ arbitrary
		LB	UB	
linear functions	$a_1 x + a_0$	1	1	1.334
quadratic functions	$a_2 x^2 + a_1 x + a_0$	1.035	1.185	1.626
cubic functions	$a_3 x^3 + a_2 x^2 + a_1 x + a_0$	1.098	1.25	1.896
polynomials of degree 4	$\sum_{i=0}^{4} a_i x^i$	1.167	1.999	2.151

Example

- We'll proof upper bound of 5/4 for cubic functions
- Notation:

$$C_i(x) := \sum_{a\in A} c_{a,i}\, x_a^{i+1}$$

$$C_i^{x'}(x) := \sum_{a\in A} c_{a,i}\, x_a\, (x'_a)^i$$

$$(\delta x_a - x_a^{\mathrm{NE}})^2 \geq 0 \quad \Rightarrow \quad C_1^x(x^{\mathrm{NE}}) \leq C_1(x^{\mathrm{NE}}) + \tfrac{1}{4} C_1(x)$$

$$C_2^{x^{\mathrm{NE}}}(x) \leq \tfrac{1}{2} C_2^x(x^{\mathrm{NE}}) + \tfrac{1}{2} C_2(x^{\mathrm{NE}})$$

$$(\tfrac{1}{2} x_a^2 - (x_a^{\mathrm{NE}})^2 + x_a\, x_a^{\mathrm{NE}})^2 \geq 0 \quad \Rightarrow \quad C_3^{x^{\mathrm{NE}}}(x) \leq \tfrac{1}{2} C_3^x(x^{\mathrm{NE}}) + \tfrac{1}{2} C_3(x^{\mathrm{NE}}) + \tfrac{1}{8} C_3(x)$$

Example Proof

$$C^{x}(x^{\mathrm{NE}}) \le C(x)$$ (smith '79)

$$C(x^{\mathrm{NE}}) \le C^{x^{\mathrm{NE}}}(x) = C_3^{x^{\mathrm{NE}}}(x) + C_2^{x^{\mathrm{NE}}}(x) + C_1^{x^{\mathrm{NE}}}(x)$$

$$\le \frac{C_3^{x}(x^{\mathrm{NE}})}{2} + \frac{C_3(x^{\mathrm{NE}})}{2} + \frac{C_3(x)}{8} + \frac{C_2^{x}(x^{\mathrm{NE}})}{2} + \frac{C_2(x^{\mathrm{NE}})}{2} + \frac{C_1^{x}(x^{\mathrm{NE}})}{2} + \frac{C_1(x^{\mathrm{NE}})}{2} + \frac{C_1(x)}{8}$$

$$\le \frac{C(x^{\mathrm{NE}})}{2} + \frac{C^{x}(x^{\mathrm{NE}})}{2} + \frac{C(x)}{8} \le \frac{1}{2}C(x^{\mathrm{NE}}) + \frac{5}{8}C(x)$$

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Nonseparable Costs

- Cost is: $$C(x) = \langle c(x), x \rangle \qquad c : X \to \mathbb{R}^A_+$$

- All works fine with:

$$\beta(c, v) := \max_{x \geq 0} \frac{\langle c(v) - c(x), x \rangle}{\langle c(v), v \rangle}$$

$$\beta(\mathcal{C}) := \sup_{c \in \mathcal{C}, v \in X} \beta(c, v)$$

Conclusion

- Worst case maybe too pessimistic
- Insights from real-world networks leads to better bounds

Open Problems

- Better ways to characterize “realistic” networks
- No fixed costs – General approach


THE
END