On the Complexity of Dualization of Monotone Disjunctive Normal Forms

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We show that the duality of a pair of monotone disjunctive normal forms of size \( n \) can be tested in \( n^{(\log n)} \) time.

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1. INTRODUCTION

Let \( f = f(x_1, \ldots, x_N) \) and \( g = g(x_1, \ldots, x_N) \) be a pair of monotone Boolean functions given by their irredundant disjunctive normal forms

\[
\begin{align*}
    f &= \bigvee_{I \in F} \bigwedge_{i \in I} x_i, \\
    g &= \bigvee_{J \in G} \bigwedge_{j \in J} x_j,
\end{align*}
\]

where \( F \) and \( G \) are the sets of the prime (minimal) implicants \( I \), \( J \subseteq \{1, \ldots, N\} \) of \( f \) and \( g \), respectively.

This paper is concerned with the following problem.

Monotone Boolean Duality. Test whether \( f \) and \( g \) are mutually dual:

\[ f(x_1, \ldots, x_N) = g(\bar{x}_1, \ldots, \bar{x}_N) \quad \text{for all } x = (x_1, \ldots, x_N) \in \{0, 1\}^N. \tag{\text{D}} \]

If \( f \) and \( g \) are not dual, find a Boolean vector \( x \in \{0, 1\}^N \) such that

\[ f(x_1, \ldots, x_N) = g(\bar{x}_1, \ldots, \bar{x}_N). \tag{\text{D}^*} \]

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Observe that if there is a pair of disjoint sets $I \in F$ and $J \in G$, then the characteristic vector of $I$ satisfies $(\mathcal{G}^*)$. In other words, any dual forms $f$ and $g$ satisfy the conditions

$$I \cap J \neq \emptyset \quad \text{for any } I \in F \text{ and } J \in G. \quad (1.1)$$

Note also that for any dual irredundant forms we have

$$\cup \{I : I \in F\} = \cup \{J : J \in G\}, \quad \max\{|I| : I \in F\} \leq |G|, \quad \max\{|J| : J \in G\} \leq |F| \quad (1.2)$$

Otherwise Eq. $(\mathcal{G}^*)$ has a trivial solution:

If there is an index $i \in [\cup \{I : I \in F\}] \triangle [\cup \{J : J \in G\}]$, say $i \in I$ for some $I \in F$ and $i \in J$ for all $J \in G$, then the characteristic vector of $I \setminus \{i\}$ satisfies $f(x) = g(x) = 0$.

Suppose that $|I| > |G|$ for some $I \in F$, and let $I'$ be a proper subset of $I$ such that $I' \cap J \neq \emptyset$ for all $J \in G$. Then $f(x) = g(x) = 0$ for the characteristic vector of $I'$. The case for $|I| < |F|$ is completely symmetric.

We shall assume henceforth that the input forms $f$ and $g$ satisfy (1.1), (1.2), and (1.3). In particular, the number of variables in $f$ and $g$ does not exceed $|F| + |G|$. For this reason, the binary length of any standard encoding of $f, g$ is polynomially related to $n = |F| + |G|$, the total number of implicants in $f$ and $g$. We call $n$ the size of $f$ and $g$.

Since the duality of $f(x)$ and $g(x)$ is equivalent to the self-duality of $yf(x) \lor zg(x) \lor yz$, where $y$ and $z$ are two additional Boolean variables, problem $(\mathcal{G}, \mathcal{G}^*)$ can be reduced in linear time to the self-dual case $f = g$ (cf. [16, p. 309]). A nother equivalent formulation of monotone Boolean duality is as follows (see, e.g., [3, 4]): given a monotone irredundant DNF $f$ and a subset $G$ of prime implicants of the dual function

$$f^d(x) = \bar{f}({\bar{x}}) = \bigwedge_{i \in F} \bigvee_{i \in I} x_i = \bigvee_{j \in F^d} \bigwedge_{j \in J} x_j,$$

either prove that $G = F^d$ or find a new prime implicant $J \in F^d \setminus G$.

Generating all prime implicants of $F^d$ is identical to generating all minimal transversals (= hitting sets) or all maximal independent sets in the hypergraph $F$. This problem is of interest for the design and analysis of combinatorial algorithms [11, 12], database theory [14], distributed systems [6, 10], combinatorial optimization [13], game theory [7, 8], artificial intelligence [15], computational learning theory [2], convex programming [8], and some other applications [4, 8].

Duality testing for arbitrary, not necessarily monotone, DNFs is NP-hard: for $g = 0$ the problem is equivalent to testing whether or not $f$ is a tautology. Unlike the general case, the complexity of monotone Boolean
duality (also known as transversal hypergraph or blocking clutter) is an open issue [6, 14, 11, 13, 3, 4]. The purpose of this paper is to show that the problem is unlikely to be NP-hard.

**Theorem 1.** Monotone Boolean duality can be solved in $n^{4\chi(n)+O(1)}$ time, where $\chi(n) = n$.

Note that

$$\chi(n) \sim \log n / \log \log n = o(\log n).$$

Theorem 1 implies that monotone Boolean duality is not NP-hard, unless any NP-complete problem can be solved in quasi-polynomial time $n^{o(\log n)}$. This provides partial evidence for the conjecture that problems reducible to monotone Boolean duality form a class properly between P and co-NP (cf. [13, 3]).

Another consequence of Theorem 1 is that, given a monotone irredundant DNF $f$ and a proper subset $G$ of prime implicants of the dual function $f^d = \bar{f}(\bar{x})$, a new prime implicant of $f^d$ can be generated in incremental time $n^{4\chi(n)+O(1)}$, where $n = |F| + |G|$. In particular, the dual DNF $f^d$ can also be computed in time $n^{4\chi(n)+O(1)}$, where $n$ is the total number of prime implicants in $f$ and $f^d$. Other applications of Theorem 1 are considered in [8].

The remainder of the paper is organized as follows. Sections 2 and 3 present a simple algorithm for monotone Boolean duality whose running time is $n^{O(\log^2 n)}$. In Sections 4 and 5, this algorithm is modified to run within the time bound $n^{4\chi(n)+O(1)}$ stated in Theorem 1.

## 2. Short Implicants and Frequently Occurring Variables

Assume that $f = 0$ for $F = \emptyset$, and $g = 1$ for $G = \{\emptyset\}$, which guarantees that $n \geq 1$ for any pair of mutually dual DNFs $f$ and $g$. We start with the following fact [5] (see also [1, 16]).

**Lemma 1.** Suppose that $f$ and $g$ are mutually dual. Then

$$E = \sum_{F} 2^{-|F|} + \sum_{G} 2^{-|G|} \geq 1. \quad (2.1)$$

**Proof.** Let $x = (x_1, \ldots, x_N)$ be a random Boolean vector uniformly distributed on $(0,1)^N$, and let $l(x) = |\{I \in F : \land_{j \in I} x_j = 1\}| + |\{J \in G : \land_{j \in J} \bar{x}_j = 1\}|$. The left-hand side of (2.1) is the expected value of $l(x)$, i.e., $E = 2^{-N} \sum l(x)$. Asuming $E < 1$, we obtain $l(x^*) = 0$.


for some $x^* \in (0,1)^N$. The latter condition is equivalent to $f(x^*) = g(x^*) = 0$, which shows the solvability of $(\mathcal{D}^*)$.

Note that for $E < 1$, a Boolean solution $x^* \in (0,1)^N$ of Eq. $(\mathcal{D}^*)$ can be found in polynomial time as follows: For $i = 1, \ldots, N$, use (2.1) to compute the expectations of $I(x^*_1, \ldots, x^*_i, 1, x^*_{i+1}, \ldots, x^*_N)$ and $I(x^*_1, \ldots, x^*_i, 0, x^*_{i+1}, \ldots, x^*_N)$ over $(x^*_{i+1}, \ldots, x^*_N) \in (0,1)^{N-i}$, and select the value of $x^*_i \in (0,1)$ so as to minimize the corresponding expectation.

Lemma 1 guarantees that any pair of dual forms $f$ and $g$ contains an implicant of only logarithmic length.

**Corollary 1.** Let $f, g$ be a pair of dual forms of size $n$, and let $m = \min(|I| : I \in F \cup G)$ be the length of a shortest implicant in $f$ and $g$. Then $m \leq \log n$.

**Proof.** The bound follows from $n2^{-m} = (|F| + |G|)2^{-m} \geq E \geq 1$.

Let $\epsilon \in (0,1]$. We say that a variable $x_i \in \{x_1, \ldots, x_n\}$ occurs in $f$ with frequency at least $\epsilon$, if $i \in I$ for at least an $\epsilon$ fraction of the implicants $I \in F$:

$$\epsilon_f = \frac{\# \{I : i \in I\}}{|F|} \geq \epsilon.$$

We shall also say that $x_i$ occurs with frequency $\geq \epsilon$ in a pair of forms $f$ and $g$ if $|F||G| \geq 1$ and $x_i$ occurs with frequency at least $\epsilon$ either in $f$ or in $g$.

**Lemma 2.** Let $f, g$ be a pair of mutually dual forms with $|F||G| \geq 1$. Then there exists a variable that occurs in $f, g$ with frequency $\geq 1/\log n$, where $n$ is the size of $f$ and $g$.

**Proof.** Since $|F||G| \geq 1$, we have $n \geq 2$ and $m \geq 1$. By Corollary 1, $F \cup G$ contains a logarithmically short implicant. Suppose without loss of generality that $m = |J^*| \leq \log n$ for some implicant $J^* \in G$ (otherwise swap $f$ and $g$). By (1.1), each implicant $I \in F$ has a nonempty intersection with $J^*$, and hence at least one of the $|J^*|$ variables $x_i, i \in J^*$, occurs in $f$ with frequency $\geq 1/|J^*|$.

**Remark 1.** The bounds of Corollary 1 and Lemma 2 are tight up to a factor of 2 because there are examples [9] of mutually dual irredudant monotone DNFs $f(x_1, \ldots, x_N)$ and $g(x_1, \ldots, x_N)$ of arbitrarily large size $n$ such that

$$\min\{|I| : I \in F \cup G\} > \frac{\log n}{2}.$$
Specifically, for each natural \( k \), the above inequalities hold for the irredundant DNFs of the dual monotone Boolean functions \( f \) and \( g \) defined by the recurrence 
\[
\begin{align*}
f_k(x_1, x_2) &= x_1 \lor x_2, \\
f_k(x_1, \ldots, x_B) &= (x_1 \lor x_2) (x_3 \lor x_4) \lor \ldots \lor (x_{N(k)-1} \lor x_{N(k)}) \lor f_k(x_{N(k)+1}, \ldots, x_{N(k)}), \\
    
\end{align*}
\] 
where \( N(k) = 2^k - 1 \).

Remark 2. Lemmas 1 and 2 and Corollary 1 hold for arbitrary, not necessarily monotone, disjunctive normal forms.

3. DUALITY TESTING IN \( n^{O(\log n)} \) TIME

We start with the following simple observation. Given a pair of monotone irredundant forms \( f(x_1, \ldots, x_N) \) and \( g(x_1, \ldots, x_N) \) and a variable \( x_i \in \{ x_1, \ldots, x_N \} \), consider the decomposition

\[
f = x_i f_0(y) \lor f_1(y), \quad g = x_i g_0(y) \lor g_1(y),
\]

where \( y = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \) and where \( f_0, f_1, g_0, g_1 \) are the monotone irredundant disjunctive normal forms with implicant sets
\[
F_0 = \{ I : i \in I, I \in F \}, \\
F_1 = \{ I : i \notin I, I \in F \}, \\
G_0 = \{ J : i \in J, J \in G \}, \\
G_1 = \{ J : i \notin J, J \in G \},
\]
respectively. Then

\[
f \text{ and } g \text{ are mutually dual if and only if } f_1 \text{ is dual to } g_0 \lor g_1 \text{ and } g_1 \text{ is dual to } f_0 \lor f_1. \tag{3.1}
\]

Proof. \( f \) and \( g \) are not dual if and only if \( x_i f_0(y) \lor f_2(y) = x_i g_0(\tilde{y}) \lor g_1(\tilde{y}) \) for some \( x \) and \( y \). The solvability of the latter equation is equivalent to the solvability of \( f_3(y) = g_0(\tilde{y}) \lor g_1(\tilde{y}) \) or \( f_2(y) \lor f_3(y) = g_2(\tilde{y}) \).

Algorithm A below recursively uses (3.1) to solve problem \((D, D^*)\) in time \( n^{O(\log n + O(1))} \) for any pair of monotone disjunctive normal forms \( f \) and \( g \) of size \( \leq n \).

Algorithm A.

Input: a pair of monotone disjunctive normal forms \( f \) and \( g \) satisfying the necessary duality condition (1.1).

1. Delete all redundant (nonminimal) implicants from \( F \) and \( G \).
2. Check conditions (1.2), (1.3), and (2.1). If any of these conditions is violated, \( f \) and \( g \) are not dual and Eq. \((D^*)\) can be solved in polynomial time.

3. If \(|F| |G| \leq 1\), the duality of \( f \) and \( g \) can be tested in \( O(1) \) time.

4. If \(|F| |G| \geq 2\), find a variable \( x_i \) that occurs in \( f, g \) with frequency \( \geq 1/\log |F| + |G| \), write \( f = x_i f_0 \lor f_1 \), \( g = x_i g_0 \lor g_1 \), and call A to solve problem \((D, D^*)\) for the two pairs of forms
   \[
   (f_1, g_0 \lor g_1),
   \]
   \[
   (g_1, f_0 \lor f_1),
   \]
   each of which satisfies (1.1). If both pairs (3.2) and (3.3) are dual, then so is \((f, g)\); otherwise, we obtain a solution of \((D^*)\).

Algorithm A either discovers that \( f \) and \( g \) are not dual and solves equation \((D^*)\) or shows the duality of \( f \) and \( g \) by reducing the original problem \((D, D^*)\) to a finite number of small subproblems with \(|F| |G| \leq 1\).

The running time of the algorithm is bounded, up to a factor of \( n^{O(3)} \), by the number \( A(f, g) \) of recursive calls in step 4.

**Lemma 4.** \( A(f, g) \leq n^{4 \log^2 n} \) for any input \( f, g \) of size \( \leq n \).

**Proof.** Since the sizes of subproblems (3.2) and (3.3) are less than that of the original problem, we can assume that, at each step of the algorithm, the splitting variable \( x_i \) occurs in the current pair of forms \( f, g \) with frequency \( \geq \varepsilon = 1/\log n \), where \( n \) is the input size. In our analysis the value of \( \varepsilon \) is “frozen” and we work with the problems of size \( \leq n \). For this class of problems, we bound \( A(f, g) \) in terms of the “volume” \( v = |F| |G| \) of the input \( f, g \).

Suppose that in step 4 of the algorithm \( x_i \) occurs with frequency at least \( \varepsilon \) in \( f \). Then we have

\[
|f_1| = |F_1| \leq |F|(1 - \varepsilon), \quad |g_0 \lor g_1| \leq |G_0 \cup G_1| \leq |G|
\]

for the pair of forms (3.2), and

\[
|f_0 \lor f_1| \leq |F_0 \cup F_1| \leq |F|, \quad |g_1| = |G_1| \leq |G| - 1,
\]

for the pair (3.3). This means that the algorithm divides the original problem of volume \( v = |F| |G| \) into two subproblems of volumes \( |f_1| |g_0 \lor g_1| \leq (1 - \varepsilon)v \) and \( |f_0 \lor f_1| |g_1| \leq |F|(|G| - 1) \leq v - 1 \). By symmetry, this is also true if the splitting variable \( x_i \) occurs with frequency \( \geq \varepsilon \) in \( g \). We have thus arrived at the recurrence

\[
A(v) \leq 1 + A((1 - \varepsilon)v) + A(v - 1), \quad A(1) = 1,
\]
where $A(v)$ is the maximum number of recursive calls of Algorithm A on any input $f, g$ of volume $\leq v$. To evaluate this recurrence, observe that 
\[ A(v - 1) \leq 1 + A((1 - \varepsilon)v - 1) + A(v - 2) \leq 1 + A((1 - \varepsilon)v) + A(v - 2), \]
which gives $A(v) \leq 2 + 2A((1 - \varepsilon)v) + A(v - 2)$. Iterating, we obtain $A(v) \leq k + kA((1 - \varepsilon)v) + A(v - k)$ for any $k \leq v$. Letting $k = [v\varepsilon]$ yields $A(v) \leq (3 + 2\varepsilon)A((1 - \varepsilon)v)$, and hence $A(v) \leq (3 + 2\varepsilon)^\log \varepsilon/v$. Since $v = |F| |G| \leq (|F| + |G|)^2/4 = n^2/4$ and $\varepsilon = 1/\log n$, the lemma follows.

4. Duality Testing in $n^{\log n}$ Time

Algorithm A reduces the duality testing problem $(\mathcal{D}, \mathcal{D}^*)$ for $(f, g)$ to the same problem for pairs of forms (3.2) and (3.3). Subproblems (3.2) and (3.3) are not independent. Suppose that subproblem (3.2) is already solved. Since discovering a Boolean solution to $f_1(y) = g_0(\bar{y}) \lor g_1(\bar{y})$ solves the original problem, we can assume without loss of generality that $f_1$ is dual to $g_0 \lor g_1$, i.e.,
\[ f_1(\bar{y}) = g_0(y) \bar{g}_1(y). \quad (4.1) \]
We can use (4.1) to simplify subproblem (3.3), which calls for computing a Boolean vector $y$ such that
\[ g_1(y) = f_0(\bar{y}) \lor f_1(\bar{y}). \quad (4.2) \]
Substituting (4.1) into (4.2) gives an equivalent equation
\[ g_1(y) = f_0(\bar{y}) \lor \bar{g}_0(y) \bar{g}_1(y). \quad (4.3) \]
It is easily seen that (4.3) has no solution with $g_0(y) = 0$. (Suppose that $g_0(y) = 0$; then (4.3) can be written as $g_1(y) = f_0(\bar{y}) \lor g_1(y)$. The latter equation implies $g_1(y) = 1$ and $f_0(\bar{y}) = 1$, which is impossible by (1.1): every implicant of $f_0$ must have a nonempty intersection with every implicant of $g_1$.) Hence (4.3) is equivalent to the system of two equations
\[ g_1(y) = f_0(\bar{y}), \quad g_0(y) = 1. \quad (4.4) \]
Let $G_0$ be the set of the prime implicants of $g_0$. The solvability of (4.4) is in turn equivalent to the solvability of at least one of the $|G_0|$ equations
\[ g_1(y[J]) = f_0(\bar{y}[J]), \quad (4.5) \]
where \( J \subseteq G_0 \) and \( y[J] \) is the vector obtained by the substitution \( y_j \leftarrow 1 \) for all \( j \in J \). (Accordingly, the \( j \)th component of \( y[J] \) is 0 for \( j \notin J \).) However, each of the \(|G_0|\) equations (4.5) is equivalent to problem \((\mathcal{D}, \mathcal{D}^*)\) for a pair of monotone DNFs \((g_i^d, f_i^d)\), where \( g_i^d \) is obtained from \( g(y) \) by setting \( y_j \leftarrow 1, j \in J \), and \( f_i^d \) is the result of the substitution \( y_j \leftarrow 0, j \in J \) into \( f_0(y) \). Observe that

\[
|G_i| \leq |G_0|, |F_i| \leq |F_0| \quad \text{for any} \quad J \subseteq G_0,
\]

and that \( g_i^d \) and \( f_i^d \) still satisfy (1.1).

We have thus obtained the following decomposition rule:

(i) Let \( f = f(x_1, \ldots, x_N) \) and \( g = g(x_1, \ldots, x_N) \) be a pair of monotone irredundant forms of volume \( v = |F|/|G| \) and let \( x_i \in \{x_1, \ldots, x_N\} \) occur in \( f \) with frequency \( e_i = \#\{i : i \in I\}/|F| \). Then in polynomial time the duality testing problem \((\mathcal{D}, \mathcal{D}^*)\) for \( f = x_i f_0(y) \lor f_i(y) \) and \( g = x_i g_0(y) \lor g_i(y) \) can be decomposed into subproblem (3.2) of volume \(|F_0|/|G| \leq (1 - e_i) |F|/|G| = (1 - e_i^d) v \), plus \(|G_0|\) subproblems (4.5) of volume at most \(|F_0|/|G| = e_i/|F|/|G| \leq e_i v \) each.

The symmetric decomposition rule for \( g \) is as follows:

(ii) In polynomial time problem \((\mathcal{D}, \mathcal{D}^*)\) for \( f \) and \( g \) can be decomposed into subproblem (3.3) of volume \( v \leq (1 - e_i^g) v \) and \(|F_0|\) additional subproblems \((\mathcal{D}, \mathcal{D}^*)\) of volume \( \leq e_i^g v \) each, where \( e_i^g = \#\{J : i \in J\}/|G| \) is the frequency of \( x_i \) in \( g \).

Finally, (3.1) directly implies that

(iii) Problem \((\mathcal{D}, \mathcal{D}^*)\) can also be decomposed into subproblems (3.2) and (3.3) of volumes \( (1 - e_i^s) v \) and \( (1 - e_i^g) v \), respectively.

We are now ready to present an algorithm that solves any problem \((\mathcal{D}, \mathcal{D}^*)\) of volume \( v \) in \( v^{(v^a + \alpha)} \) time by recursively combining decomposition rules (i)–(iii).

**Algorithm B.**

**Input:** a pair of monotone disjunctive normal forms \( f \) and \( g \) satisfying the necessary duality condition (1.1).

1. Delete all redundant implicants from \( F \) and \( G \) and set \( v \leftarrow |F|/|G| \).
2. Check conditions (1.2) and (1.3). If these conditions are violated, \( \text{E.g.} (\mathcal{D}^*) \) can be solved in polynomial time.
3. If \( \min(|F|/|G|) \leq 2 \) problem \((\mathcal{D}, \mathcal{D}^*)\) can be solved in polynomial time.
4. Select any variable $x_i$ such that

$$\varepsilon_i^f = \frac{\#\{i : i \in I\}}{|F|} > 0 \quad \text{and} \quad \varepsilon_i^g = \frac{\#\{i : i \in J\}}{|G|} > 0.$$ 

Comment: The existence of such a variable follows from (1.2).

5. Define

$$\varepsilon(v) = 1/\chi(v), \quad \text{where} \quad \chi^t = v. \quad (4.6)$$

If $\varepsilon_i^f \leq \varepsilon(v)$, apply rule (i) to obtain

$$B(v) \leq 1 + B((1 - \varepsilon_i^f)v) + |G_0|B(\varepsilon_i^f v), \quad (4.7)$$

where $B(v)$ is the maximum number of recursive calls of Algorithm B on any input $f, g$ of volume $\leq v$. Similarly, if $\varepsilon_i^f > \varepsilon(v)$ but $\varepsilon_i^g \leq \varepsilon(v)$, apply rule (ii) to get

$$B(v) \leq 1 + B((1 - \varepsilon_i^g)v) + |F_0|B(\varepsilon_i^g v). \quad (4.8)$$

6. Finally, if $\min(\varepsilon_i^f, \varepsilon_i^g) > \varepsilon(v)$, use rule (iii), which gives

$$B(v) \leq 1 + B((1 - \varepsilon_i^f)v) + B((1 - \varepsilon_i^g)v). \quad (4.9)$$

5. PROOF OF THEOREM 1

**Lemma 5.** Let $B(v)$ be the maximum number of recursive calls of Algorithm B on any input $f, g$ of volume $\leq v$. Then

$$B(v) \leq v^{\chi(v)} \quad (5.1)$$

**Proof.** First note that step 3 of Algorithm B implies $B(v) = 1$ for $\min(|F|, |G|) \leq 2$. In particular, $B(v) = 1$ for $0 \leq v \leq 8$. Assume that

$$\min(|F|, |G|) \geq 3. \quad (5.2)$$

In order to show (5.1) by induction on $v = 8, 9, \ldots$, observe that by (5.2) and (4.7)

$$B(v) \leq 1 + B((1 - \varepsilon_i^f)v) + |G_0|B(\varepsilon_i^f v)$$

$$\leq 1 + B((1 - \varepsilon_i^f)v) + |G|B(\varepsilon_i^f v)$$

$$= 1 + B((1 - \varepsilon_i^f)v) + \frac{v}{|F|}B(\varepsilon_i^f v)$$

$$\leq 1 + B((1 - \varepsilon_i^f)v) + \frac{v}{3}B(\varepsilon_i^f v)$$

$$\leq B((1 - \varepsilon_i^f)v) + \frac{v}{2}B(\varepsilon_i^f v).$$
In other words, if \( \varepsilon_1 \leq \varepsilon(v) \) and the algorithm uses (i) in step 5, then
\[
B(v) \leq B((1 - \varepsilon) v) + \frac{\varepsilon}{2} B(\varepsilon v)
\text{ for some } \varepsilon \in (0, \varepsilon(v)].
\] (5.3)

By induction,
\[
B((1 - \varepsilon) v) + \frac{\varepsilon}{2} B(\varepsilon v) \leq [(1 - \varepsilon)v]^{x(1-\varepsilon) v} + \frac{\varepsilon}{2}[\varepsilon v]^{x(\varepsilon) v}
\leq [(1 - \varepsilon)v]^{x(v)} + \frac{\varepsilon}{2}[\varepsilon v]^{x(v)}
= \psi(\varepsilon)v^{x(v)},
\] (5.4)
where the last inequality follows from the monotonicity of \( x(\cdot) \) and where
\[
\psi(\varepsilon) = (1 - \varepsilon)^{x(v)} + \frac{\varepsilon}{2}x(\varepsilon).
\]

Since \( \psi(\varepsilon) \) is convex in \( \varepsilon \), it follows that \( \psi(\varepsilon) \leq \max(\psi(0), \psi(\varepsilon(v))) \) for any \( \varepsilon \in (0, \varepsilon(v)] \). However, \( \psi(0) = 1 \), and
\[
\psi(\varepsilon(v)) = (1 - \varepsilon(v))^{x(v)} + \frac{\varepsilon}{2}x(v)
= \left( 1 - \frac{1}{x(v)} \right)^{x(v)} + \frac{\varepsilon}{2} \left( \frac{1}{x(v)} \right)^{x(v)}
= \left( 1 - \frac{1}{x(v)} \right)^{x(v)} + \frac{\varepsilon}{2} \leq \frac{1}{e} + \frac{\varepsilon}{2} < 1
\]
(see (4.6)). Hence \( \psi(\varepsilon) \leq 1 \) for all \( \varepsilon \in (0, \varepsilon(v)] \), which along with (5.3) and (5.4) completes the inductive proof of (5.1) for the case \( \varepsilon_1 \leq \varepsilon(v) \) (see step 5 of Algorithm B). By symmetry, (5.1) also holds for \( \varepsilon_2 \leq \varepsilon(v) \).

It remains to show (5.1) under the assumption \( \min(\varepsilon_1, \varepsilon_2) \geq \varepsilon(v) \); see step 6 of the algorithm. By (4.9),
\[
B(v) \leq 1 + B((1 - \varepsilon_1) v) + B((1 - \varepsilon_2) v) \leq 1 + 2B((1 - \varepsilon(v)) v).
\]

Using the monotonicity of \( x(\cdot) \), we obtain by induction on \( v = 8, 9, \ldots \)
\[
1 + 2B((1 - \varepsilon(v)) v) \leq 1 + 2[(1 - \varepsilon(v)) v]^{x(1-\varepsilon(v)) v}
\leq 1 + 2[(1 - \varepsilon(v)) v]^{x(v)}
= 1 + 2\left( 1 - \frac{1}{x(v)} \right)^{x(v)}
\leq 1 + \frac{2}{e} x(v) \leq v x(v),
\]
and (5.1) follows. \( \blacksquare \)
Since \((2^n)^{2^n} > 2^n\), we have \(\chi(n) < 2^n\). Now from
\[ v = |F| + |G| \leq (|F| + |G|)^2/4 = n^2/4 < n^2 \]
there follows that \(\chi(v) < \chi(n^2) < 2\chi(n)\). Hence by Lemma 5 the number of recursive calls of Algorithm B
on any input \(f, g\) of size \(n\) does not exceed \(v^{\chi(v)} < n^5\). This shows the
bound stated in Theorem 1.

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REFERENCES

2. N. H. Bshouty, R. Cleve, S. Kannan, and C. Tamon, Oracles and queries that are
sufficient for exact learning, in “Proceedings of the 7th Annual ACM Conference on
3. J. C. Bioch and T. Ibaraki, Complexity of identification and dualization of positive
4. T. Eiter and G. Gottlob, Identifying the minimal transversals of a hypergraph and related
6. H. Garcia-Molina and D. Barbara, How to assign votes in a distributed system, J. Assoc.
8. V. Gurvich and L. Khachiyan, “Generating the Irredundant Conjunctive and Disjunctive
Variable in Dual Monotone DNFs,” Technical Report LCSR-TR-252, Department of
on Parallel and Distributed Processing, Dallas, TX, 1991,” pp. 150–157.
12. E. Lawler, J. K. Lenstra, and A. H. K. Rinnoy Kan, Generating all maximal independent
14. H. Mannila and K.-J. Räihä, Design by example: An application of Armstrong relations,