FINDING A WALRASIAN EQUILIBRIUM IS EASY FOR A FIXED NUMBER OF AGENTS

FEDERICO ECHENIQUE AND ADAM WIERMAN

Abstract. In this work, we study the complexity of finding a Walrasian equilibrium. Our main result gives an algorithm which can compute an approximate Walrasian equilibrium in an exchange economy with general, but well-behaved utility functions, in time that is polynomial in the number of goods when the number of agents is held constant. This result has applications to macroeconomics and finance, where applications of Walrasian equilibrium theory tend to deal with many goods but a fixed number of agents.

1. Introduction

The problem of computing Walrasian equilibria has been studied over multiple decades. The earliest studies include Scarf (1977) and Todd (1976); but the literature has been reinvigorated by the recent efforts of the computer science community. Some recent surveys include Vazirani (2007), Codenotti and Varadarajan (2007), and Codenotti, Pemmaraju, and Varadarajan (2004).

The basic message that has emerged from the literature is negative: computing a Walrasian equilibrium tends to be "hard" for general settings. For example, Chen, Dai, Du, and Teng (2009) prove that finding a Walrasian equilibrium is hard (PPAD-complete and hard to approximate) even in economies with piece-wise linear concave utilities that are separable by goods. Similarly, Codenotti, Saberi, Varadarajan, and Ye (2006) prove that the problem is also hard in economies in which agents have Leontief preferences.[1]

We thanks Chris Umans for his advice on our paper. Echenique is affiliated with the Division of the Humanities and Social Sciences, California Institute of Technology; Wierman is affiliated with Computers and Mathematical Sciences, California Institute of Technology; emails: `fede,adamw@caltech.edu` .

[1]We emphasize that Codenotti, Saberi, Varadarajan, and Ye (2006) reduce arbitrary two-player games, in which one player has n strategies and the other has m strategies, into an economy with $n+m$ agents and goods. Thus they consider problems in which both goods and agents grow at the same rate.

In contrast, for more limited settings, there exist positive results. In particular, there are studies presenting computationally tractable instances of Walrasian equilibrium, but they require very special assumptions on the markets in question. For example, the results assume that the economies satisfy the gross substitutes condition, or that all utilities are linear, or of the CES functional form (with certain values of the CES parameter) (Devanur, Papadimitriou, Saberi, and Vazirani, 2002; Codenotti, Pemmaraju, and Varadarajan, 2005; Jain, 2004).[2] Other results focus on economies with a representative agent, through the device of a Fisher equilibrium (Jain and Vazirani, 2007). These results are deep and ingenious, but the assumptions placed on their models leave out most economic applications of general equilibrium theory.

As is evident from the above, there is a gap between the generality of the hardness results and the specificity of the instances for which finding an equilibrium is tractable. The goal of the current work is to find a middle ground where computing a Walrasian equilibrium is tractable, but which can still capture settings that include economic applications of the theory.

To that end, our focus is on the setting of many goods but a fixed number of agents. Our main result (Theorem 1) exhibits an algorithm for finding an approximate Walrasian equilibrium that runs in polynomial-time in the number of goods when the number of agents is fixed. Importantly, our result applies to general, but well-behaved, utility functions (see Section 2.1) and to settings where there are infinitely many agents but where the number of "types" of agents is fixed (Section 3).

The restriction to a fixed number of agents still allows many applications in economics. Specifically, general equilibrium theory is used very heavily in macroeconomics and finance. In fact, currently, most applications of the theory are in these two areas. In these settings, the theory is applied to large economies, in the sense of having infinitely many goods, but the number of agents remains fixed. This is because, in macroeconomic and financial applications, the time horizon is usually infinite, which implies that there are infinitely many different goods. Additionally, there is often uncertainty, which gives rise to an infinite dimensional commodity space. On the other hand, the models typically assume finitely many (long-lived) agents or types of agents.

[2]Segal (2007) presents positive results on the communication complexity of equilibria.

We emphasize macro and finance *models* because it is hard to doubt the importance of the computational complexity of a model; and researchers in these fields very often need algorithms to find equilibria. The relevance of complexity for actual economies, and for economics, is not guaranteed, as we explain elsewhere (Echenique, Golovin, and Wierman, 2011).

Macro and finance aside, one may still be interested in economies with both a large number of goods and agents. Our results are still applicable, as long as the heterogeneity of the agents is limited. Perhaps the most interesting limitation of heterogeneity is an economy with "endowment classes:" agents are fully heterogeneous in their preferences but not in their sources of income. In such a "class society," endowments can be classified into a fixed number of types, and as a consequence agents belong to different classes depending on how they derive their incomes. If, additionally, utilities are homothetic, we can aggregate consumers into a representative consumer for each class, and thus effectively assume that there are fixed number of agents. We explain in the paper how this procedure generates the model of Fisher equilibria, which has received attention recently in the literature on computing equilibria (as discussed above). As a consequence, our algorithm finds Fisher equilibria efficiently.

The closest papers in the literature to ours are Deng, Papadimitriou, and Safra (2002) and Devanur and Kannan (2008), who show that the problem of finding Walrasian equilibrium is easy when the number of agents is bounded and utilities are, respectively, linear or piecewise linear and separable by goods. Our result allows for general concave and non-separable utilities, and rules out boundary solutions to the consumers' maximization problem (see the discussion on page 5). The separability by goods means that marginal utilities can be treated independently. This property is exploited in an algorithm to find equilibria. Our approach is different, using a combinatorial version of the Negishi approach to proving existence of Walrasian equilibria.

2. Main result

We study the standard model of an exchange economy with n agents and l goods. In such a model, agents are endowed with non-negative quantities of each good; they derive income from selling their endowments at the prevailing prices; and use the income to purchase a consumption bundle. In a model of an exchange economy, all economic activity reduces to pure barter, and there is no production of new goods. Our results, however, extend naturally to economies with production.

We would simply need to assume that firms' production technologies are convex.

Before discussing our result, we introduce the basic notation and definitions for the model.

2.1. Basic definitions.

We adopt the following notational conventions: $\mathbf{R}^m_+$ denotes the positive orthant of the m-dimensional Euclidean space; $\mathbf{R}^m_{++}$ is the set of vectors of $\mathbf{R}^m$ that are strictly positive in all its components. We use the norm defined as $\|x\| = \sup_i |x_i|$, and understand the distance between two vectors x and y to be $\|x - y\|$.

A function $u : \mathbf{R}^m_+ \to \mathbf{R}$ is C^1 if there is an open set $U \supseteq \mathbf{R}^m_+$ and a function $f : U \to \mathbf{R}$ that is differentiable and has a continuous derivative, such that f and u coincide on $\mathbf{R}^m_+$. The following definitions will be useful: Say that u is

- *monotonic* if $x \leq y$ and $x \neq y$ imply that $u(x) < u(y)$;
- *(strictly) quasiconcave* if for any $x, y \in \mathbf{R}^m_+$, $x \neq y$ and $\alpha \in (0,1)$, $u(x) \leq u(y) \Rightarrow u(x)(<) \leq u(\alpha x + (1-\alpha)y)$;
- *concave* if for any $x, y \in \mathbf{R}^m_+$, $x \neq y$ and $\alpha \in (0,1)$, $\alpha u(x) + (1-\alpha)u(y) \leq u(\alpha x + (1-\alpha)y)$;
- *homothetic* if for any $x, y \in \mathbf{R}^l_+$, $u(y) = u(x)$ implies that $u(\alpha y) = u(\alpha x)$, for all scalar $\alpha \in \mathbf{R}_+$.

An *exchange economy* is a tuple $(\omega_i, u_i)_{i=1}^n$ where $\omega_i \in \mathbf{R}^l_+$ and $u_i : \mathbf{R}^l_+ \to \mathbf{R}$. The number l is the number of goods in the economy. The number of agents is n, and each one is characterized by two objects: a vector of *endowments* ω_i, and a *utility function* u_i.

An *allocation* in $(\omega_i, u_i)_{i=1}^n$ is a vector $x \in \mathbf{R}^{nl}_+$ for which $\sum_{i=1}^n x_i = \sum_{i=1}^n \omega_i$.

A *Walrasian equilibrium* in $(\omega_i, u_i)_{i=1}^n$ is a pair (p, x) where

(1) $p \in \mathbf{R}^l_{++}$, i.e., a price vector,
(2) $x = (x_i)_{i=1}^n \in \mathbf{R}^{nl}_+$ is an allocation, i.e., supply equals demand,
(3) and, for all i, $p \cdot \omega_i = p \cdot x_i$, and

$$u_i(y) > u_i(x_i) \Rightarrow p \cdot y > p \cdot x_i,$$

i.e., agents maximize utility when consuming x_i.

2.2. Main result.

The main result of this paper is to present an algorithm that computes an approximation of a Walrasian equilibrium in time that is polynomial in the number of goods when the number of agents is fixed.

Before presenting the result, we first describe what we mean by an approximation of a Walrasian equilibrium. Then we describe the assumptions that we impose on exchange economies, and state our results.

An approximate equilibrium consists of a price and an allocation in which agents are utility maximizing, supply equals demand, and agents' expenditures are approximately equal their incomes. Formally, A *Walrasian ε-equilibrium* is a pair (p, x) where $p \in \mathbf{R}^l_+$, x is an allocation, and for all i

$$u_i(y) > u_i(x_i) \Rightarrow p \cdot y > p \cdot x_i$$

and $|p \cdot \omega_i - p \cdot x_i| < \varepsilon$. One can, instead, desire the approximation to imply ε-maximization, or that expenditure equals income but supply is only approximately equal to demand. As we explain in Section 2.3, our approach is useful for those kinds of approximations as well.

The class of exchange economies that we consider is defined as follows. Let E be a family of exchange economies. Each economy $(u_i, \omega_i)_{i=1}^n$ in E has the same number of agents, n. They may differ in the number of goods. We assume that all economies in E satisfy the following conditions:

(1) (all goods exist) $\sum_{i=1}^n \omega_i \in \mathbf{R}^l_{++}$;
(2) (regular utilities) u_i is C^1, concave, and strictly monotonic;
(3) (boundary condition) If $x \in \mathbf{R}^l_+$, $y \in \mathbf{R}^l_{++}$, and for some good s, $x_s = 0$, then $u(x) < u(y)$;
(4) (normalization) for all $x \in \mathbf{R}^{nl}_+$ such that $\sum_{i=1}^n \omega_i = \sum_{i=1}^n x_i$, $u_i(x) \in [0, 1]$.

Our assumptions on E deserve some discussion. The assumptions placed on utilities are ubiquitous in economic models. Many general equilibrium models assume that utilities are regular, and an "Inada" condition equivalent to our boundary condition: the role of this is assumption is to rule out that agents' optimal consumption choice has zero consumption of some goods. The normalization assumption puts a uniform bound on utilities evaluated within an allocation.[3] Concavity is stronger than the assumption of quasiconcavity in textbook treatments of general equilibrium theory, but still commonly assumed in applications. Concavity is a requirement of our analysis in two ways: First, concavity is required by the Negishi approach for proving existence, which is the basis of our analysis. Second, we also use concavity to bound the degree of approximation in our algorithm away from the number of goods. It is fair to say that our assumptions coincide with

[3]Note that this bound is not global, only over the bundles achieved in some allocation.

the hypotheses of the Second Welfare Theorem (a crucial ingredient for us), with a strengthening of quasiconcavity to concavity of utilities.

We are now ready to state our main result.

Theorem 1. *Let $\varepsilon > 0$. There is an algorithm that, for any economy in E, finds a Walrasian ε-equilibrium in time polynomial in l when n is fixed.*

2.3. **Other notions of approximate equilibrium.** It is important to note that approximations of Walrasian equilibria have been studied before. The recent literature on CS and economics focuses, understandably, on approximate equilibria since exact equilibria can only have approximate discrete analogues. The older literature on general equilibrium theory in economics also looks at approximations of equilibria, for completely different reasons. Our notion of Walrasian ε-equilibrium is somewhat different from the notions studied before and so it is important to relate it to those notions.

We now show how, with a small modification, our algorithm finds approximations that are similar to the ones studied in the CS/Econ literature, as well as in the general equilibrium literature from the 60s and 70s.

The first definition of approximation seeks to relax how exactly consumers optimize. It imposes ε-maximization of utility: Let $\varepsilon > 0$. An *ε-approximate equilibrium* in an exchange economy $(u_i, \omega_i)_{i=1}^n$ is a pair (p, x) where $p \in \mathbf{R}^l_+$, x is an allocation, and for all i

$$p \cdot y \leq p \cdot \omega_i \Rightarrow u_i(y) \leq u_i(x_i) + \varepsilon,$$

and $|p \cdot \omega_i - p \cdot x_i| < \varepsilon$.

The notion of ε-approximate equilibrium is close to the one studied in Deng, Papadimitriou, and Safra (2002). They require that the consumers are ε-maximizing utility when consuming the bundles mandated by the equilibrium, but they assume that demand is only approximately equal to supply. In our definition, demand is exactly equal to supply. On the other hand, in our definition consumers are only approximately spending their incomes.[4]

If we relax the requirement that demand equals supply, we obtain the following notion of approximate equilibrium, which is similar in spirit (but stronger) to the one in Starr (1969).

An *strong ε-approximate equilibrium* in an exchange economy $(u_i, \omega_i)_{i=1}^n$ is a pair (p, x) where $p \in \mathbf{R}^l_+$, $x \in \mathbf{R}^{nl}_+$ with $\| \sum_i x_i - \sum_i \omega_i \| < \varepsilon$, and

[4]Approximate budget exhaustion is assumed in part of the literature on core convergence, see for example Anderson (1978).

for all i

$$p \cdot y \leq p \cdot \omega_i \Rightarrow u_i(y) \leq u_i(x_i),$$

and $p \cdot \omega_i = p \cdot x_i$.

We need to impose an additional assumption on the economies in E. Suppose that there is $\Theta > 0$ and $\pi > 0$ such that, for all $(u_i, \omega_i)_{i=1}^n$ in E,

$$\sup_{p \in \Delta} \left(p \cdot \sum_{i=1}^n \omega_i \right) \leq \Theta,$$

and if x is an allocation in $(u_i, \omega_i)_{i=1}^n$, then $D_s u_i(x_i) > \pi$. The first component of the assumption simply rules our arbitrarily large endowments. The number of different goods in the economies in E may grow, but the total "mass" in the economy must remain bounded. When endowments are bounded, it is easy to see that marginal utilities must be bounded below (using condition 3 in the definition of E). We require that the bound, π, be uniform across the economies in E.

The role of the bound Θ is to control how small the welfare weights in an economy may be (see Lemma 7 in the proof of Theorem 2). Using both bounds, Θ and π, we can also control how small prices can be in equilibrium. These two magnitudes: the bounds on welfare weights and prices, lie behind the following result.

Theorem 2. *Let $\varepsilon > 0$. There is an algorithm that, for any economy in E, finds an ε-approximate equilibrium, and a strong ε-approximate equilibrium, in time polynomial in l.*

3. Applications to replica economies, representative consumers, endowment classes, and Fisher equilibria

We have presented our main result (Theorem 1) in a context with a fixed number of agents, but it also has applications more generally. As we emphasized in the introduction, many economic models assume that there are many goods, but a fixed number of agents. It is nonetheless very interesting to study economies with a large number of agents as well as goods. In this section we limit agent heterogeneity in ways that are a bit more subtle than assuming there is a fixed number of them. But, in all these cases, we show that when agent heterogeneity is limited, finding a Walrasian equilibrium is easy.

First, and most immediately, our result applies directly to "replica economies:" these are economies with many agents, where each agent is a copy of some prototypical (small) set of types of agents. Replica economies is one of the most important model of large economies in economics. It plays a fundamental role in results on core convergence.[5] For our purposes, however, the model is more restrictive than we need for our result to apply. Next, we adopt at the (arguably realistic) idea that agents may be fully heterogeneous in their preferences, but not in their endowments.

In our second class of models, we discuss the existence of a representative consumer. We give a sufficient condition for an economy to admit a representative consumer, and observe that our result immediately gives an algorithm for approximating Walrasian equilibria. Clearly, representative consumers only exist under very stringent assumptions, but there are nevertheless many important models in economics that assume their existence (famously, many models in macroeconomics assume a representative consumer).

The most important instance of representative consumers may correspond to Fisher equilibria. Some of the most positive recent results on the complexity of economic models have been related to Fisher equilibria (Jain and Vazirani (2007)). We show that the results on Fisher equilibria are a special case of our result, the reason being that their models are essentially models of a representative consumer.

In third place, we assume that an economy may have many agents but that each of them belongs to one of a small number of "endowment classes." For example, some agents are laborers (endowed with labor), while others are endowed with land, and others with capital. If we can partition agents into a small number of classes, then—together with a strengthening of our assumption on preferences—we can in effect work with a model with a small number of agents, even if the actual number is large. The idea we are trying to exploit is that consumers may be heterogeneous in their preferences, but not in their sources of income. Under homotheticity, we can invoke some classical aggregation theorems to effectively work with a small set of agents.

3.1. **Replica economies.** We present a direct application of our result to a model with an unbounded number of agents and replica economies. The main purpose of this section is to flesh out the idea that, in a world with limited heterogeneity on the part of agents, approximate Walrasian equilibria may be easy to find.

Consider an exchange economy $(\omega_i, u_i)_{i=1}^n$ under the assumptions we established in Section 2. The Kth *replica* of $(\omega_i, u_i)_{i=1}^n$ is the exchange economy $(\omega_{i,k}, u_{i,k})_{i=1,\ldots,n,k=1,\ldots,K}$ where for all i, j and k we have that

[5]See Chapter 18 in MasColell, Whinston, and Green (1995). The theorem by Debreu and Scarf is one of the most beautiful results in general equilibrium theory; see Debreu and Scarf (1963).

$\omega_{ik} = \omega_{jk}$ and $u_{ik} = u_{jk}$. In a replica economy, each agent is indexed by a pair of numbers i and k: i denotes the "type" of agent and k denotes the "serial number" of the agent, among those of his type.

It is trivial to verify that a Walrasian ε-equilibrium of $(\omega_i, u_i)_{i=1}^n$ is also a Walrasian ε-equilibrium of $(\omega_{i,k}, u_{i,k})_{i=1,\ldots,n,k=1,\ldots,K}$, for any k.[6]

Our algorithm gives an approximate Walrasian equilibrium of $(\omega_i, u_i)_{i=1}^n$. This is also an approximate Walrasian equilibrium of $(\omega_{i,k}, u_{i,k})_{i=1,\ldots,n,k=1,\ldots,K}$, for any K.

3.2. **Representative consumers and Fisher equilibria.** Consider a collection of n agents. We shall assume that each one of them has a continuous, strictly monotone, and strictly quasiconcave utility function u_i. Let $d_i(p, m)$ denote the solution to the problem of maximizing u_i over $x_i \in \mathbf{R}^l_+$ such that $p \cdot x_i \leq m$; the function d_i is the *demand function* generated by u_i.

Theorem (Samuelson's Aggregation Theorem). *Let $W : \mathbf{R}^n \to \mathbf{R}$ be strictly increasing. If, for every $p, \omega \in \mathbf{R}^l_{++}$, $\delta_1^*(p,\omega), \ldots, \delta_n^*(p,\omega)$ solves the problem*

$$\max_{\delta \in \Delta} W\left(u_1(d_1(p, p \cdot \delta_1 \omega)), \ldots, u_n(d_n(p, p \cdot \delta_n \omega))\right);$$

then there is a continuous, strictly monotonic and concave function u, generating a demand function d such that

$$d(p, p \cdot \omega) = \sum_{i=1}^n d_i(p, p \cdot \delta_i^*(p,\omega)\omega),$$

for all $p, \omega \in \mathbf{R}^l_{++}$. Further, u takes the form

$$u(x) = \begin{array}{l} \max W(u_1(x_1), \ldots, u_n(x_n)) \\ s.t. \ \sum_i x_i = x. \end{array}$$

Corollary 3. *Fix $\alpha_1, \ldots, \alpha_n \in \Delta$ and suppose that u_i is homothetic, in addition to the previously made assumptions. Then the utility function u defined by*

$$u(x) = \begin{array}{l} \max \Pi(u_i(x_i))^{\alpha_i} \\ s.t. \ \sum_i x_i = x. \end{array}$$

generates a demand function d such that

$$d(p, p \cdot \omega) = \sum_{i=1}^n d_i(p, p \cdot \delta_i^*(p,\omega)\omega),$$

[6]The mode is used to establish that the core of the replicated economies is, in the limit, equal to the set of Walrasian equilibrium allocation of the original economy $(\omega_i, u_i)_{i=1}^n$.

for all $p, \omega \in \mathbf{R}^l_{++}$.

Given a set of n agents, with continuous, homothetic, monotone and strictly concave utility functions u_i, we can define a *Fisher equilibrium* as follows. Suppose there is 1 units of each of l goods, and that each agent i is endowed with α_i units of "money" (unit of account). Let $1 = \sum_i \alpha_i$. Then a *Fisher equilibrium* is a vector of prices $p \in \Delta$ such that

$$\sum_i d_i(p, \alpha_i) = (1, \ldots, 1)$$

(that is, supply equals demand).

Now, a Fisher equilibrium is a special case of a Walrasian equilibrium in a model in which there is a representative consumer. By Corollary 3, if $\omega = (1, \ldots, 1)$ then a price vector p is a Fisher equilibrium if and only if it is a Walrasian equilibrium in the economy with a single (representative) consumer, with utility function u as defined in Corollary 3. Since this is an economy with a single consumer, our Theorem 1 delivers a fast algorithm for approximating Walrasian equilibria.

Corollary 3 is a result due to Eisenberg (1961), and used in the literature on Fisher equilibria. The connection between Fisher equilibria and the Negishi approach is already remarked upon in Codenotti and Varadarajan (2007).

3.3. **Fixed endowment classes.** Consider a collection E of exchange economies $(\omega_i, u_i)_{i=1}^n$. Suppose now that both n and the number of goods can differ among the members of E. Assume that utilities and endowments satisfy all the assumptions of Section 2, and that in addition utilities are homothetic.

We limit the heterogeneity among agents in E by limiting how endowments differ among agents. For an exchange economy $(\omega_i, u_i)_{i=1}^n$, an *endowment class* is a set $P \subseteq \{1, \ldots, n\}$, together with vectors $\omega \in \mathbf{R}^l_+$ and $(\alpha_i)_{i \in P}$ such that $\alpha_i \geq 0$, $\sum_{i \in P} \alpha_i = 1$, and $\omega_i = \alpha_i \omega$. Now, suppose that there is a fixed number K such that for every $(\omega_i, u_i)_{i=1}^n$ in E, there are at most K endowment classes that partition the set of agents $\{1, \ldots, n\}$.

By Corollary 3, the homotheticity of utilities allow for the existence of a representative consumer for each of the K endowment classes. We can now find a Walrasian equilibrium for the economy populated by such representative consumers. From an equilibrium allocation and prices, one finds a final equilibrium allocation by solving the convex problem in Samuelson's Theorem. There is a fixed number of such problems to solve.

4. Proofs

We now present the proof of Theorem 1. Note that throughout the following we assume that for any concave maximization problem in $\mathbf{R}^l$ there is an algorithm that finds a solution in time polynomial in l. Thus, we are in effect reducing the calculation of a Walrasian equilibrium to a polynomial number of concave maximization. Importantly, these maximizations must be solved approximately; however we ignore this for the exposition in order to improve its clarity.

The key idea in the proof is to make use of the Negishi approach for proving existence of Walrasian equilibrium. We outline the Negishi approach and explain how we need to adapt it before proceeding with the proof.

4.0.1. *Proof overview: the Negishi approach.* Negishi's approach to equilibrium existence consists of exploiting the Second Welfare Theorem (SWT) to prove the existence of equilibria. We need the following definition:

An *Walrasian equilibrium with transfers* is a triple (p, x, T), where:

- $p \in \mathbf{R}^l_+$ (a vector of prices);
- $T \in \mathbf{R}^n$ and $\sum_{i=1}^n T_i = 0$ (a vector of transfers);
- x is an allocation (supply equals demand);
- for all i

$$u_i(y) > u_i(x_i) \Rightarrow p \cdot y > p \cdot \omega_i + T_i$$

  and $p \cdot x_i = p \cdot \omega_i + T_i$ (agents are maximizing utility).

Note that a Walrasian equilibrium in the usual sense is just a Walrasian equilibrium with transfers (p, x, T) in which $T = 0$; and that a Walrasian ε-equilibrium is one where $\|T\| \leq \varepsilon$.

Negishi (1960) proceeds by using welfare weights as stand-in for Pareto optimal allocations. For every vector of weights $(\lambda_1, \ldots, \lambda_n)$, where λ_i is the weight on agent i, the SWT establishes the existence of a Walrasian equilibrium with transfers (p, x, T) in which x maximizes $\sum_i \lambda_i u_i(x_i)$. One can then map λ into a vector of welfare weights that tries to "undo" the compensations introduced by the transfers T: these welfare weights would seek to reward agents who have transfers $T_i < 0$ and punish agents with transfers $T_i > 0$. Such a map will not have a fixed point λ for which the corresponding transfers are not all zero: it "moves" all λ with positive or negative transfers.

Negishi then uses Kakutani's fixed point theorem to establish the existence of a "neutral" vector of welfare weights: weights that induce zero transfers. The crucial feature of his approach is that the fixed point argument is done in a space of dimension n, not l.[7]

We adopt the main idea in Negishi's approach, but we need to deal with several complications. First, Kakutani's fixed point theorem is not constructive. Instead we base our algorithm on Sperner's lemma, the combinatorial underpinning of Kakutani's (and Brouwer's) theorem. In particular, we look directly for zero T, and do not explicitly use a fixed-point argument (although these ideas are related, of course).

Second, and more importantly, as the number of goods changes, the mapping from welfare weights to transfers can change in ways that are difficult to control. Our algorithm needs to be robust to such changes: this is perhaps the main difficulty in applying Negishi's approach as a computational devise. Here we exploit the concavity of utilities to obtain a Lipschitz bound on the mapping from welfare weights to transfers. The bound allows us to approximate zero transfers in a way that is independent of the number of goods in the economy. The number of goods only enters the problem when one obtains a Walrasian equilibrium with transfers for given welfare weights, and we only perform this computation a fixed number of times.

4.0.2. *Overview of Sperner's lemma.* The following is copied from Border (1989) with very minor adaptations.

A collection of vectors $x_0, \ldots, x_m$ in $\mathbf{R}^n$ is *affinely independent* if $\sum_{i=0}^m \theta_i x_i = 0$ and $\sum_{i=0}^m \theta_i = 0$ implies that $\theta_0 = \theta_1 = \ldots = \theta_m = 0$

A *m-simplex* is the set of all strictly positive convex combinations of an affinely independent set of $m+1$ vectors. A *closed m-simplex* is the convex hull of an affinely independent set of $m+1$ vectors. Given the affinely independent vectors $x_0, \ldots, x_m$, the simplex $\Delta(x_0, \ldots, x_m)$ is the set

$$\Delta(x_0, \ldots, x_m) = \left\{ \sum_{i=0}^m \theta_i x_i : \theta_i > 0, i = 0, \ldots, m; \sum_{i=0}^m \theta_i = 1 \right\}.$$

Here, each x_i is a *vertex* of $\Delta(x_0, \ldots, x_m)$ and each k-simplex $\Delta(x_{i_0}, \ldots, x_{i_k})$ is a *face* of $\Delta(x_0, \ldots, x_m)$. The *diameter* of a simplex is the largest distance between any two of its vertexes.

For each $y = \sum_{i=0}^m \theta_i x_i$ in the closure of $\Delta(x_0, \ldots, x_m)$, let $\chi(y) = \{i : \theta_i > 0\}$. Note that if $\chi(y) = \{i_0, \ldots, i_k\}$ then $y \in \Delta(x_{i_0}, \ldots, x_{i_k})$.

Denote by e_i the vector in $\mathbf{R}^n$ which has all its coordinates 0, except for a 1 in its ith coordinate. The *standard n-simplex* is the simplex

[7]For this reason is has been used in the study of Walrasian equilibrium with infinite dimensional commodity spaces. See, for example, Bewley (1991), Magill (1981) or Mas-Colell (1986).

$\Delta(e_1, \ldots, e_n)$, denoted simply as Δ. Note that

$$\overline{\Delta} = \{x \in \mathbf{R}^n : x \geq 0 \wedge \sum_i x_i = 1\}.$$

A *simplicial subdivision* of $\overline{\Delta}$ is a collection $A_1, \ldots, A_J$ of simplexes such that $\overline{\Delta} = \cup_{j=1}^J \overline{A}_j$ and for each $j \neq h$ $\overline{A}_j \cap \overline{A}_h$ is either empty or the closure of a common face. The *mesh* of a simplicial subdivision is the largest diameter of any its simplexes.

Fix a simplicial subdivision $A_1, \ldots, A_J$ of Δ. Let V denote the collection of all the vertices of $A_1, \ldots, A_J$. A function $f : V \to \{1, \ldots, n\}$ for which $f(v) \in \chi(v)$ for all $v \in V$ is called a *proper labeling* of the simplicial subdivision.

Theorem (Sperner's Lemma). *Let $A_1, \ldots, A_J$ be a simplicial subdivision of $\overline{\Delta}$, and f a proper labeling of this subdivision. Then there is (an odd number of) A_j such that f achieves all the values $\{1, \ldots, n\}$ on the vertices of A_j.*

See Border (1989) for a proof of Sperner's lemma. A simplex A_j for which f achieves all the values $\{1, \ldots, n\}$ is called *completely labeled*.

4.0.3. *Three simple lemmas.* Let $\varepsilon > 0$. We shall work with the norm $\|x\| = \sup_i |x_i|$.

For each $\lambda \in \overline{\Delta}$, consider the problem $\Pi(\lambda)$:

$$\max \textstyle\sum_{i=1}^n \lambda_i u_i(x_i)$$
$$\text{s.t.} \begin{cases} \sum_{i=1}^n x_i \leq \sum_{i=1}^n \omega_i \\ x_i \geq 0. \end{cases}$$

Let $x(\lambda)$ be a solution to the above problem. Since $\lambda \in \overline{\Delta}$ there is at least one h with $\lambda_h > 0$. By the Inada condition, we have that $x_h(\lambda) \in \mathbf{R}^l_{++}$. Let

$$p(\lambda) = \lambda_h D u_i(x_h(\lambda)).$$

Note that the first-order conditions of the problem $\Pi(\lambda)$ imply that the definition of $p(\lambda)$ does not depend on the chosen consumer h with $\lambda_h > 0$. Define, for all i,

$$T_i(\lambda) = p(\lambda) \cdot (x_i(\lambda) - \omega_i).$$

Define function $g : \overline{\Delta} \to \mathbf{R}^n$ by $g(\lambda)_i = T_i(\lambda)$.

Lemma 4. *For $\lambda, \lambda' \in \Delta$, $\|g(\lambda) - g(\lambda')\| \leq (n-1)\|\lambda - \lambda'\|$.*

Proof. The concavity of u_i implies that $Du_i(x)(\omega - x) \geq u(\omega) - u(x)$ (Rockafellar (1970) Theorem 23.2), so

$$\begin{aligned} T_i(\lambda) &= p(\lambda) \cdot (x_i(\lambda) - \omega_i) \\ &= \lambda_i Du_i(x_i(\lambda))(x_i(\lambda) - \omega) \\ &\leq \lambda_i(u_i(x_i(\lambda)) - u_i(\omega_i)) \leq \lambda_i(M - m). \end{aligned}$$

On the other hand, $\sum_j T_j(\lambda) = 0$ so

$$T_i(\lambda) = -\sum_{j \neq i} T_j(\lambda) \geq -\sum_{j \neq i}(M - m)\lambda_j \geq -(n-1)(M-m)\|\lambda\|.$$

Thus, $|T_i(\lambda)| \leq (n-1)(M-m)\|\lambda\|$. □

Lemma 4 implies that g is Lipschitz continuous with constant $K = (n-1)(M-m)$.

Lemma 5. *If* $\lambda_i = 0$ *then* $g(\lambda)_i \leq 0$.

Proof. If $x(\lambda)$ is a solution to $\Pi(\lambda)$ then $x_i(\lambda) = 0$, as utility functions are strictly monotonic. Then $T_i(\lambda) = -p(\lambda) \cdot \omega_i \leq 0$. □

Observe that $(x(\lambda), p(\lambda), T(\lambda))$ is a Walrasian equilibrium with transfers, as $p(\lambda)$ is chosen to satisfy the first-order conditions of all agents with $\lambda_i > 0$, and agents with $\lambda_i = 0$ are maximizing utility trivially.

Choose a simplicial subdivision of $\overline{\Delta}$ of mesh $\frac{\varepsilon}{K(n-1)^2}$. This means that if λ and λ' are vertexes in the same subsimplex, then $\|\lambda - \lambda'\| < \frac{\varepsilon}{K(n-1)^2}$. Note that, in that case, Lemma 4 and the continuity of g implies that $\|g(\lambda) - g(\lambda')\| < \frac{\varepsilon}{(n-1)^2}$.

If there is a vertex for which $g(\lambda) \leq 0$ then we are done, as we would have found λ such that $T_i(\lambda) \leq 0$. In that case, $\sum_{i=1}^n T_i(\lambda) = 0$ implies that $T = 0$ and we have found a Walrasian equilibrium. Suppose then that there is no such vertex. That is, for every vertex λ of the simplicial subdivision, there is an agent i with $g(\lambda)_i > 0$.

Define a labeling of the subsimplex as follows. Let the label of a vertex λ be the i for which the transfer to i in $g(\lambda)$ is largest; i.e. i is the largest component of $g(\lambda)$ (note it is strictly positive). If there is more than one, choose the smallest such i. By Lemma 5, the labeling thus constructed is a proper labeling because if $i \notin \chi(\lambda)$ then $\lambda_i = 0$ implies that $g(\lambda)_i \leq 0$.

By Sperner's lemma, there is a completely labeled subsimplex, say $\lambda^1, \ldots, \lambda^n$, where λ^i is labeled i. Let $\eta^i = g(\lambda^i)$. By construction,

$$\|\eta^i - \eta^j\| \leq \frac{\varepsilon}{(n-1)^2}.$$

Lemma 6. $\|\eta^i\| < \varepsilon$.

Proof. We shall prove that $\eta^i_i \leq \varepsilon/(n-1)$. This suffices to prove the lemma because if $\eta^i_j > 0$ then $\eta^i_j \leq \eta^i_i \leq \varepsilon$; and if $\eta^i_j < 0$ then $\sum_h \eta^i_h = 0$ implies that

$$\eta^i_j \geq -\sum_{h:\eta^i_h>0} \eta^i_h \geq -(n-1)\eta^i_i \geq -\varepsilon,$$

as η^i_i is the largest value of a component of η^i; so $\left|\eta^i_j\right| \leq \varepsilon$.

Suppose then, towards a contradiction, that there is i with $\eta^i_i > \varepsilon/(n-1)$. Since $\sum_j \eta^i_j = 0$, there is j with $\eta^i_j < -\varepsilon/(n-1)^2$. Then,

$$\|\eta^j - \eta^i\| \geq \left|\eta^j_j - \eta^i_j\right| \geq \left|\eta^i_j\right| > \frac{\varepsilon}{(n-1)^2}$$

where the first inequality is by definition of $\|\|$, the second because $\eta^j_j > 0$ and $\eta^i_j < 0$. But $\|\eta^j - \eta^i\| > \frac{\varepsilon}{(n-1)^2}$ contradicts the construction of the subsimplex. □

To sum up, the algorithm is as follows:

(1) Compute K from the given utilities and endowments.
(2) Construct a simplicial subdivision S of $\overline{\Delta}$ of mesh $\frac{\varepsilon}{K(n-1)^2}$.
(3) For each vertex λ of S, calculate $g(\lambda)$. If $g(\lambda) \leq 0$ then stop (and report an exact Walrasian equilibrium). If there is i with $g(\lambda)_i > 0$ then label the vertex by the smallest i of those with largest $g(\lambda)_i$.
(4) Find a completely labeled subsimplex, say $\lambda^1, \ldots, \lambda^n$. This can be done by exhaustive search or by some version of "Scarf's algorithm."
(5) Report $g(\lambda^1)$ as a Walrasian ε-equilibrium.

4.1. **Proof of Theorem 2.** The following lemma is needed to prove both statements in the theorem. It allows the algorithm to run on a subsimplex in which welfare weights are bounded away from zero.

Lemma 7. *Suppose that E satisfies the assumption of bounded endowments. Let $\eta > 0$ be such that $-\Theta + (n-1)\eta < 0$. Then for all $\lambda \in \Delta$, $\lambda_i \leq \eta$ implies that $g(\lambda)_i < 0$.*

Proof. Let $\lambda \in \Delta \setminus \Delta^\eta$. Let i be such that $\lambda_i \leq \eta$. Consider $\hat{\lambda}$, defined by

$$\hat{\lambda}_j = \begin{cases} \lambda_j + \lambda_i/(n-1) & \text{if } j \neq i \\ 0 & \text{if } j = i. \end{cases}$$

Then (by the same argument as in Lemma 5), $g(\hat{\lambda})_i = -p(\hat{\lambda})\omega_i$.

Now, $\|\hat{\lambda} - \lambda\| = \lambda_i \leq \eta$, so Lemma 4 implies that $\|g(\hat{\lambda}) - g(\lambda)\| \leq (n-1)\eta$. Then $\left|g(\hat{\lambda})_i - g(\lambda)_i\right| \leq (n-1)\eta$; so $g(\hat{\lambda})_i = -p(\hat{\lambda})\omega_i$ implies that

$$g(\lambda)_i \leq -p(\hat{\lambda})\omega_i + (n-1)\eta \leq -\Theta + (n-1)\eta < 0$$

□

Let $\Delta^\eta = \{\lambda \in \Delta : \lambda_i > \eta\}$.

We now use the algorithm in the previous section, but modified to run on the simplex $\overline{\Delta}^\eta$. Note that for any boundary λ, $\lambda \in \overline{\Delta}^\eta \setminus \Delta^\eta$, for all i with $\lambda_i = \eta$ we have that $g(\lambda)_i < 0$. The labeling described in the algorithm is therefore a proper labeling.

Let $\varepsilon' > 0$ be such that $\varepsilon' < \varepsilon\eta$. The algorithm outputs a Walrasian ε'-equilibrium (x, p). We shall prove that (x, p) is an ε-approximate equilibrium.

Observe that η, and therefore ε', depend on $(n-1)$ and Θ. It is therefore constant across the economies in E.

Let (x, p) be the outcome of the algorithm. Note that since $\lambda_i \geq \eta > 0$ for all i we have that $x_i \in \mathbf{R}^l_{++}$. Let x_i^* be a solution to

$$\begin{aligned} &\max u_i(\tilde{x}_i) \\ &s.t.\ p \cdot \tilde{x} \leq p \cdot \omega_i. \end{aligned}$$

We shall prove that $u_i(\tilde{x}_i) - u_i(x_i) < \varepsilon$. First, if $p \cdot x \geq p \cdot \omega_i$ there is nothing to prove, as the desired conclusion follows from the definition of Walrasian ε-equilibrium. Let us then assume that $p \cdot x < p \cdot \omega_i$.

By the concavity of u_i,

$$\begin{aligned} u_i(x_i^*) - u_i(x_i) &\leq Du_i(x_i) \cdot (x_i^* - x_i) \\ &= (1/\lambda_i)p \cdot (x_i^* - x_i) \\ &\leq (1/\eta)p \cdot \omega_i^* - p \cdot x_i \\ &\leq (1/\eta)\varepsilon' < \varepsilon. \end{aligned}$$

We now proceed to prove that the algorithm finds a strong ε-approximate equilibrium.

Let η be as above, and let $\varepsilon' > 0$ be such that

$$\varepsilon' < \frac{\varepsilon\eta\pi}{n}.$$

The algorithm on the $\overline{\Delta}^\eta$ simplex outputs a Walrasian ε'-equilibrium (x, p). Observe that η and ε', depend on $(n-1)$, π and Θ. It is therefore constant across the economies in E.

Let x_i^* be a solution to

$$\max u_i(\tilde{x}_i)$$
$$s.t.\ p \cdot \tilde{x} \leq p \cdot \omega_i.$$

This defines $x^* \in \mathbf{R}_+^{nl}$. We shall prove that (x^*, p) is a strong ε-approximate equilibrium.

Define, for $1 \leq s \leq l$, $y^s \in \mathbf{R}_+^l$ by

$$y_h^s = \begin{cases} x_h & \text{if } h \neq s \\ x_h + \frac{p \cdot (x_i^* - x_i)}{p_h} & \text{if } h = s. \end{cases}$$

and $\theta_s = \frac{p_s \cdot (x_{si}^* - x_{si})}{p \cdot (x_i^* - x_i)}$ (here x_{si}^* denotes the amount of good s in bundle x^*, and similarly for x_{si}). So $\sum_s \theta_s = 1$. Then it is easy to verify that

$$x^* = \sum_{s=1}^{l} \theta_s y^s.$$

Note that $p = \lambda_i Du_i(x_i)$, so Lemma 7 implies that, for any good s, $p_s \geq \eta\pi$. Then, using the expression of x^* in terms of the vectors y^s we obtain:

$$\|x_i^* - x_i\| \leq \sup\{\frac{|p \cdot (x_i^* - x_i)|}{p_h}\} \leq \sup\{\frac{|p \cdot \omega_i - p \cdot x_i|}{\eta\pi}\} < \frac{\varepsilon'}{\eta\pi}$$

Then

$$\|x^* - x\| \leq \sum_i \|x_i^* - x_i\| < n\frac{\varepsilon'}{\eta\pi} < \varepsilon,$$

where the last inequality follows from the choice of ε'.

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