Computational Game Theory | **Spring Semester, 2011/2012**

Basic Network Creation Games - Survey

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1 Survey introduction

This paper contains a survey of the basic network creation games. It consists of a presentation of the topic and results as they were shown in the original article. The survey does not intend to replace the article, but to show a brief review of the topic and the results in the article. In addition to the review of the article, the paper contains a personal additions section with a brief personal thought of me about the setup, and it also contains a proof I have made about the existance of swap equilibrium in a general graph, for both Max-games and Sum-games.

2 Basic Network Creation Games

2.1 Introduction

The topic of basic network creation games was introduced in an article in 2011 by Alon et. al.

Network creation games are games where there are N players that build themselves a network to serve them best reaching the other players. The various versions of the game differ by the rules of the games.

The model that the article discusses is:

1. N agents that want to connect each other
2. Each agent wants to minimize its usage cost (sum or max - later will be explained)
3. Each agent can only perform edge swaps, which means swapping incident edge with another incident edge if it decreases its usage cost

Usage cost for the sum version is defined for a vertex v simply as the sum of the distances to all other vertices. The graph is in sum equilibrium if for every edge vw, swapping vw with vw' doesn't decrease the sum of distances of v. For the max version, the usage cost is defined by the *local diameter* of the vertex. Local diameter of a vertex v is the maximum distance between v to any other vertex in the graph. The graph is in max equilibrium if exists for every edge vw, swapping vw with vw' doesn't decrease the local diameter of v. The graph is in max equilibrium if the property holds for every vertex. The graphs that were designed in the article have 2 additional properties, they are *insertion-stable* and *deletion-crtical*. Insertion-stable means that adding another edge does not decrease the local diameter of either endpoint. Deletion-critical means that removing any edge decreases the local diameter of some endpoint. If a graph is both deletion-critical and insertion-stable, it is definitely in max-equilibrium.

The difference of that model from previous models presented is the space of valid steps in game (swaps only) and the equality of all edges (all valued as 1).

2.2 Motivation

The advantage of the problem as it is formed in the article is that the search for equilibrium is polynomially bound (by the number of swaps). The problem of finding nash-equilibria in network creation games was unrealistic since the problem for agent to determine if he is in nash equilibrium is NP-Complete itself.

Another motivation to form the model as it is, was to force fixed usage cost as 1, since many similar models of network creation games have gotten to a state where the various bounds that were proved on them, depended on the range of α.

The writers of the article state that finding bounds on this simpler model, will help resolving the same problems in the similar models of network creation games.

2.3 The results in the article

The article has proven numerous bounds on the model.

First, it discussed the problem of finding equilibrium trees. In Sum-games, it was proven that only one equilbrium tree exists with diameter 2, and therefore it's the star. They disproved the existance of diameter 3 equilibrium tree.

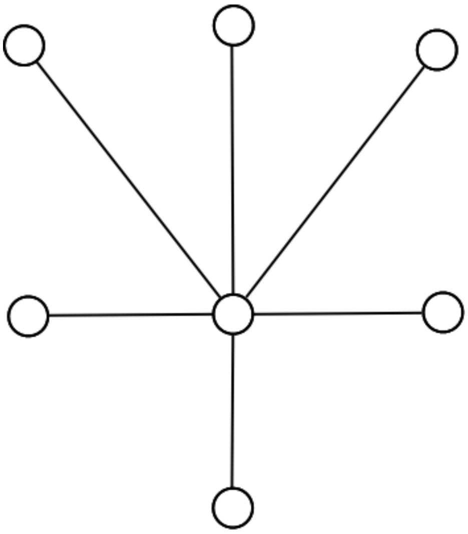

Figure 1: Example of star (of diameter 2)

In Max-Games, the writers proved that the diameter of an equilibrium tree, is at most of size 3 and showed the existance of such tree. In this case, a star also a max-equilibrium tree, hence a tree of diameter 2 exists.

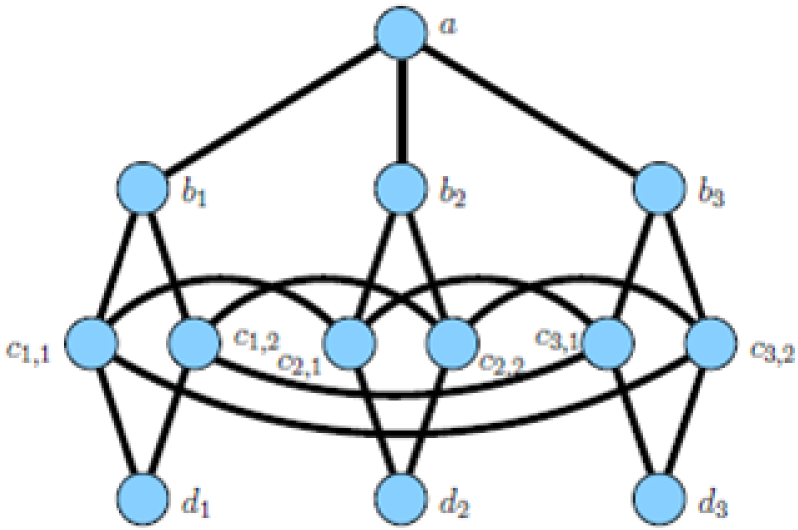

Figure 2: Diameter 3 Max-equilibrium

In the field of general networks, the writers proved that a diameter 3 sum-equilibrium graph exists, and ruled out a conjecture made by Albers et. al, that all sum-equilibrium graphs are of diameter 2. They also proved that the diameter in sum-equilibrium graph is $2^{O(\sqrt{logn})}$.

For sum-games in general networks, the writers proved the existance of a graph of $\theta(\sqrt{n})$ diameter

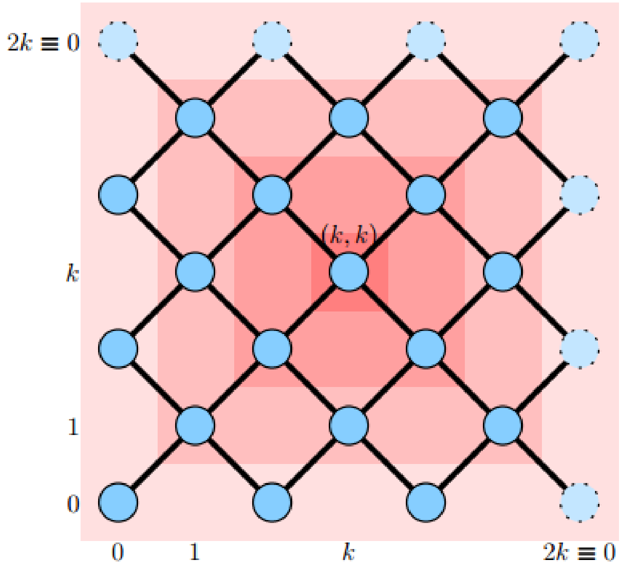

Figure 3: Max-equilibrium graph with $O(\sqrt{n})$ diameter

The graph in figure 3 is a sketch of an example of a $2k * 2k$ graph where to local diameter of every vertex is k. It was shown in the article that the graph is of local diaemeter k and also that it is both deletion-critical and insertion-stable.

3 Personal additions

3.1 The influence of the model

The model that was presented in the article is interesting in the point of view of simplifying a long studied problem and the writers have been able to prove solid bounds in the article. Though, I did not find an evidence to the relation between that model to the previously discussed models in network creation games. It does make sense that one player can swap links that incident to him, but the fact that all links cost the same without any consideration of distance or any general term of cost is outside the scope of the original problem in my opinion.

3.2 The existance of equilibrium

The article attempted to prove different bound regarding swap equilibrium in different setups that included the type of the equilibrium graph (tree or general graph) and also the type of the game which differed between Max-game and Sum-game.

I will attempt to prove further theorems regarding the reachabillity of equilibrium given a graph and a game. Note that the upcoming assume that the model as it was described, under the assumption that the model doesn't allow deletions of edges (swaps with the same edge), as it becomes trivial. I will prove that given a graph, there exists a graph with the same amount of edges and vertices, which is swap equilibrium. For the proof, first I will complete to prove a lemma that was shown in the paper, without a proof.

Lemma 0.1 *For a vertex v of local diameter 2, swapping an incident edge does not improve the sum of distances from v, nor the local diameter.*

Proof Assume that we want to swapp vw with vw'. d(v,w') decreases to 1 (previous distance was 2), but d(v,w) which was 1 increases to 2. The distances to all other vetices ($V - \{w'\}$) cannot improve as compared with the previous state since only the edge vw' was added, and it cannot shorten path to other vertices, since they were of length 2 already. Thererfore, swaps doesn't improve the local diameter and thefore the max and sum equilibrium holds.

Theorem 0.2 *Given a graph G = (V,E) and Sum-game, there exists an equilibrium graph $G' = (V', E')$ where $|V'| = |V|$ and $|E'| = |E| \iff |E| \geq |V| - 1$ or $|V| = 1$ or $|E| = 0$*

Proof $\Rightarrow$ Assume that exists equilibrium graph as stated and for the sake of contradiction that $|E| < |V| - 1$, then the graph is not in the form of a tree, and must consist of at least 2 connected components. A vertex v for component A w.l.o.g, would prefer swap to a vertex in other component, otherwise its sum of distances to other vertices is ∞, as well as the local diameter and that contradicts the fact that the graph is in swap equilibrium.

$\Leftarrow$ Given a graph G with $|E| \geq |V| - 1$, the equilibrium graph can be constructed as a base of 2-diameter star with edges added. The graph is in swap equilibrium both in Max-game and Sum-game, as the diameter of star is 2 (it is constructed of $|V| - 1$ edges), and by adding additional edges the diameter of the vertices cannot decrease. No vertex would prefer swap, as it cannot decrease its local diameter or sum of distances to other vertices, according to the lemma.

Special cases where equilibrium exists are when $|E| = 0$ and even though distances from each vertex to another are ∞, no swaps can be performed and also when $|V|$ = 1 where trivially the graph is in equilibrium. □